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# Decomposition of symmetric multivariate density function

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**Abstract.** For a $T$-variate density function, the present article defines the quasi-symmetry of order $k$ ($< T$) and the marginal symmetry of order $k$, and gives the theorem that the density function is $T$-variate permutation symmetric if and only if it is quasi-symmetric and marginal symmetric of order $k$. The theorem is illustrated for the multivariate normal density function.

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## §1. Introduction

For analysis of square contingency tables, it is known that the symmetry model holds if and only if both the quasi-symmetry and marginal homogeneity models hold (for example, see Caussinus [3], Tomizawa and Tahata [6]). For multi-way contingency tables, Bhapkar and Darroch [1] defined the complete symmetry, quasi-symmetry and marginal symmetry models, and showed that the complete symmetry model holds if and only if both the quasi-symmetry and marginal symmetry models hold.

By the way, a similar decomposition for bivariate density function (instead of cell probabilities) is given by Tomizawa, Seo and Minaguchi [5]. Let $X$ and $Y$ be two continuous random variables with a density function $f(x, y)$. The density function $f(x, y)$ is said to be symmetric if we have

$$f(x, y) = f(y, x) \quad \text{for every } (x, y) \in \mathbf{R}^2;$$

see Tong [7]. Tomizawa, et al. [5] defined quasi-symmetry and marginal homogeneity for the density function, and gave the theorem that the density function $f(x, y)$ is symmetric if and only if it is both quasi-symmetric and marginal homogeneous.

Let the support of $f(x, y)$ denote $K^2$, where

$$K^2 = \{(x, y) : f(x, y) > 0\}.$$

We assume that the support of $f(x, y)$ is an open connected set in $\mathbf{R}^2$. Also, let $\theta(s_1, s_2; t_1, t_2)$ be the odds-ratio for $X$-values $s_1, s_2$ and $Y$-values $t_1, t_2$; namely,

$$\theta(s_1, s_2; t_1, t_2) = \frac{f(s_1, t_1)f(s_2, t_2)}{f(s_2, t_1)f(s_1, t_2)}.$$

Then the density function $f(x, y)$ is said to be quasi-symmetric if we have

$$\theta(s_1, s_2; t_1, t_2) = \theta(t_1, t_2; s_1, s_2)$$

for any $(s_i, t_j) \in K^2$. Thus this indicates that the density function is symmetric with respect to the odds-ratio. The density function $f(x, y)$ is said to be marginal homogeneous if we have

$$f_X(t) = f_Y(t) \quad \text{for every } t \in \mathbf{R},$$

where $f_X(t)$ and $f_Y(t)$ are the marginal density functions of $X$ and $Y$, respectively. Now, we are interested in extending the decomposition of the symmetric density function in multivariate case.

In this article, we define the quasi-symmetry and marginal symmetry for multivariate density function, and decompose the symmetry into quasi-symmetry and marginal symmetry. Section 2 provides the decomposition for trivariate density function. Section 3 extends the decomposition to multivariate density function. Section 4 illustrates our decompositions for normal distributions. Section 5 describes some comments.

## §2. Decomposition of trivariate density function

Let $X_1, X_2$ and $X_3$ be three continuous random variables with a density function $f(x_1, x_2, x_3)$. The density function $f(x_1, x_2, x_3)$ is said to be permutation symmetric ($S^3$) if for each permutation $(\pi_1, \pi_2, \pi_3)$ of $(1, 2, 3)$ and every $(x_1, x_2, x_3) \in \mathbf{R}^3$, we have

$$f(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = f(x_1, x_2, x_3);$$

see Tong [7], and Fang, Kotz and Ng [4].

Let $f_{X_1}(x_1), f_{X_2}(x_2)$ and $f_{X_3}(x_3)$ be the marginal density functions of $X_1, X_2$ and $X_3$, respectively. For the density function $f(x_1, x_2, x_3)$, we shall define marginal symmetry of order 1 (denoted by $M_1^3$) by

$$M_1^3 : f_{X_1}(t) = f_{X_2}(t) = f_{X_3}(t) \quad \text{for every } t \in \mathbf{R}.$$

Also, we define marginal symmetry of order 2 (denoted by $M_2^3$) by

$$M_2^3 : f_{X_1X_2}(s,t) = f_{X_1X_2}(t,s) = f_{X_1X_3}(s,t) = f_{X_2X_3}(s,t) \text{ for every } (s,t) \in \mathbf{R}^2.$$

Thus, $M_2^3$ indicates that each of marginal distributions of $(X_1, X_2)$, $(X_1, X_3)$ and $(X_2, X_3)$ has a same bivariate density function being symmetric. Note that $M_2^3$ implies $M_1^3$.

Let the support of $f(x_1, x_2, x_3)$ denote $K^3$, where

$$K^3 = \{(x_1, x_2, x_3) : f(x_1, x_2, x_3) > 0, a < x_i < b, i = 1, 2, 3, -\infty \le a < b \le \infty\}.$$

We assume that the support of $f(x_1, x_2, x_3)$ is an open connected set in $\mathbf{R}^3$. Generally, we can express the density function as

$$\begin{aligned} f(x_1, x_2, x_3) = {}& \mu\alpha_1(x_1)\alpha_2(x_2)\alpha_3(x_3) \times \\ & \beta_{12}(x_1, x_2)\beta_{13}(x_1, x_3)\beta_{23}(x_2, x_3)\gamma(x_1, x_2, x_3), \end{aligned} \tag{2.1}$$

where $(x_1, x_2, x_3) \in K^3$, and for an arbitrary fixed value $c \in (a, b)$,

$$\begin{gathered} \alpha_1(c) = 1, \quad \beta_{12}(c, x_2) = \beta_{12}(x_1, c) = 1, \\ \gamma(c, x_2, x_3) = \gamma(x_1, c, x_3) = \gamma(x_1, x_2, c) = 1, \end{gathered}$$

with similar properties of $\alpha_2, \alpha_3, \beta_{13}$ and $\beta_{23}$. The terms $\alpha_i$ correspond to main effects of the variable $X_i$, $\beta_{ij}$ to interaction effects of $X_i$ and $X_j$, and $\gamma$ to interaction effect of $X_1$, $X_2$ and $X_3$. Namely

$$\begin{aligned} & \mu = f(c,c,c), \\ & \alpha_1(x_1) = \frac{f(x_1,c,c)}{f(c,c,c)}, \quad \alpha_2(x_2) = \frac{f(c,x_2,c)}{f(c,c,c)}, \quad \alpha_3(x_3) = \frac{f(c,c,x_3)}{f(c,c,c)}, \\ & \beta_{12}(x_1,x_2) = \frac{f(x_1,x_2,c)f(c,c,c)}{f(x_1,c,c)f(c,x_2,c)}, \\ & \beta_{13}(x_1,x_3) = \frac{f(x_1,c,x_3)f(c,c,c)}{f(x_1,c,c)f(c,c,x_3)}, \\ & \beta_{23}(x_2,x_3) = \frac{f(c,x_2,x_3)f(c,c,c)}{f(c,x_2,c)f(c,c,x_3)}, \\ & \gamma(x_1,x_2,x_3) = \frac{f(x_1,x_2,x_3)f(x_1,c,c)f(c,x_2,c)f(c,c,x_3)}{f(c,c,c)f(x_1,x_2,c)f(x_1,c,x_3)f(c,x_2,x_3)}. \end{aligned}$$

The term $\alpha_1(x_1)$ indicates the odds of density function with respect to $X_1$-values with $(X_2, X_3) = (c, c)$. Note that

$$\begin{aligned}\beta_{12}(x_1, x_2) &= \left(\frac{f(x_1, x_2, c)}{f(c, x_2, c)}\right) \Big/ \left(\frac{f(x_1, c, c)}{f(c, c, c)}\right) \\ &= \left(\frac{f(x_1, x_2, c)}{f(x_1, c, c)}\right) \Big/ \left(\frac{f(c, x_2, c)}{f(c, c, c)}\right),\end{aligned}$$

and

$$\begin{aligned}\gamma(x_1, x_2, x_3) &= \left(\frac{f(x_1, x_2, x_3) f(c, c, x_3)}{f(x_1, c, x_3) f(c, x_2, x_3)}\right) \Big/ \left(\frac{f(x_1, x_2, c) f(c, c, c)}{f(x_1, c, c) f(c, x_2, c)}\right) \\ &= \left(\frac{f(x_1, x_2, x_3) f(c, x_2, c)}{f(x_1, x_2, c) f(c, x_2, x_3)}\right) \Big/ \left(\frac{f(x_1, c, x_3) f(c, c, c)}{f(x_1, c, c) f(c, c, x_3)}\right) \\ &= \left(\frac{f(x_1, x_2, x_3) f(x_1, c, c)}{f(x_1, x_2, c) f(x_1, c, x_3)}\right) \Big/ \left(\frac{f(c, x_2, x_3) f(c, c, c)}{f(c, x_2, c) f(c, c, x_3)}\right).\end{aligned}$$

Thus, $\beta_{12}(x_1, x_2)$ indicates the odds-ratio of density function with respect to $(X_1, X_2)$-values with $X_3 = c$. Also $\gamma(x_1, x_2, x_3)$ indicates the ratio of odds-ratios of density function, i.e., the ratio of odds-ratio with respect to $(X_1, X_2)$-values with $X_3 = x_3$ to that with $X_3 = c$ (or the ratio of odds-ratio with respect to $(X_i, X_j)$-values with $X_k = x_k$ to that with $X_k = c$, where $(i, j, k) = (1, 3, 2)$ and $(2, 3, 1)$).

The density function is $S^3$ if and only if it is expressed as the form (2.1) with

$$S^3 : \begin{cases} \alpha_1(x_1) = \alpha_2(x_1) = \alpha_3(x_1), \\ \beta_{12}(x_1, x_2) = \beta_{12}(x_2, x_1) = \beta_{13}(x_1, x_2) = \beta_{23}(x_1, x_2), \\ \gamma(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = \gamma(x_1, x_2, x_3). \end{cases}$$

We shall define quasi-symmetry of order 1 (denoted by $Q_1^3$), and order 2 (denoted by $Q_2^3$). We define $Q_1^3$ by (2.1) with

$$Q_1^3 : \begin{cases} \beta_{12}(x_1, x_2) = \beta_{12}(x_2, x_1) = \beta_{13}(x_1, x_2) = \beta_{23}(x_1, x_2), \\ \gamma(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = \gamma(x_1, x_2, x_3). \end{cases}$$

Thus $Q_1^3$ indicates

$$\begin{aligned}&\theta(s_1, s_2; t_1, t_2; u) = \theta(t_1, t_2; s_1, s_2; u) \\ &= \theta(s_1, s_2; u; t_1, t_2) = \theta(t_1, t_2; u; s_1, s_2) \\ &= \theta(u; s_1, s_2; t_1, t_2) = \theta(u; t_1, t_2; s_1, s_2),\end{aligned}$$

where $(s_i, t_j, u) \in K^3$ and so on, and

$$\theta(s_1, s_2; t_1, t_2; u) = \frac{f(s_1, t_1, u) f(s_2, t_2, u)}{f(s_2, t_1, u) f(s_1, t_2, u)},$$
$$\theta(s_1, s_2; u; t_1, t_2) = \frac{f(s_1, u, t_1) f(s_2, u, t_2)}{f(s_2, u, t_1) f(s_1, u, t_2)},$$
$$\theta(u; s_1, s_2; t_1, t_2) = \frac{f(u, s_1, t_1) f(u, s_2, t_2)}{f(u, s_2, t_1) f(u, s_1, t_2)};$$

because we can see

$$\theta(s_1, s_2; t_1, t_2; u) = \frac{\theta(c, s_1; c, t_1; u)\theta(c, s_2; c, t_2; u)}{\theta(c, s_2; c, t_1; u)\theta(c, s_1; c, t_2; u)},$$

and so on. Therefore $Q_1^3$ indicates that the density function is symmetric with respect to the odds-ratio.

Also, we define $Q_2^3$ by (2.1) with

$$Q_2^3 : \gamma(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = \gamma(x_1, x_2, x_3).$$

Thus $Q_2^3$ indicates

$$\begin{aligned} \frac{\theta(s_1, s_2; t_1, t_2; u_1)}{\theta(s_1, s_2; t_1, t_2; u_2)} &= \frac{\theta(t_1, t_2; s_1, s_2; u_1)}{\theta(t_1, t_2; s_1, s_2; u_2)} \\ = \frac{\theta(s_1, s_2; u_1; t_1, t_2)}{\theta(s_1, s_2; u_2; t_1, t_2)} &= \frac{\theta(t_1, t_2; u_1; s_1, s_2)}{\theta(t_1, t_2; u_2; s_1, s_2)} \\ = \frac{\theta(u_1; s_1, s_2; t_1, t_2)}{\theta(u_2; s_1, s_2; t_1, t_2)} &= \frac{\theta(u_1; t_1, t_2; s_1, s_2)}{\theta(u_2; t_1, t_2; s_1, s_2)}, \end{aligned}$$

where $(s_i, t_j, u_k) \in K^3$ and so on; because

$$\frac{\theta(s_1, s_2; t_1, t_2; u_k)}{\theta(s_1, s_2; t_1, t_2; c)} = \frac{\gamma(s_1, t_1, u_k)\gamma(s_2, t_2, u_k)}{\gamma(s_2, t_1, u_k)\gamma(s_1, t_2, u_k)}.$$

Therefore $Q_2^3$ indicates that the density function is symmetric with respect to the ratio of odds-ratios. We point out that each of $S^3$, $Q_1^3$ and $Q_2^3$ does not depend on the value of $c$ fixed. It is obviously that $Q_1^3$ implies $Q_2^3$. Note that the alternative way of expressing $Q_1^3$ is

$$Q_1^3 : f(x_1, x_2, x_3) = \theta_1(x_1)\theta_2(x_2)\theta_3(x_3)v(x_1, x_2, x_3),$$

where $v$ is positive and permutation symmetric function, i.e., $v(x_{\pi_1}, x_{\pi_2}, x_{\pi_3}) = v(x_1, x_2, x_3)$. We obtain the following theorem.

**Theorem 1.** *For k fixed ($k = 1, 2$), the trivariate density function $f(x_1, x_2, x_3)$ is $S^3$ if and only if it is both $Q_k^3$ and $M_k^3$.*

Referring to Bhapkar and Darroch [1] for discrete probabilities in multi-way contingency tables, we can prove theorem for multivariate density function as follows.

*Proof.* Consider the case of $k = 1$. If a density function is $S^3$, then it satisfies $Q_1^3$ and $M_1^3$. Assume that it is both $Q_1^3$ and $M_1^3$, and then we shall show that it satisfies $S^3$.

Let $f^*(x_1, x_2, x_3)$ be the density function which satisfies both $Q_1^3$ and $M_1^3$. Since $f^*(x_1, x_2, x_3)$ satisfies $Q_1^3$, we see

$$\log f^*(x_1, x_2, x_3) = \log \theta_1(x_1) + \log \theta_2(x_2) + \log \theta_3(x_3) + \log v(x_1, x_2, x_3),$$

where $v$ is positive and permutation symmetric function. Let the density $g(x_1, x_2, x_3)$ be $c^{-1}v(x_1, x_2, x_3)$ with $c = \iiint v(x_1, x_2, x_3)dx_1dx_2dx_3$. Also, since $f^*(x_1, x_2, x_3)$ satisfies $M_1^3$, we see

$$f^*_{X_1}(t) = f^*_{X_2}(t) = f^*_{X_3}(t) = \mu(t) \quad \text{for } t \in \mathbf{R}, \tag{2.2}$$

where $f^*_{X_1}(t), f^*_{X_2}(t)$ and $f^*_{X_3}(t)$ are the marginal density functions of $X_1, X_2$ and $X_3$, respectively. Consider the arbitrary density function $f(x_1, x_2, x_3)$ satisfying $M_1^3$ with

$$f_{X_1}(t) = f_{X_2}(t) = f_{X_3}(t) = \mu(t) \quad \text{for } t \in \mathbf{R}, \tag{2.3}$$

where $f_{X_1}(t), f_{X_2}(t)$ and $f_{X_3}(t)$ are the marginal density functions of $X_1, X_2$ and $X_3$, respectively. From (2.2) and (2.3), we see

$$\iiint \{f(x_1, x_2, x_3) - f^*(x_1, x_2, x_3)\} \times \log\left(\frac{f^*(x_1, x_2, x_3)}{g(x_1, x_2, x_3)}\right) dx_1dx_2dx_3 = 0. \tag{2.4}$$

Using the equation (2.4), we obtain

$$I(f, g) = I(f^*, g) + I(f, f^*),$$

where

$$I(h_1, h_2) = \iiint h_1(x_1, x_2, x_3) \log\left(\frac{h_1(x_1, x_2, x_3)}{h_2(x_1, x_2, x_3)}\right) dx_1dx_2dx_3.$$

For $g$ fixed, we see

$$\min_f I(f, g) = I(f^*, g),$$

and then $f^*$ uniquely minimizes $I(f, g)$.

Let $f^{**}(x_1, x_2, x_3) = f^*(x_1, x_3, x_2)$. In a similar way, we also see

$$\iiint \{f(x_1, x_2, x_3) - f^{**}(x_1, x_2, x_3)\} \log\left(\frac{f^{**}(x_1, x_2, x_3)}{g(x_1, x_2, x_3)}\right) dx_1 dx_2 dx_3 = 0,$$

where $f(x_1, x_2, x_3)$ is $M_1^3$ with (2.3). Thus, we obtain

$$I(f, g) = I(f^{**}, g) + I(f, f^{**}).$$

For $g$ fixed, we see

$$\min_f I(f, g) = I(f^{**}, g),$$

and then $f^{**}$ uniquely minimizes $I(f, g)$. Therefore, we see $f^*(x_1, x_2, x_3) = f^{**}(x_1, x_2, x_3)$. Thus, $f^*(x_1, x_2, x_3) = f^*(x_1, x_3, x_2)$.

Also, in a similar way, we obtain

$$f^*(x_1, x_2, x_3) = f^*(x_2, x_1, x_3) = f^*(x_2, x_3, x_1) = f^*(x_3, x_1, x_2) = f^*(x_3, x_2, x_1).$$

Therefore, we have $f^*(x_1, x_2, x_3) = f^*(x_{\pi_1}, x_{\pi_2}, x_{\pi_3})$. Namely $f^*(x_1, x_2, x_3)$ satisfies $S^3$. The case of $k = 2$ can be proved in a similar way as the case of $k = 1$. So the proof is completed.

## §3. Decomposition of multivariate density function

Let $X_1, \ldots, X_T$ be $T$ continuous random variables with a density function $f(x_1, \ldots, x_T)$. The density function $f(x_1, \ldots, x_T)$ is said to be permutation symmetric ($S^T$) if for each permutation $(\pi_1, \ldots, \pi_T)$ of $(1, \ldots, T)$ and every $(x_1, \ldots, x_T) \in \mathbf{R}^T$, we have

$$f(x_{\pi_1}, \ldots, x_{\pi_T}) = f(x_1, \ldots, x_T);$$

see Tong [7] and Fang et al. [4].

Let the support of $f(x_1, \ldots, x_T)$ denote $K^T$, where

$$K^T = \{(x_1, \ldots, x_T) : f(x_1, \ldots, x_T) > 0, \\ a < x_i < b, i = 1, \ldots, T, -\infty \le a < b \le \infty\}.$$

We assume that the support of $f(x_1, \ldots, x_T)$ is an open connected set in $\mathbf{R}^T$. Generally, we can express the density function as

$$f(x_1, \ldots, x_T) = \alpha \Big[\prod_{i_1=1}^{T} \alpha_{i_1}(x_{i_1})\Big]\Big[\prod\prod_{1 \le i_1 < i_2 \le T} \alpha_{i_1 i_2}(x_{i_1}, x_{i_2})\Big] \times \cdots \\ \times \Big[\prod \cdots \prod_{1 \le i_1 < \cdots < i_{T-1} \le T} \alpha_{i_1 \ldots i_{T-1}}(x_{i_1}, \ldots, x_{i_{T-1}})\Big] \cdot \alpha_{1 \ldots T}(x_1, \ldots, x_T), \tag{3.1}$$

where $(x_1, \dots, x_T) \in K^T$, and for an arbitrary fixed value $c \in (a, b)$,

$$\{\alpha_i(c) = \alpha_{i_1 i_2}(c, x_{i_2}) = \cdots = \alpha_{1\dots T}(x_1, \dots, x_{T-1}, c) = 1\}.$$

Then, the density function $f(x_1, \dots, x_T)$ being $S^T$ is also expressed as (3.1) with

$$S^T : \alpha_{i_1\dots i_m}(x_{i_1}, \dots, x_{i_m}) = \alpha_{i_1\dots i_m}(x_{\pi_{i_1}}, \dots, x_{\pi_{i_m}}) = \alpha_{j_1\dots j_m}(x_{i_1}, \dots, x_{i_m})$$
$$(m = 1, \dots, T; 1 \le i_1 < \cdots < i_m \le T; 1 \le j_1 < \cdots < j_m \le T),$$

where $(\pi_{i_1}, \dots, \pi_{i_m})$ is permutation of $(i_1, \dots, i_m)$.

For $k = 1, \dots, T-1$, we shall define quasi-symmetry of order $k$ (denoted by $Q_k^T$) by (3.1) with

$$Q_k^T : \alpha_{i_1\dots i_m}(x_{i_1}, \dots, x_{i_m}) = \alpha_{i_1\dots i_m}(x_{\pi_{i_1}}, \dots, x_{\pi_{i_m}}) = \alpha_{j_1\dots j_m}(x_{i_1}, \dots, x_{i_m})$$
$$(m = k+1, \dots, T; 1 \le i_1 < \cdots < i_m \le T; 1 \le j_1 < \cdots < j_m \le T).$$

Also, for $k = 1, \dots, T-1$, we shall define marginal symmetry of order $k$ (denoted by $M_k^T$) by

$$M_k^T : f_{X_{i_1}\dots X_{i_k}}(x_{i_1}, \dots, x_{i_k}) = f_{X_{i_1}\dots X_{i_k}}(x_{\pi_{i_1}}, \dots, x_{\pi_{i_k}}) = f_{X_{j_1}\dots X_{j_k}}(x_{i_1}, \dots, x_{i_k})$$
$$(1 \le i_1 < \cdots < i_k \le T; 1 \le j_1 < \cdots < j_k \le T),$$

where $f_{X_{i_1}\dots X_{i_k}}$ is the marginal density function of $(X_{i_1}, \dots, X_{i_k})$. Then we obtain the following theorem.

**Theorem 2.** *For $k$ fixed $(k = 1, \dots, T-1)$, the multivariate density function $f(x_1, \dots, x_T)$ is $S^T$ if and only if it is both $Q_k^T$ and $M_k^T$.*

The proof of Theorem 2 is omitted because it is obtained in a similar way to the proof of Theorem 1.

## §4. Symmetry of multivariate normal density function

**Example 1.** Consider a $T$-dimensional random vector $\boldsymbol{X} = (X_1, \dots, X_T)'$ having a normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_T)'$ and covariance matrix $\boldsymbol{\Sigma}$. The density function is

$$(4.1)\quad f(x_1, \dots, x_T) = \frac{1}{(2\pi)^{\frac{T}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\Big\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\Big\}.$$

Denote $\boldsymbol{\Sigma}^{-1}$ by $\boldsymbol{A} = (a_{ij})$ with $a_{ij} = a_{ji}$. Then the density function can be expressed as

$$f(x_1, \dots, x_T) = C \exp\Big\{ -\frac{1}{2} H \Big\},$$

where $C$ is positive constant and

$$H = \sum_{s=1}^{T} a_{ss}x_s^2 + \sum_{s \neq t} a_{st}x_s x_t - 2\sum_{s=1}^{T}\sum_{t=1}^{T} a_{st}\mu_s x_t.$$

By setting $c = 0$ without loss of generality, we see

$$\alpha_i(x_i) = \exp\Big\{ -\frac{1}{2}(a_{ii}x_i^2 - 2\sum_{s=1}^{T} a_{si}\mu_s x_i)\Big\} \quad (i = 1, \ldots, T),$$

$$\alpha_{ij}(x_i, x_j) = \exp\left(-a_{ij}x_i x_j\right) \quad (i < j), \tag{4.2}$$

and for $m = 3, \ldots, T$,

$$\alpha_{i_1 \ldots i_m}(x_{i_1}, \ldots, x_{i_m}) = 1 \quad (1 \leq i_1 < \cdots < i_m \leq T).$$

Therefore the density function (4.1) is $Q_k^T$ for $k = 2, \ldots, T-1$. Also from (4.2), the density function (4.1) is $Q_1^T$ if and only if $\{a_{ij}\ (= a_{ji})\}$ are constant (e.g., equals $w$) for all $i < j$; namely, $\boldsymbol{\Sigma}^{-1}$ has the form

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{D} + w\boldsymbol{e}\boldsymbol{e}', \tag{4.3}$$

where $\boldsymbol{D}$ is the $T \times T$ diagonal matrix, $\boldsymbol{e}$ is the $T \times 1$ vector of 1 elements, and $w$ is scalar. Although the detail is omitted, then $\boldsymbol{\Sigma}$ has the form

$$\boldsymbol{\Sigma} \;=\; \boldsymbol{D}^{-1} + d\boldsymbol{D}^{-1}\boldsymbol{e}\boldsymbol{e}'\boldsymbol{D}^{-1},$$

where $d$ is scalar. Therefore, the density function (4.1) is $Q_1^T$ if and only if $\boldsymbol{\Sigma}$ has the form

$$\boldsymbol{\Sigma} = \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_T \end{pmatrix} + d \begin{pmatrix} b_1 \\ \vdots \\ b_T \end{pmatrix} \begin{pmatrix} b_1, \ldots, b_T \end{pmatrix}. \tag{4.4}$$

Let $V(X_i) = \sigma_i^2$ $(i = 1, \ldots, T)$ and let $\rho_{ij}$ be the correlation coefficient of $X_i$ and $X_j$ $(i \neq j)$ with $|\rho_{ij}| < 1$. Assume that
(i) $\sigma_1^2 = \cdots = \sigma_T^2$ $(= \sigma^2)$ and $\rho_{ij} = \rho$ $(i < j)$.
Then

$$\boldsymbol{\Sigma} = \sigma^2(1-\rho)\Big(\boldsymbol{E} + \frac{\rho}{1-\rho}\boldsymbol{e}\boldsymbol{e}'\Big),$$

where $\boldsymbol{E}$ is the $T \times T$ identity matrix. This satisfies the form (4.4) of $\boldsymbol{\Sigma}$. Therefore the density function (4.1) with condition (i) is $Q_1^T$.

Next, assume that
(ii) $\sigma_1^2 = \cdots = \sigma_T^2$ $(= \sigma^2)$.

From (4.4), then $Q_1^T$ holds if and only if

$$\begin{cases} \sigma^2 = b_i + db_i^2 & (i = 1, \dots, T), \\ \sigma^2 \rho_{ij} = db_i b_j & (i < j), \end{cases}$$

hold, namely, $b_1 = \cdots = b_T$ since $|\rho_{ij}| < 1$. Therefore the density function (4.1) with condition (ii) is $Q_1^T$ if and only if $\rho_{ij} = \rho$ for all $i < j$ hold.

Also, assume that

(iii) $\rho_{ij} = \rho$ ($\neq 0$) for all $i < j$.

Then we see

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_T \end{pmatrix} \big( (1-\rho)\boldsymbol{E} + \rho \boldsymbol{e}\boldsymbol{e}' \big) \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_T \end{pmatrix}.$$

Although the detail is omitted, we can see

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{1-\rho} \left( \begin{pmatrix} \sigma_1^{-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_T^{-2} \end{pmatrix} + \frac{1}{m} \begin{pmatrix} \sigma_1^{-1} \\ \vdots \\ \sigma_T^{-1} \end{pmatrix} \begin{pmatrix} \sigma_1^{-1}, \dots, \sigma_T^{-1} \end{pmatrix} \right),$$

where $m = -(1-\rho)/\rho - T$. Therefore from (4.3), the density function (4.1) with condition (iii) is $Q_1^T$ if and only if $\sigma_1^2 = \cdots = \sigma_T^2$ holds.

Assume that

(iv) $\rho_{ij} = 0$ for all $i < j$.

Then the density function (4.1) is $Q_1^T$ because $\alpha_{ij}(x_i, x_j) = 1$ in (4.2) with $a_{ij} = 0$ for $i < j$.

We shall consider the relationship between the density function (4.1) and $M_k^T$ ($k = 1, \dots, T-1$). Obviously, the density function (4.1) is $M_1^T$ if and only if $\mu_1 = \cdots = \mu_T$ and $\sigma_1^2 = \cdots = \sigma_T^2$ hold. Also, for each $k$ ($k = 2, \dots, T-1$), it is $M_k^T$ if and only if $\mu_1 = \cdots = \mu_T$, $\sigma_1^2 = \cdots = \sigma_T^2$, and $\rho_{ij} = \rho$ for all $i < j$. Thus, from Theorem 2 we can see that the density function (4.1) with $\mu_1 = \cdots = \mu_T$ and $\sigma_1^2 = \cdots = \sigma_T^2$ is $S^T$ if and only if it is $Q_1^T$. Also, from Theorem 2, the density function (4.1) is $S^T$ if and only if $\mu_1 = \cdots = \mu_T$, $\sigma_1^2 = \cdots = \sigma_T^2$ and $\rho_{ij} = \rho$ for all $i < j$ hold.

**Example 2.** Consider a $T$-dimensional random vector $\boldsymbol{U} = (U_1, \dots, U_T)'$ having a multinominal distribution with

$$\mathrm{P}(U_1 = u_1, \dots, U_T = u_T | N) = \frac{N!}{u_1! \cdots u_T! (N - \sum_{i=1}^T u_i)!} \pi_1^{u_1} \cdots \pi_T^{u_T} (1 - \sum_{i=1}^T \pi_i)^{N - \sum_{i=1}^T u_i},$$

where $u_i$ is nonnegative integer with $0 \le u_i \le N$. Let

$$\boldsymbol{\pi} = (\pi_1, \dots, \pi_T)', \quad \hat{\boldsymbol{\pi}} = (\hat{\pi}_1, \dots, \hat{\pi}_T)',$$

where $\hat{\pi}_i = u_i/N$. Also let $\boldsymbol{X} = \sqrt{N}(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi})$. Then it is well-known that $\boldsymbol{X}$ has asymptotically (as $N \to \infty$) a $T$-variate normal distribution with mean $T \times 1$ zero vector $\mathbf{0} = (0, \dots, 0)'$ and covariance matrix

$$\boldsymbol{\Sigma} = \boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}', \tag{4.5}$$

where

$$\boldsymbol{D} = \begin{pmatrix} \pi_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_T \end{pmatrix};$$

see, e.g., Bishop, Fienberg and Holland [2]. So we shall consider the properties of normal distribution having covariance matrix (4.5). We see that $\boldsymbol{\Sigma}$ in (4.5) satisfies the form (4.4) obtained in Example 1. Therefore the density function of normal distribution $N(\mathbf{0}, \boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$ is always $Q_1^T$. Also, it is $Q_k^T$ ($k = 2, \dots, T-1$).

The marginal distribution of $X_i$ in $\boldsymbol{X} = (X_1, \dots, X_T)'$ is $N(0, \pi_i(1-\pi_i))$ for $i = 1, \dots, T$. Therefore the density function of $N(\mathbf{0}, \boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$ is $M_1^T$ if and only if $\pi_1 = \cdots = \pi_T$ holds.

Also two dimensional marginal distribution of $(X_i, X_j)$ for $i < j$ has the mean zero vector and the covariance matrix

$$\begin{pmatrix} \pi_i(1-\pi_i) & -\pi_i\pi_j \\ -\pi_i\pi_j & \pi_j(1-\pi_j) \end{pmatrix}.$$

Thus, the density function of $N(\mathbf{0}, \boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$ is $M_2^T$ if and only if $\pi_1 = \cdots = \pi_T$ holds. In a similar way, it is $M_k^T$ if and only if $\pi_1 = \cdots = \pi_T$ holds ($k = 3, \dots, T-1$).

Therefore we can see from Theorem 2 that the density function of $N(\mathbf{0}, \boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}')$ is $S^T$ if and only if it is $M_k^T$ ($k = 1, \dots, T-1$), because it always satisfies $Q_k^T$.

## §5. Comments

When an arbitrary density function $f(x_1, \dots, x_T)$ is not permutation symmetric, Theorem 2 may be useful for knowing the reason, i.e., for $k$ fixed, which structure of quasi-symmetry of order $k$ and marginal symmetry of order $k$ is lacking.

We point out that for a $T$-variate normal distribution, if the variances of $X_1, \dots, X_T$ are the same and the correlation coefficients of $X_i$ and $X_j$ for all

$i < j$ are the same, then the density functions is quasi-symmetric of order 1, i.e., $Q_1^T$ (as seen in Example 1); however, the converse always does not hold. Indeed, the normal density function with covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{D} - \boldsymbol{\pi}\boldsymbol{\pi}'$ (in Example 2) is always $Q_1^T$ even when the variances of $X_1, \ldots, X_T$ are not the same and the correlation coefficients of $X_i$ and $X_j$ are not the same for $1 \leq i < j \leq T$.

Finally we note that it is difficult to illustrate the decomposition of symmetry for the elliptical distribution instead of the normal distribution in Example of Section 4 because the $\{\alpha_i(x_i)\}$ and $\{\alpha_{ij}(x_i, x_j)\}$ are expressed as the ratio of density functions.

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