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A projective bimodule resolution and the Hochschild cohomology for a cluster-tilted algebra of type $\mathbb{D}_4$

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Abstract. In this paper we give explicit projective bimodule resolutions for algebras in a class of some special biserial algebras, which contains a cluster-tilted algebra of type $\mathbb{D}_4$. As a main result we completely determine the dimensions of the Hochschild cohomology groups for these algebras.

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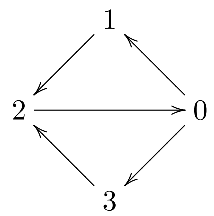
§1. Introduction

Let $\mathcal{Q}$ be the following quiver with four vertices 0, 1, 2, 3 and five arrows:



For $i = 0, 1, 2$, let e_i be the trivial path at the vertex i, and f_1 the trivial path at the vertex 3. For our convenience, f_i also denotes the trivial path at the vertex i for $i = 0, 2$. (Hence we may write $e_j = f_j$ for $j = 0, 2$.) For $i = 0, 1$, let a_i be the arrow from i to $i + 1$, and a_2 the arrow from 2 to 0. Moreover let b_0 the arrow from 0 to 3, and b_1 the arrow from 3 to 2. For our convenience, b_2 also denotes the arrow from 2 to 0. (Thus we may write $a_2 = b_2$.)

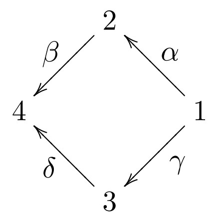
Throughout this paper, all indices i of e_i, f_i, a_i and b_i are considered as modulo 3. Hence it follows that, for each $i \in \mathbb{Z}$, a_i starts at e_i and ends with e_{i+1}, whereas b_i starts at f_i and ends with f_{i+1}. We write paths from left to right.

The purpose of the paper is to describe the Hochschild cohomology groups for a class of algebras Λ_n which contains some cluster-tilted algebra of Dynkin type $\mathbb{D}_4$. Let K be an algebraically closed field, and let n be a non-negative integer. Let I_n denote the ideal in the path algebra $K\mathcal{Q}$ generated by the following uniform elements:

$$(a_0a_1a_2)^n a_0a_1 - b_0b_1, \quad (a_ia_{i+1}a_{i+2})^n a_ia_{i+1}, \quad b_ib_{i+1} \quad \text{for } i = 1, 2.$$

Denote the algebra $K\mathcal{Q}/I_n$ by Λ_n. We immediately see that these algebras Λ_n are special biserial algebras of [SW], but are not self-injective algebras, since the indecomposable projective right Λ_n-modules corresponding to e_1 and f_1 have the isomorphic socles.

In [BHL], Bastian, Holm and Ladkani introduced specific quivers, called "standard forms" for derived equivalences, and proved that any cluster-tilted algebra of Dynkin type $\mathbb{D}$ is derived equivalent to a cluster-tilted algebra whose quiver is a standard form. If $n = 0$ then our algebra Λ_0 $(= K\mathcal{Q}/I_0)$ is a cluster-tilted algebra of type $\mathbb{D}_4$ and its quiver $\mathcal{Q}$ is precisely one of standard forms. Also, by [ABS, Example 3.6], Λ_0 is isomorphic to the trivial extension $B \ltimes \operatorname{Ext}^2_B(D(B), B)$ of the titled algebra $B = K\mathcal{Q}'/I'$ of type $\mathbb{D}_4$, where $D(B) = \operatorname{Hom}_K(B, K)$ and $\mathcal{Q}'$ is the quiver



and I' is the ideal generated by $\alpha\beta - \gamma\delta$.

Recently, in the papers [ES, SS, ST], the Hochschild cohomology of certain self-injective special biserial algebras have been studied, where the authors constructed some sets $\mathcal{G}^i$ $(i \geq 0)$ found in [GSZ] to provide minimal projective bimodule resolutions. In this paper, following these approaches, we give sets $\mathcal{G}^i$ for the right Λ_n-module $\Lambda_n/\mathfrak{r}_n$ where $\mathfrak{r}_n$ is the radical of Λ_n, and then use them to provide a projective bimodule resolution of Λ_n; see Section 2. The sets $\mathcal{G}^i$ also appear in the papers [A, GHMS, GS]. As a main consequence we give the dimension of the Hochschild cohomology group $\mathrm{HH}^i(\Lambda_n)$ $(i \geq 0)$, completely, for all $n \geq 0$ (Theorem 4.10). In particular, we get the dimensions for the Hochschild cohomology groups $\mathrm{HH}^i(\Lambda_0)$ for the cluster-tilted algebra Λ_0 of type $\mathbb{D}_4$ (Corollary 4.12).

Throughout this paper, for any arrow c, we denote the trivial path corresponding to the origin of c by $\mathfrak{o}(c)$, and the trivial path corresponding to the terminus of c by $\mathfrak{t}(c)$. Therefore we have $\mathfrak{o}(a_i) = e_i$, $\mathfrak{o}(b_i) = f_i$, $\mathfrak{t}(a_i) = e_{i+1}$, and $\mathfrak{t}(b_i) = f_{i+1}$ for all $i \in \mathbb{Z}$. For our algebra Λ_n we denote its radical by $\mathfrak{r}_n$ and its enveloping algebra $\Lambda_n^{\mathrm{op}} \otimes_K \Lambda_n$ by Λ_n^{e}. Also we write $\otimes_K$ as $\otimes$, for simplicity.

§2. A minimal projective resolution of $\Lambda_n/\mathfrak{r}_n$

In this section we construct sets $\mathcal{G}^i$ $(i \geq 0)$ of [GSZ] for the right Λ_n-module $\Lambda_n/\mathfrak{r}_n$, from which we can get an explicit minimal projective resolution of $\Lambda_n/\mathfrak{r}_n$.

Let $A = KQ/I$ be any finite-dimensional algebra with Q a finite quiver and I an admissible ideal. Denote the radical of A by $\mathfrak{r}$. Let $\mathcal{G}^0$ be the set of vertices of Q, $\mathcal{G}^1$ the set of arrows of Q, and $\mathcal{G}^2$ a minimal set of generators of I. In [GSZ], Green, Solberg and Zacharia showed that for each $i \geq 3$ there is a subset $\mathcal{G}^i$ of KQ such that every $x \in \mathcal{G}^i$ is an uniform element and we have a minimal projective resolution of $A/\mathfrak{r}$

$$\cdots \xrightarrow{\delta^4} P^3 \xrightarrow{\delta^3} P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \xrightarrow{\delta^0} A/\mathfrak{r} \to 0$$

satisfying the following:

(a) For $i \geq 0$, $P^i = \bigoplus_{x \in \mathcal{G}^i} \mathfrak{t}(x)A$.

(b) For $x \in \mathcal{G}^i$, there are unique elements $r_y, s_z \in KQ$, where $y \in \mathcal{G}^{i-1}$ and $z \in \mathcal{G}^{i-2}$, such that $x = \sum_{y \in \mathcal{G}^{i-1}} y r_y = \sum_{z \in \mathcal{G}^{i-2}} z s_z$.

(c) For $i \geq 1$, the differential $\delta^i : P^i \to P^{i-1}$ is defined by

$$\mathfrak{t}(x)\lambda \longmapsto \sum_{y \in \mathcal{G}^{i-1}} r_y \mathfrak{t}(x)\lambda \quad \text{for } x \in \mathcal{G}^i \text{ and } \lambda \in A,$$

where r_y are elements in the expression (b).

Fix an integer $n \geq 0$. We now construct sets $\mathcal{G}^i$ $(i \geq 0)$ for the right Λ_n-module $\Lambda_n/\mathfrak{r}_n$. First let

$$R^0 = e_0 = f_0, \quad S^0 = e_1, \quad T^0 = f_1, \quad \text{and} \quad U^0 = e_2 = f_2,$$

and put $\mathcal{G}^0 = \{R^0, S^0, T^0, U^0\}$. To define further sets $\mathcal{G}^i$, we introduce the following elements in $K\mathcal{Q}$:

Definition 2.1. We inductively define the elements as follows:

(i) For $i \geq 0$

$$R^{6i+6} = R^{6i+5}(a_2a_0a_1)^n a_2, \qquad R_j^{6i+1} = \begin{cases} R^{6i}a_0 & \text{if } j = 0 \\ R^{6i}b_0 & \text{if } j = 1, \end{cases}$$

$$R^{6i+2} = R_0^{6i+1}(a_1a_2a_0)^n a_1 - R_1^{6i+1}b_1, \qquad R^{6i+3} = R^{6i+2}a_2 = R^{6i+2}b_2,$$

$$R_j^{6i+4} = \begin{cases} R^{6i+3}(a_0a_1a_2)^n a_0 & \text{if } j = 0 \\ R^{6i+3}b_0 & \text{if } j = 1, \end{cases} \qquad R^{6i+5} = R_0^{6i+4}a_1 - R_1^{6i+4}b_1.$$

(ii) $S^1 = S^0a_1 (= a_1)$ and for $i \geq 0$

$$S_j^{6i+6} = \begin{cases} S^{6i+5}(a_0a_1a_2)^n a_0 & \text{if } j = 0 \\ S^{6i+5}b_0 & \text{if } j = 1, \end{cases} \qquad S^{6i+7} = S_0^{6i+6}a_1 - S_1^{6i+6}b_1,$$

$$S^{6i+2} = S^{6i+1}(a_2a_0a_1)^n a_2, \qquad S_j^{6i+3} = \begin{cases} S^{6i+2}a_0 & \text{if } j = 0 \\ S^{6i+2}b_0 & \text{if } j = 1, \end{cases}$$

$$S^{6i+4} = S_0^{6i+3}(a_1a_2a_0)^n a_1 - S_1^{6i+3}b_1, \qquad S^{6i+5} = S^{6i+4}a_2 = S^{6i+4}b_2.$$

(iii) $T^1 = T^0b_1 (= b_1)$ and for $i \geq 0$

$$T_j^{6i+6} = \begin{cases} T^{6i+5}a_0 & \text{if } j = 0 \\ T^{6i+5}b_0 & \text{if } j = 1, \end{cases} \qquad T_j^{6i+7} = T_0^{6i+6}(a_1a_2a_0)^n a_1 - T_1^{6i+6}b_1,$$

$$T^{6i+2} = T^{6i+1}a_2 = T^{6i+1}b_2, \qquad T_j^{6i+3} = \begin{cases} T^{6i+2}(a_0a_1a_2)^n a_0 & \text{if } j = 0 \\ T^{6i+2}b_0 & \text{if } j = 1, \end{cases}$$

$$T^{6i+4} = T_0^{6i+3}a_1 - T_1^{6i+3}b_1, \qquad T^{6i+5} = T^{6i+4}(a_2a_1a_0)^n a_2.$$

(iv) For $i \geq 0$

$$U^{6i+6} = U_0^{6i+5}(a_1a_2a_0)^n a_1 - U_1^{6i+5}b_1, \qquad U^{6i+1} = U^{6i}a_2 = U^{6i}b_2,$$

$$U_j^{6i+2} = \begin{cases} U^{6i+1}(a_0a_1a_2)^n a_0 & \text{if } j = 0 \\ U^{6i+1}b_0 & \text{if } j = 1, \end{cases} \qquad U^{6i+3} = U_0^{6i+2}a_1 - U_1^{6i+2}b_1,$$

$$U^{6i+4} = U^{6i+3}(a_2a_0a_1)^n a_2, \qquad U_j^{6i+5} = \begin{cases} U^{6i+4}a_0 & \text{if } j = 0 \\ U^{6i+4}b_0 & \text{if } j = 1. \end{cases}$$

Now, for $i \geq 1$, we define

$$\mathcal{G}^i = \begin{cases} \{R^i, S_0^i, S_1^i, T_0^i, T_1^i, U^i\} & \text{if } i \equiv 0 \pmod 3, \\ \{R_0^i, R_1^i, S^i, T^i, U^i\} & \text{if } i \equiv 1 \pmod 3, \\ \{R^i, S^i, T^i, U_0^i, U_1^i\} & \text{if } i \equiv 2 \pmod 3. \end{cases} \tag{2.1}$$

Then it is straightforward to show that $\mathcal{G}^i$ ($i \geq 0$) satisfy the conditions (a), (b), and (c) above.

Remark 2.2. As explained above, the set (2.1) provides a minimal projective resolution $(P^\bullet, \delta)$ of $\Lambda_n/\mathfrak{r}_n$ defined by (a)–(c). We immediately see that $(P^\bullet, \delta)$ is not a periodic projective resolution, but $\operatorname{Ker} \delta^1$ has a periodic projective resolution of period $\begin{cases} 3 & \text{if } n = 0 \\ 6 & \text{if } n > 0. \end{cases}$

In particular, if $n = 0$, then $(P^\bullet, \delta)$ is a linear resolution, so that we get the following proposition.

Proposition 2.3. *The algebra Λ_0 is a Koszul algebra.*

§3. A projective bimodule resolution of Λ_n

In this section we give an explicit minimal projective bimodule resolution of Λ_n:

$$(Q^\bullet, \partial): \quad \cdots \xrightarrow{\partial^4} Q^3 \xrightarrow{\partial^3} Q^2 \xrightarrow{\partial^2} Q^1 \xrightarrow{\partial^1} Q^0 \xrightarrow{\partial^0} \Lambda_n \to 0$$

for any $n \geq 0$. First we define the projective Λ_n-Λ_n-bimodule Q^i ($i \geq 0$) by using the sets $\mathcal{G}^i$ in (2.1) as follows:

$$Q^i = \bigoplus_{g \in \mathcal{G}^i} \Lambda_n \mathfrak{o}(g) \otimes \mathfrak{t}(g) \Lambda_n.$$

Then for $i \geq 0$ we easily see that $\mathfrak{o}(g_1) \otimes \mathfrak{t}(g_1) \neq \mathfrak{o}(g_2) \otimes \mathfrak{t}(g_2)$, that is, $\Lambda_n \mathfrak{o}(g_1) \otimes \mathfrak{t}(g_1) \Lambda_n \not\simeq \Lambda_n \mathfrak{o}(g_2) \otimes \mathfrak{t}(g_2) \Lambda_n$ as Λ_n-Λ_n-bimodules for all $g_1, g_2 \in \mathcal{G}^i$ with $g_1 \neq g_2$. Actually Q^i can be written as follows:

$$Q^0 = (\Lambda_n e_0 \otimes e_0 \Lambda_n) \oplus (\Lambda_n e_1 \otimes e_1 \Lambda_n) \oplus (\Lambda_n f_1 \otimes f_1 \Lambda_n) \oplus (\Lambda_n e_2 \otimes e_2 \Lambda_n),$$

and for $i \geq 0$

$$\begin{aligned} Q^{3i+3} &= (\Lambda_n e_0 \otimes e_0 \Lambda_n) \oplus (\Lambda_n e_1 \otimes e_1 \Lambda_n) \oplus (\Lambda_n e_1 \otimes f_1 \Lambda_n) \\ &\quad \oplus (\Lambda_n f_1 \otimes e_1 \Lambda_n) \oplus (\Lambda_n f_1 \otimes f_1 \Lambda_n) \oplus (\Lambda_n e_2 \otimes e_2 \Lambda_n), \\ Q^{3i+1} &= (\Lambda_n e_0 \otimes e_1 \Lambda_n) \oplus (\Lambda_n e_0 \otimes f_1 \Lambda_n) \oplus (\Lambda_n e_1 \otimes e_2 \Lambda_n) \\ &\quad \oplus (\Lambda_n f_1 \otimes e_2 \Lambda_n) \oplus (\Lambda_n e_2 \otimes e_0 \Lambda_n), \\ Q^{3i+2} &= (\Lambda_n e_0 \otimes e_2 \Lambda_n) \oplus (\Lambda_n e_1 \otimes e_0 \Lambda_n) \oplus (\Lambda_n f_1 \otimes e_0 \Lambda_n) \\ &\quad \oplus (\Lambda_n e_2 \otimes e_1 \Lambda_n) \oplus (\Lambda_n e_2 \otimes f_1 \Lambda_n). \end{aligned}$$

Now we need to give maps $\partial^i : Q^i \to Q^{i-1}$ of Λ_n-Λ_n-bimodules. Let $\partial^0 : Q^0 \to \Lambda_n$ be the multiplication. For $i \geq 1$ we define ∂^i as follows:

Definition 3.1. For $i \geq 1$ we define the map $\partial^i : Q^i \to Q^{i-1}$ by the following: For $j \geq 0$

(a) For $l = 0, 1, 2$ and $r = 0, 1$

$$\partial^1 : \begin{cases} e_l \otimes e_{l+1} & \mapsto \; e_l \otimes a_l - a_l \otimes e_{l+1}, \\ f_r \otimes f_{r+1} & \mapsto \; f_r \otimes b_r - b_r \otimes f_{r+1}. \end{cases}$$

(b) For $l = 0, 1, 2$

$$\partial^{6j+6} : \begin{cases} e_l \otimes e_l & \mapsto \begin{cases} e_0 \otimes (a_2 a_0 a_1)^n a_2 \\ \quad + \left(\sum_{k=0}^{3n-1} a_0 a_1 a_2 \cdots a_k \otimes a_k \cdots a_{3n-1}\right) \\ \quad\quad +(a_0 a_1 a_2)^n a_0 \otimes e_0 - b_0 \otimes e_0 & \text{if } l = 0, \\[1em] e_1 \otimes (a_0 a_1 a_2)^n a_0 \\ \quad + \left(\sum_{k=1}^{3n} a_1 a_2 a_3 \cdots a_k \otimes a_k \cdots a_{3n}\right) \\ \quad\quad +(a_1 a_2 a_0)^n a_1 \otimes e_1 & \text{if } l = 1, \\[1em] e_2 \otimes (a_1 a_2 a_0)^n a_1 \\ \quad + \left(\sum_{k=2}^{3n+1} a_2 a_3 a_4 \cdots a_k \otimes a_k \cdots a_{3n+1}\right) \\ \quad\quad +(a_2 a_0 a_1)^n a_2 \otimes e_2 - e_2 \otimes b_1 & \text{if } l = 2, \end{cases} \\[1em] e_1 \otimes f_1 & \mapsto \; e_1 \otimes b_0 + a_1 \otimes f_1, \\ f_1 \otimes e_1 & \mapsto \; f_1 \otimes a_0 + b_1 \otimes e_1, \\ f_1 \otimes f_1 & \mapsto \; f_1 \otimes b_0 + b_1 \otimes f_1. \end{cases}$$

(c) For $l = 0, 1, 2$ and $r = 0, 1$

$$\partial^{6j+7} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} e_0 \otimes a_0 - a_0 \otimes e_1 + b_0 \otimes e_1 & \text{if } l = 0, \\ e_1 \otimes a_1 - a_1 \otimes e_2 - e_1 \otimes b_1 & \text{if } l = 1, \\ e_2 \otimes a_2 - a_2 \otimes e_0 & \text{if } l = 2, \end{cases} \\[1em] f_r \otimes f_{r+1} & \mapsto \begin{cases} e_0 \otimes b_0 + b_0 \otimes f_1 - (a_0 a_1 a_2)^n a_0 \otimes f_1 & \text{if } r = 0, \\ -f_1 \otimes b_1 - b_1 \otimes e_2 + f_1 \otimes (a_1 a_2 a_0)^n a_1 & \text{if } r = 1. \end{cases} \end{cases}$$

(d) For $l = 0, 1, 2$ and $r = 1, 2$

$$\partial^{6j+2} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} e_0 \otimes (a_1 a_2 a_0)^n a_1 \\ \quad + \left(\sum_{k=0}^{3n-1} a_0 a_1 a_2 \cdots a_k \otimes a_{k+2} \cdots a_{3n+1} \right) \\ \quad\quad + (a_0 a_1 a_2)^n a_0 \otimes e_2 \\ \quad\quad\quad - (e_0 \otimes b_1 + b_0 \otimes e_2) \quad \text{if } l = 0, \\ \\ e_l \otimes (a_{l+1} a_{l+2} a_l)^n a_{l+1} \\ + \left(\sum_{k=l}^{3n+l-1} a_l a_{l+1} a_{l+2} \cdots a_k \otimes a_{k+2} \cdots a_{3n+l+1} \right) \\ \quad\quad + (a_l a_{l+1} a_{l+2})^n a_l \otimes e_{l+2} \quad \text{if } l = 1, 2, \end{cases} \\ \\ f_r \otimes f_{r+2} & \mapsto f_r \otimes b_{r+1} + b_r \otimes f_{r+2}. \end{cases}$$

(e) For $l = 0, 1, 2$

$$\partial^{6j+3} : \begin{cases} e_l \otimes e_l & \mapsto \begin{cases} e_0 \otimes a_2 - a_0 \otimes e_0 + b_0 \otimes e_0 & \text{if } l = 0, \\ e_1 \otimes a_0 - a_1 \otimes e_1 & \text{if } l = 1, \\ e_2 \otimes a_1 - a_2 \otimes e_2 - e_2 \otimes b_1 & \text{if } l = 2, \end{cases} \\ \\ e_1 \otimes f_1 & \mapsto e_1 \otimes b_0 - (a_1 a_2 a_0)^n a_1 \otimes f_1, \\ f_1 \otimes e_1 & \mapsto f_1 \otimes (a_0 a_1 a_2)^n a_0 - b_1 \otimes e_1, \\ f_1 \otimes f_1 & \mapsto f_1 \otimes b_0 - b_1 \otimes f_1. \end{cases}$$

(f) For $l = 0, 1, 2$ and $r = 0, 1$

$$\partial^{6j+4} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} e_0 \otimes (a_0 a_1 a_2)^n a_0 \\ \quad + \left(\sum_{k=0}^{3n-1} a_0 a_1 a_2 \cdots a_k \otimes a_{k+1} \cdots a_{3n} \right) \\ \quad\quad + (a_0 a_1 a_2)^n a_0 \otimes e_1 - b_0 \otimes e_1 \quad \text{if } l = 0, \\ \\ e_1 \otimes (a_1 a_2 a_0)^n a_1 \\ \quad + \left(\sum_{k=1}^{3n} a_1 a_2 a_3 \cdots a_k \otimes a_{k+1} \cdots a_{3n+1} \right) \\ \quad\quad + (a_1 a_2 a_0)^n a_1 \otimes e_2 - e_1 \otimes b_1 \quad \text{if } l = 1, \\ \\ e_2 \otimes (a_2 a_0 a_1)^n a_2 \\ \quad + \left(\sum_{k=2}^{3n+1} a_2 a_3 a_4 \cdots a_k \otimes a_{k+1} \cdots a_{3n+2} \right) \\ \quad\quad + (a_2 a_0 a_1)^n a_2 \otimes e_0 \quad \text{if } l = 2, \end{cases} \\ \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} e_0 \otimes b_0 - b_0 \otimes f_1 + a_0 \otimes f_1 & \text{if } r = 0, \\ -f_1 \otimes b_1 + b_1 \otimes e_2 + f_1 \otimes a_1 & \text{if } r = 1. \end{cases} \end{cases}$$

(g) For $l = 0, 1, 2$ and $r = 1, 2$

$$\partial^{6j+5}: \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} e_0 \otimes a_1 - a_0 \otimes e_2 - e_0 \otimes b_1 + b_0 \otimes e_2 & \text{if } l = 0, \\ e_1 \otimes a_2 - a_1 \otimes e_0 & \text{if } l = 1, \\ e_2 \otimes a_0 - a_2 \otimes e_1 & \text{if } l = 2, \end{cases} \\ \\ f_r \otimes f_{r+2} & \mapsto \begin{cases} f_1 \otimes (a_2 a_0 a_1)^n a_2 - b_1 \otimes e_0 & \text{if } r = 1, \\ e_2 \otimes b_0 - (a_2 a_0 a_1)^n a_2 \otimes f_1 & \text{if } r = 2. \end{cases} \end{cases}$$

Remark 3.2. As in Remark 2.2, we denote by $(P^\bullet, \delta)$ the minimal projective resolution of $\Lambda_n/\mathfrak{r}_n$ given by the sets $\mathcal{G}^i$ in (2.1). Then, it is easy to see that, for each $i \geq 0$, the map $h_i : \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} Q^i \to P^i$ determined by $h_i\,(\mathfrak{o}(g) \otimes_{\Lambda_n} \mathfrak{o}(g) \otimes_K \mathfrak{t}(g)) = \mathfrak{t}(g)$ $(g \in \mathcal{G}^i)$ is an isomorphism of right Λ_n-modules such that the square

$$\begin{array}{ccc} \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} Q^{i+1} & \xrightarrow{\mathrm{id}\otimes_{\Lambda_n}\partial^{i+1}} & \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} Q^i \\ {\scriptstyle h_{i+1}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle h_i} \\ P^{i+1} & \xrightarrow{\delta^{i+1}} & P^i \end{array}$$

is commutative. Since $(P^\bullet, \delta)$ is a minimal projective resolution of $\Lambda_n/\mathfrak{r}_n$, it follows that $(\Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} Q^\bullet, \mathrm{id} \otimes_{\Lambda_n} \partial)$ is a minimal projective resolution of $\Lambda_n/\mathfrak{r}_n$.

Now we have the following theorem. The proof is same as that of [ES, Theorem 2.4] and that of [ST, Theorem 1.6]; see also [A, BE, GHMS, O, SS]. But we include a proof for the convenience of the reader.

Theorem 3.3. $(Q^\bullet, \partial)$ *is a minimal projective bimodule resolution of* Λ_n.

Proof. By direct computations we have $\mathrm{Im}\,\partial^{i+1} \subseteq \mathrm{Ker}\,\partial^i$ for all $i \geq 0$, so that $(Q^\bullet, \partial)$ is a complex.

We verify the converse inclusion $\mathrm{Ker}\,\partial^i \subseteq \mathrm{Im}\,\partial^{i+1}$ for all $i \geq 0$, which implies the exactness of $(Q^\bullet, \partial)$. Suppose for contradiction that $\mathrm{Ker}\,\partial^m \not\subseteq \mathrm{Im}\,\partial^{m+1}$ for some $m \geq 0$. Then for some simple Λ_n-Λ_n-bimodule $S \otimes T$ (where S is a simple left Λ_n-module, and T is a simple right Λ_n-module) we get the non-zero composite

$$f : \mathrm{Ker}\,\partial^m \longrightarrow \mathrm{Ker}\,\partial^m/\mathrm{Im}\,\partial^{m+1} \longrightarrow S \otimes T.$$

Also, since $(\Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} Q^\bullet, \mathrm{id} \otimes_{\Lambda_n} \partial)$ is a minimal projective resolution of $\Lambda_n/\mathfrak{r}_n$ by Remark 3.2, we get the isomorphism of right Λ_n-modules

$$\begin{aligned} F : \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} \mathrm{Im}\,\partial^{m+1} &\xrightarrow{\simeq} \mathrm{Im}\,(\mathrm{id} \otimes_{\Lambda_n} \partial^{m+1}) = \mathrm{Ker}\,(\mathrm{id} \otimes_{\Lambda_n} \partial^m) \\ &\xrightarrow{\simeq} \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} \mathrm{Ker}\,\partial^m \end{aligned}$$

satisfying $F(x \otimes y) = x \otimes y$ for $x \in \Lambda_n/\mathfrak{r}_n$ and $y \in \operatorname{Im} \partial^{m+1}$. We now consider the non-zero composite

$$G : \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} Q^{m+1} \xrightarrow{\mathrm{id} \otimes_{\Lambda_n} \partial^{m+1}} \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} \operatorname{Im} \partial^{m+1}$$
$$\xrightarrow{F} \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} \operatorname{Ker} \partial^m \xrightarrow{\mathrm{id} \otimes_{\Lambda_n} f} \Lambda_n/\mathfrak{r}_n \otimes_{\Lambda_n} S \otimes T.$$

Then we easily have $G = \mathrm{id} \otimes_{\Lambda_n} (f\partial^{m+1})$. But clearly $f\partial^{m+1} = 0$, so it follows that $G = 0$. This is a contradiction. Hence $(Q^\bullet, \partial)$ is exact.

The minimality of $(Q^\bullet, \partial)$ follows from the fact that $\partial^i(\mathfrak{o}(g) \otimes \mathfrak{t}(g))$ lies in the radical of Q^{i-1} for all $i \geq 1$ and $g \in \mathcal{G}^i$. □

Remark 3.4. We immediately see that the projective resolution $(Q^\bullet, \partial)$ is not periodic, but $\operatorname{Ker} \partial^1$ has a periodic minimal projective bimodule resolution of period $\begin{cases} 3 & \text{if } \operatorname{char} K = 2 \text{ and } n = 0 \\ 6 & \text{otherwise.} \end{cases}$

Remark 3.5. By Happel [H], the number of the indecomposable projective summand of Q^i which is isomorphic to $\Lambda_n u \otimes v\Lambda_n$ equals the dimension of $\operatorname{Ext}^i_{\Lambda_n}(S_u, S_v)$ for each $i \geq 0$, where $u, v \in \{e_0, e_1, e_2, f_1\}$, and S_u, S_v denote the simple Λ_n-modules corresponding to u and v, respectively. Therefore we have for $j \geq 0$ that

$$\dim_K \operatorname{Ext}^{3j}_{\Lambda_n}(S_u, S_v) = \begin{cases} 1 & \text{if } (u,v) = (e_i, e_i) \text{ with } i = 0,1,2,\ (e_1, f_1),\ (f_1, e_1), \text{ or } (f_1, f_1), \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim_K \operatorname{Ext}^{3j+1}_{\Lambda_n}(S_u, S_v) = \begin{cases} 1 & \text{if } (u,v) = (e_i, e_{i+1}) \text{ with } i = 0,1,2,\ (e_0, f_1), \text{ or } (f_1, e_2), \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim_K \operatorname{Ext}^{3j+2}_{\Lambda_n}(S_u, S_v) = \begin{cases} 1 & \text{if } (u,v) = (e_i, e_{i+2}) \text{ with } i = 0,1,2,\ (f_1, e_0), \text{ or } (e_2, f_1), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we see for $i \geq 0$ that the dimension of $\operatorname{Ext}^i_{\Lambda_n}(S_u, S_v)$ coincides with the number of the elements $g \in \mathcal{G}^i$ such that $g = ugv$.

Remark 3.6. In [GHMS], a minimal projective bimodule resolution for any finite-dimensional Koszul algebra is constructed by using the sets $\mathcal{G}^i$. Hence for $n = 0$ we could have applied [GHMS, Theorem 2.1] to get a minimal projective bimodule resolution of Λ_0.

§4. Hochschild cohomology group of Λ_n

In this section we find the dimensions the Hochschild cohomology groups of Λ_n by using the minimal projective bimodule resolution $(Q^\bullet, \partial)$ of Theorem 3.3.

Applying the functor $\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(-, \Lambda_n)$ to $(Q^\bullet, \partial)$, we have the complex

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^0, \Lambda_n) \xrightarrow{\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^1, \Lambda_n)} \mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^1, \Lambda_n) \xrightarrow{\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^2, \Lambda_n)}$$

$$\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^2, \Lambda_n) \xrightarrow{\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^3, \Lambda_n)} \mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^3, \Lambda_n) \xrightarrow{\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^4, \Lambda_n)} \cdots .$$

Then, for $i \geq 0$, the ith Hochschild cohomology group $\mathrm{HH}^i(\Lambda_n)$ of Λ_n is given by the K-space

$$\mathrm{HH}^i(\Lambda_n) := \mathrm{Ext}^i_{\Lambda_n^{\mathrm{e}}}(\Lambda_n, \Lambda_n) = \mathrm{Ker}\,\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^{i+1}, \Lambda_n)/\mathrm{Im}\,\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^i, \Lambda_n).$$

Note that, for each $j = 2, 3, \ldots, 7$ and $k \geq 0$, since $\partial^j = \partial^{6k+j}$ holds, it follows that $\mathrm{HH}^{6k+j}(\Lambda_n) \simeq \mathrm{HH}^j(\Lambda_n)$.

4.1. The basis for $\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^i, \Lambda_n)$

We start by giving a K-basis for $\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^i, \Lambda_n)$ for $i \geq 0$. It is well-known that for each $i \geq 0$ the map

$$\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^i, \Lambda_n) \to \bigoplus_{g \in \mathcal{G}^i} \mathfrak{o}(g)\Lambda_n\mathfrak{t}(g); \quad f \mapsto \sum_{g \in \mathcal{G}^i} f(\mathfrak{o}(g) \otimes \mathfrak{t}(g))$$

is an isomorphism of K-spaces. Therefore, by a computation of a K-basis for $\bigoplus_{g \in \mathcal{G}^i} \mathfrak{o}(g)\Lambda_n\mathfrak{t}(g)$, we have the following lemma. Here we recall that all indices i of e_i, f_i, a_i and b_i are considered as modulo 3, and $f_0 = e_0$, $f_2 = e_2$ and $a_2 = b_2$, by our convention.

Lemma 4.1. *We have the following K-basis of* $\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^i, \Lambda_n)$ *for* $i \geq 0$:

(a) *For* $k = 0, 1, 2$ *and* $m = 0, 1, \ldots, n$, *let* $\alpha_k^m : Q^0 \to \Lambda_n$ *be the map of* Λ_n-Λ_n*-bimodules determined by: for* $l = 0, 1, 2$

$$\alpha_k^m : \begin{cases} e_l \otimes e_l & \mapsto \begin{cases} (a_k a_{k+1} a_{k+2})^m & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases} \\ f_1 \otimes f_1 & \mapsto 0 \end{cases}$$

and let $\beta : Q^0 \to \Lambda_n$ *be the map of* Λ_n-Λ_n*-bimodules determined by*

$$\beta : \begin{cases} e_l \otimes e_l & \mapsto \quad 0 \quad \text{for } l = 0, 1, 2 \\ f_1 \otimes f_1 & \mapsto \quad f_1. \end{cases}$$

Then

$$\{\alpha_k^m, \beta \mid k = 0, 1, 2;\ m = 0, 1, \ldots, n\}$$

defines a K-basis of $\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^0, \Lambda_n)$.

(b) *For $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$, let $\phi_k^m : Q^{3j+3}(= Q^3) \to \Lambda_n$ be the map of Λ_n-Λ_n-bimodules defined by: for $l = 0, 1, 2$*

$$\phi_k^m : \begin{cases} e_l \otimes e_l & \mapsto \quad \begin{cases} (a_k a_{k+1} a_{k+2})^m & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases} \\ f_1 \otimes f_1 & \mapsto \quad 0 \\ e_1 \otimes f_1 & \mapsto \quad 0 \\ f_1 \otimes e_1 & \mapsto \quad 0, \end{cases}$$

and let $\psi : Q^{3j+3}(= Q^3) \to \Lambda_n$ be the map of Λ_n-Λ_n-bimodules determined by

$$\psi : \begin{cases} e_l \otimes e_l & \mapsto \quad 0 \quad \text{for } l = 0, 1, 2 \\ f_1 \otimes f_1 & \mapsto \quad f_1 \\ e_1 \otimes f_1 & \mapsto \quad 0 \\ f_1 \otimes e_1 & \mapsto \quad 0. \end{cases}$$

Then

$$\{\phi_k^m, \psi \mid k = 0, 1, 2;\ m = 0, 1, \ldots, n\}$$

gives a K-basis of $\mathrm{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^{3j+3}, \Lambda_n)$.

(c) *For $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$, let $\mu_k^m : Q^{3j+1}(= Q^1) \to \Lambda_n$ be a map of Λ_n-Λ_n-bimodules determined by: for $l = 0, 1, 2$ and $r = 0, 1$*

$$\mu_k^m : \begin{cases} e_l \otimes e_{l+1} & \mapsto \quad \begin{cases} (a_k a_{k+1} a_{k+2})^m a_k & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases} \\ f_r \otimes f_{r+1} & \mapsto \quad 0, \end{cases}$$

and, for $s = 0, 1$, let $\nu_s : Q^{3j+1}(= Q^1) \to \Lambda_n$ be a map of Λ_n-Λ_n-bimodules defined by: for $l = 0, 1, 2$ and $r = 0, 1$

$$\nu_s : \begin{cases} e_l \otimes e_{l+1} & \mapsto \quad 0 \\ f_r \otimes f_{r+1} & \mapsto \quad \begin{cases} b_s & \text{if } r = s \\ 0 & \text{if } r \neq s. \end{cases} \end{cases}$$

Then

$$\{\mu_k^m, \nu_s \mid k = 0, 1, 2;\ m = 0, 1, \ldots, n;\ s = 0, 1\}$$

defines a K-basis for $\operatorname{Hom}_{\Lambda_n^e}(Q^{3j+1}, \Lambda_n)$.

(d) *Let* $\eta : Q^{3j+2}(= Q^2) \to \Lambda_n$ *be a map of* Λ_n-Λ_n*-bimodules defined by: for* $l = 0, 1, 2$ *and* $r = 1, 2$

$$\eta : \begin{cases} e_l \otimes e_{l+2} & \mapsto \quad \begin{cases} (a_0 a_1 a_2)^n a_0 a_1 & \text{if } l = 0 \\ 0 & \text{if } l = 1, 2 \end{cases} \\ \\ f_r \otimes f_{r+2} & \mapsto \quad 0. \end{cases}$$

Moreover, if $n > 0$*, then for* $k = 0, 1, 2$ *and* $m = 0, 1, \ldots, n-1$*, let* $\theta_k^m : Q^{3j+2}(= Q^2) \to \Lambda_n$ *be a map of* Λ_n-Λ_n*-bimodules defined by: for* $l = 0, 1, 2$ *and* $r = 1, 2$

$$\theta_k^m : \begin{cases} e_l \otimes e_{l+2} & \mapsto \quad \begin{cases} (a_k a_{k+1} a_{k+2})^m a_k a_{k+1} & \text{if } l = k \\ 0 & \text{if } l \neq k \end{cases} \\ \\ f_r \otimes f_{r+2} & \mapsto \quad 0. \end{cases}$$

If $n = 0$*, then* $\{\eta\}$ *is a K-basis of* $\operatorname{Hom}_{\Lambda_0^e}(Q^{3j+2}, \Lambda_0)$*, and if* $n > 0$*, then*

$$\{\eta, \theta_k^m \mid k = 0, 1, 2;\ m = 0, 1, \ldots, n-1\}$$

defines a K-basis for $\operatorname{Hom}_{\Lambda_n^e}(Q^{3j+2}, \Lambda_n)$.

As an immediate consequence, we have the following corollary:

Corollary 4.2. *For $i \geq 0$, the dimension of* $\operatorname{Hom}_{\Lambda_n^e}(Q^i, \Lambda_n)$ *is given as follows*:

$$\dim_K \operatorname{Hom}_{\Lambda_n^e}(Q^i, \Lambda_n) = \begin{cases} 3n + 4 & \text{if } i \equiv 0 \pmod 3 \\ 3n + 5 & \text{if } i \equiv 1 \pmod 3 \\ 3n + 1 & \text{if } i \equiv 2 \pmod 3). \end{cases}$$

4.2. The images of $\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$

Now we find the images of basis elements in Lemma 4.1 under the map $\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$. By direct computations we have the following.

Lemma 4.3. *For $j \geq 0$ we have the following*:

(a) *For $k = 0,1,2$ and $m = 0,1,\ldots,n$ we have $\alpha_k^m\partial^1 = \phi_k^m\partial^{6j+7}$, and for $l = 0,1,2$, and $r = 0,1$,*

$$\phi_k^m\partial^{6j+7} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} (a_k a_{k+1} a_{k+2})^m a_k & \text{if } l \equiv k \pmod 3 \\ 0 & \text{if } l \equiv k+1 \pmod 3 \\ -(a_{k+2} a_k a_{k+1})^m a_{k+2} & \text{if } l \equiv k+2 \pmod 3 \end{cases} \\ \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} b_0 & \text{if } r = k = m = 0 \\ -b_1 & \text{if } r = 1,\ k = 2 \text{ and } m = 0 \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Furthermore, for $l = 0,1,2$, and $r = 0,1$, we get

$$\beta\partial^1 : \begin{cases} e_l \otimes e_{l+1} & \mapsto 0 \\ \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} -b_0 & \text{if } r = 0 \\ b_1 & \text{if } r = 1 \end{cases} \end{cases}$$

and

$$\psi\partial^{6j+7} : \begin{cases} e_l \otimes e_{l+1} & \mapsto 0 \\ \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} b_0 & \text{if } r = 0 \\ -b_1 & \text{if } r = 1. \end{cases} \end{cases}$$

(b) *For $k = 0,1,2$ and $m = 0,1,\ldots,n$, we have: for $l = 0,1,2$, and $r = 1,2$,*

$$\mu_k^m\partial^{6j+2} : \begin{cases} e_l \otimes e_{l+2} \mapsto & \\ \quad \begin{cases} (n+1)(a_0a_1a_2)^n a_0 a_1 & \text{if } l = k = m = 0, \text{ or} \\ & \quad \text{if } m = l = 0 \text{ and } k = 1 \\ n(a_0a_1a_2)^n a_0 a_1 & \text{if } k = 2 \text{ and } m = l = 0 \\ 0 & \text{if } 0 < m \leq n, \text{ or if } l = 1,2 \end{cases} \\ \\ f_r \otimes f_{r+2} \mapsto 0. \end{cases}$$

Also, for $s = 0,1$, we have: for $l = 0,1,2$, and $r = 1,2$,

$$\nu_s\partial^{6j+2} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} -(a_0a_1a_2)^n a_0 a_1 & \text{if } l = 0 \\ 0 & \text{if } l = 1,2 \end{cases} \\ \\ f_r \otimes f_{r+2} & \mapsto 0. \end{cases}$$

(c) *For* $g \in \mathcal{G}^{6j+3}$ *we have* $\eta\partial^{6j+3}(\mathfrak{o}(g)\otimes\mathfrak{t}(g)) = 0$. *Also, if* $n > 0$, *then for* $k = 0,1,2$ *and* $m = 0,1,\ldots,n-1$ *we have*: *for* $l = 0,1,2$

$$\theta_k^m\partial^{6j+3} : \begin{cases} e_l \otimes e_l & \mapsto \begin{cases} (a_k a_{k+1} a_{k+2})^{m+1} & \text{if } l \equiv k \pmod 3 \\ 0 & \text{if } l \equiv k+1 \pmod 3 \\ -(a_{k+2} a_k a_{k+1})^{m+1} & \text{if } l \equiv k+2 \pmod 3 \end{cases} \\ e_1 \otimes f_1 & \mapsto 0 \\ f_1 \otimes e_1 & \mapsto 0 \\ f_1 \otimes f_1 & \mapsto 0. \end{cases}$$

(d) *For* $k = 0,1,2$ *and* $m = 0,1,\ldots,n$, *we get*: *for* $l = 0,1,2$ *and* $r = 0,1$

$$\phi_k^m\partial^{6j+4} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \\ \quad \begin{cases} (n+1)(a_k a_{k+1} a_{k+2})^n a_k & \text{if } l \equiv k \pmod 3 \text{ and } m = 0 \\ n(a_{k+1} a_{k+2} a_k)^n a_{k+1} & \text{if } l \equiv k+1 \pmod 3 \text{ and } m = 0 \\ (n+1)(a_{k+2} a_k a_{k+1})^n a_{k+2} & \text{if } l \equiv k+2 \pmod 3 \text{ and } m = 0 \\ 0 & \text{if } 0 < m \le n \end{cases} \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} b_0 & \text{if } r = k = m = 0 \\ b_1 & \text{if } r = 1,\ k = 2 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

and

$$\psi\partial^{6j+4} : \begin{cases} e_l \otimes e_{l+1} & \mapsto 0 \\ f_r \otimes f_{r+1} & \mapsto \begin{cases} -b_0 & \text{if } r = 0 \\ -b_1 & \text{if } r = 1. \end{cases} \end{cases}$$

(e) *For* $k = 0,1,2$, *and* $m = 0,1,\ldots,n$, *we have*: *for* $l = 0,1,2$ *and* $r = 1,2$

$$\mu_k^m\partial^{6j+5} :$$

$$\begin{cases} e_l \otimes e_{l+2} \quad \mapsto \\ \quad \begin{cases} (a_k a_{k+1} a_{k+2})^m a_k a_{k+1} & \text{if } l = k \text{ and } 0 \le m \le n-1, \\ & \quad \text{or if } l = k = 0 \text{ and } m = n \\ -(a_{k+2} a_k a_{k+1})^m a_{k+2} a_k & \text{if } l \equiv k+2 \pmod 3 \text{ and } 0 \le m \le n-1, \\ & \quad \text{or if } l = 0,\ k = 1 \text{ and } m = n \\ 0 & \text{otherwise} \end{cases} \\ f_r \otimes f_{r+2} \quad \mapsto \quad 0. \end{cases}$$

Moreover, for $s = 0, 1$, we have that: for $l = 0, 1, 2$ and $r = 1, 2$,

$$\nu_s \partial^{6j+5} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} -(a_0a_1a_2)^n a_0a_1 & \text{if } s = l = 0 \\ (a_0a_1a_2)^n a_0a_1 & \text{if } s = 1 \text{ and } l = 0 \\ 0 & \text{otherwise} \end{cases} \\ f_r \otimes f_{r+2} & \mapsto \quad 0. \end{cases}$$

(f) *For $g \in \mathcal{G}^{6j+6}$ we have $\eta\partial^{6j+6}(\mathfrak{o}(g)\otimes\mathfrak{t}(g)) = 0$. Furthermore, if $n > 0$, then for $g \in \mathcal{G}^{6j+6}$, $k = 0, 1, 2$, and $m = 0, 1, \ldots, n-1$ we get $\theta_k^m\partial^{6j+6}(\mathfrak{o}(g) \otimes \mathfrak{t}(g)) = 0$.*

In the rest of the paper, we consider the lower indices k of μ_k^m, ϕ_k^m and θ_k^m as modulo 3. From the lemma above, we immediately have the following:

Corollary 4.4. (a) *For $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$*

$$\alpha_k^m\partial^1 = \phi_k^m\partial^{6j+7} = \begin{cases} \mu_0^0 - \mu_2^0 + \nu_0 & \text{if } k = 0 \text{ and } m = 0 \\ \mu_2^0 - \mu_1^0 - \nu_1 & \text{if } k = 2 \text{ and } m = 0 \\ \mu_k^m - \mu_{k+2}^m & \text{otherwise.} \end{cases}$$

Also, $\beta\partial^1 = \nu_1 - \nu_0$ and $\psi\partial^{6j+7} = \nu_0 - \nu_1$.

(b) *For $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$*

$$\mu_k^m\partial^{6j+2} = \begin{cases} (n+1)\eta & \text{if } k = m = 0, \text{ or if } k = 1 \text{ and } m = 0 \\ n\eta & \text{if } m = 0 \text{ and } k = 2 \\ 0 & \text{if } 0 < m \le n, \end{cases}$$

and $\nu_s\partial^{6j+2} = -\eta$ for $s = 0, 1$.

(c) *$\eta\partial^{6j+3} = 0$, and if $n > 0$ then $\theta_k^m\partial^{6j+3} = \phi_k^{m+1} - \phi_{k+2}^{m+1}$ for $k = 0, 1, 2$ and $m = 0, 1, \ldots, n-1$.*

(d) *For $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$*

$$\phi_k^m\partial^{6j+4} = \begin{cases} (n+1)\mu_0^n + n\mu_1^n + (n+1)\mu_2^n + \nu_0 & \text{if } k = m = 0 \\ (n+1)\mu_1^n + n\mu_2^n + (n+1)\mu_0^n & \text{if } k = 1 \text{ and } m = 0 \\ (n+1)\mu_2^n + n\mu_0^n + (n+1)\mu_1^n + \nu_1 & \text{if } k = 2 \text{ and } m = 0 \\ 0 & \text{if } 0 < m \le n \end{cases}$$

and $\psi\partial^{6j+4} = -\nu_0 - \nu_1$.

(e) *For* $k = 0, 1, 2$ *and* $m = 0, 1, \dots, n$

$$\mu_k^m \partial^{6j+5} = \begin{cases} \theta_k^m - \theta_{k+2}^m & \text{if } 0 \leq m \leq n-1 \\ \eta & \text{if } k = 0 \text{ and } m = n \\ -\eta & \text{if } k = 1 \text{ and } m = n \\ 0 & \text{if } k = 2 \text{ and } m = n \end{cases}$$

and for $s = 0, 1$

$$\nu_s \partial^{6j+5} = \begin{cases} -\eta & \text{if } s = 0 \\ \eta & \text{if } s = 1. \end{cases}$$

(f) $\eta \partial^{6j+6} = 0$, *and if* $n > 0$ *then* $\theta_k^m \partial^{6j+6} = 0$ *for* $k = 0, 1, 2$ *and* $m = 0, 1, \dots, n-1$.

4.3. A basis of $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$

We now find a K-basis of $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$ for $i \geq 0$. Keeping the notations from the previous subsections we have the following lemma.

Lemma 4.5. *For* $i \geq 1$ *we have the following* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$: *For* $j \geq 0$

(a) (1) *If* $n = 0$, *then* $\{\mu_0^0 - \mu_2^0 + \nu_0, \mu_1^0 - \mu_0^0, \mu_2^0 - \mu_1^0 - \nu_1\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+1}, \Lambda_n)$.

(2) *If* $n > 0$, *then* $\{\mu_0^0 - \mu_2^0 + \nu_0, \mu_1^0 - \mu_0^0, \mu_2^0 - \mu_1^0 - \nu_1, \mu_0^m - \mu_2^m, \mu_1^m - \mu_0^m \mid m = 1, \dots, n\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+1}, \Lambda_n)$.

(b) $\{\eta\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+2}, \Lambda_n)$.

(c) (1) *If* $n = 0$, *then* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+3}, \Lambda_n) = 0$.

(2) *If* $n > 0$, *then* $\{\phi_0^m - \phi_2^m, \phi_1^m - \phi_0^m \mid m = 1, \dots, n\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+3}, \Lambda_n)$.

(d) (1) *If* $\operatorname{char} K \mid 3n+2$, *then* $\{-\mu_1^n + \mu_2^n + \nu_0, \mu_2^n - \mu_0^n + \nu_1, \nu_0 + \nu_1\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+4}, \Lambda_n)$.

(2) *If* $\operatorname{char} K \nmid 3n+2$, *then* $\{\mu_2^n, -\mu_1^n + \mu_2^n + \nu_0, \mu_2^n - \mu_0^n + \nu_1, \nu_0 + \nu_1\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+4}, \Lambda_n)$.

(e) (1) *If* $n = 0$, *then* $\{\eta\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+5}, \Lambda_n)$.

(2) *If* $n > 0$, *then* $\{\theta_0^m - \theta_2^m, \theta_1^m - \theta_0^m, \eta \mid m = 0, 1, \dots, n-1\}$ *is a* K*-basis of* $\operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+5}, \Lambda_n)$.

(f) $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^{6j+6}, \Lambda_n) = 0$.

Proof. (a) We first consider a K-basis of $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^1, \Lambda_n)$. Let x be any element in $\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^0, \Lambda_n)$. Then, by Lemma 4.1 (a), x can be written in the form

$$x = \left(\sum_{k=0}^{2}\sum_{m=0}^{n} p_k^m \alpha_k^m\right) + q\beta$$

where $p_k^m, q \in K$ for $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$. By Corollary 4.4 (a) we get

$$\begin{aligned}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^1, \Lambda_n)(x) &= \left(\sum_{k=0}^{2}\sum_{m=0}^{n} p_k^m (\alpha_k^m \partial^1)\right) + q(\beta\partial^1)\\
&= \begin{cases} p_0^0(\mu_0^0 - \mu_2^0 + \nu_0) + p_1^0(\mu_1^0 - \mu_0^0) \\ \qquad + p_2^0(\mu_2^0 - \mu_1^0 - \nu_1) + q(\nu_1 - \nu_0) & \text{if } n = 0 \\[2ex] p_0^0(\mu_0^0 - \mu_2^0 + \nu_0) + p_1^0(\mu_1^0 - \mu_0^0) + p_2^0(\mu_2^0 - \mu_1^0 - \nu_1) \\ \qquad + q(\nu_1 - \nu_0) + \displaystyle\sum_{k=0}^{2}\sum_{m=1}^{n} p_k^m(\mu_k^m - \mu_{k+2}^m) & \text{if } n > 0 \end{cases}\end{aligned} \tag{4.1}$$

$$= \begin{cases} (p_0^0 - q)(\mu_0^0 - \mu_2^0 + \nu_0) + (p_1^0 - q)(\mu_1^0 - \mu_0^0) \\ \quad + (p_2^0 - q)(\mu_2^0 - \mu_1^0 - \nu_1) & \text{if } n = 0 \\[2ex] (p_0^0 - q)(\mu_0^0 - \mu_2^0 + \nu_0) + (p_1^0 - q)(\mu_1^0 - \mu_0^0) \\ \quad + (p_2^0 - q)(\mu_2^0 - \mu_1^0 - \nu_1) \\ \qquad + \displaystyle\sum_{m=1}^{n}\Big((p_0^m - p_2^m)(\mu_0^m - \mu_2^m) + (p_1^m - p_2^m)(\mu_1^m - \mu_0^m)\Big) & \text{if } n > 0. \end{cases}$$

Therefore $\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^1, \Lambda_n)(x) = x\partial^1$ belongs to

$$\begin{cases} K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) & \text{if } n = 0 \\[2ex] K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) \\ \qquad \bigoplus \left(\displaystyle\bigoplus_{m=1}^{n}\Big(K(\mu_0^m - \mu_2^m) \bigoplus K(\mu_1^m - \mu_0^m)\Big)\right) & \text{if } n > 0. \end{cases}$$

Conversely it is obvious by Corollary 4.4 (a) that, for $m = 1, \ldots, n$, the elements $\mu_0^0 - \mu_2^0 + \nu_0$, $\mu_1^0 - \mu_0^0$, $\mu_2^0 - \mu_1^0 - \nu_1$, $\mu_0^m - \mu_2^m$, and $\mu_1^m - \mu_0^m$ are in

$\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^1, \Lambda_n)$. Hence we get

$$\begin{aligned}
&\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^1, \Lambda_n)\\
&= \begin{cases} K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) & \text{if } n = 0 \\ K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) & \\ \qquad \bigoplus \left(\bigoplus_{m=1}^{n} \Big(K(\mu_0^m - \mu_2^m) \bigoplus K(\mu_1^m - \mu_0^m) \Big) \right) & \text{if } n > 0. \end{cases}
\end{aligned}$$

This shows the desired results in this case.

A similar computation gives the desired basis of $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^{6j+7}, \Lambda_n)$ for $j \geq 0$. Therefore the statement (a) is proved.

(b) Let $x \in \operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(Q^{6j+1}, \Lambda_n)$. Then, by Lemma 4.1 (c), x can be written in the form

$$x = \left(\sum_{k=0}^{2} \sum_{m=0}^{n} p_k^m \mu_k^m \right) + q_0\nu_0 + q_1\nu_1$$

where $p_k^m, q_0, q_1 \in K$ for $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$. Then we have by Corollary 4.4 (b) that

$$\begin{aligned}
\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^{6j+2}, \Lambda_n)(x) &= \left(\sum_{k=0}^{2} \sum_{m=0}^{n} p_k^m (\mu_k^m \partial^{6j+2}) \right) + q_0(\nu_0\partial^{6j+2}) + q_1(\nu_1\partial^{6j+2}) \\
&= \Big((n+1)p_0^0\eta + (n+1)p_1^0\eta + np_2^0\eta \Big) + (-q_0\eta) + (-q_1\eta) \\
&= \Big((n+1)p_0^0 + (n+1)p_1^0 + np_2^0 - q_0 - q_1 \Big)\eta \qquad (4.2) \\
&\in K\eta.
\end{aligned}$$

Conversely, we get $\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^{6j+2}, \Lambda_n)(-\nu_0) = \eta$ by Corollary 4.4 (b). So it follows that $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^{6j+2}, \Lambda_n) = K\eta$. Hence the statement (b) is proved.

Similar observations show the remaining cases (c)–(f). □

As an immediate consequence we have the dimension of $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^i, \Lambda_n)$ for $i \geq 1$.

Corollary 4.6. *For $i \geq 1$ the dimension of* $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^i, \Lambda_n)$ *is as follows*:

$$\dim_K \operatorname{Im}\operatorname{Hom}_{\Lambda_n^{\mathrm{e}}}(\partial^i, \Lambda_n) = \begin{cases} 0 & \textit{if } i \equiv 0 \pmod 6 \\ 2n+3 & \textit{if } i \equiv 1 \pmod 6 \\ 1 & \textit{if } i \equiv 2 \pmod 6 \\ 2n & \textit{if } i \equiv 3 \pmod 6 \\ 3 & \textit{if } i \equiv 4 \pmod 6 \textit{ and } \operatorname{char} K \mid 3n+2 \\ 4 & \textit{if } i \equiv 4 \pmod 6 \textit{ and } \operatorname{char} K \nmid 3n+2 \\ 2n+1 & \textit{if } i \equiv 5 \pmod 6. \end{cases}$$

4.4. A basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$

Now we find a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$ for each $i \geq 0$. In the following we note that if $\operatorname{char} K \mid 3n+2$, then $\operatorname{char} K \neq 3$.

Lemma 4.7. *For $j \geq 0$ we have the following K-basis:*

(a) (1) *If $n = 0$, then $\{\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^1, \Lambda_n)$.*

(2) *If $n > 0$, then $\{\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta, \alpha_0^m + \alpha_1^m + \alpha_2^m \mid m = 1, \dots, n\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^1, \Lambda_n)$.*

(b) (1) *If $n = 0$, then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 - \psi\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+7}, \Lambda_n)$.*

(2) *If $n > 0$, then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 - \psi, \phi_0^m + \phi_1^m + \phi_2^m \mid m = 1, \dots, n\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+7}, \Lambda_n)$.*

(c) (1) *If $n = 0$, then $\{\mu_0^0 + \nu_0, \mu_0^0 - \mu_2^0 + \nu_0, \mu_1^0 - \mu_0^0, \mu_2^0 - \mu_1^0 - \nu_1\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+2}, \Lambda_n)$.*

(2) *If $n > 0$, then $\{\mu_0^0 + (n+1)\nu_0, \mu_0^0 - \mu_2^0 + \nu_0, \mu_1^0 - \mu_0^0, \mu_2^0 - \mu_1^0 - \nu_1, \mu_0^m, \mu_0^m - \mu_2^m, \mu_1^m - \mu_0^m \mid m = 1, \dots, n\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+2}, \Lambda_n)$.*

(d) (1) *If $n = 0$, then $\{\eta\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+3}, \Lambda_n)$.*

(2) *If $n > 0$, then $\{\theta_0^m + \theta_1^m + \theta_2^m, \eta \mid m = 0, 1, \dots, n-1\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+3}, \Lambda_n)$.*

(e) (1) *If $\operatorname{char} K \mid 3n+2$ and $n = 0$ (so $\operatorname{char} K = 2$), then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 + \psi\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+4}, \Lambda_n)$.*

(2) *If $\operatorname{char} K \mid 3n+2$ and $n > 0$ (so $\operatorname{char} K \neq 3$), then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 + \psi, \phi_0^m + \phi_1^m + \phi_2^m, \phi_0^m - \phi_2^m, \phi_1^m - \phi_0^m \mid m = 1, \dots, n\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+4}, \Lambda_n)$.*

(3) *If $\operatorname{char} K \nmid 3n+2$ and $n = 0$, then $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+4}, \Lambda_n) = 0$.*

(4) *If $\operatorname{char} K \nmid 3n+2$ and $n > 0$, then $\{\phi_0^m, \phi_0^m - \phi_2^m, \phi_1^m - \phi_0^m \mid m = 1, \dots, n\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+4}, \Lambda_n)$.*

(f) (1) *If $\operatorname{char} K \mid 3n+2$ and $n = 0$ (so $\operatorname{char} K = 2$), then $\{\mu_0^0 + \mu_1^0 + \mu_2^0, -\mu_1^0 + \mu_2^0 + \nu_0, \mu_2^0 - \mu_0^0 + \nu_1, \nu_0 + \nu_1\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+5}, \Lambda_n)$.*

(2) *If $\operatorname{char} K \mid 3n+2$ and $n > 0$ (so $\operatorname{char} K \neq 3$), then $\{\mu_0^m + \mu_1^m + \mu_2^m, -\mu_1^n + \mu_2^n + \nu_0, \mu_2^n - \mu_0^n + \nu_1, \nu_0 + \nu_1 \mid m = 0, 1, \dots, n\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+5}, \Lambda_n)$.*

(3) *If $\operatorname{char} K \nmid 3n+2$ and $n = 0$, then $\{\mu_2^0, -\mu_1^0 + \mu_2^0 + \nu_0, \mu_2^0 - \mu_0^0 + \nu_1, \nu_0 + \nu_1\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+5}, \Lambda_n)$.*

(4) *If $\operatorname{char} K \nmid 3n+2$ and $n > 0$, then $\{\mu_0^m + \mu_1^m + \mu_2^m, \mu_2^n, -\mu_1^n + \mu_2^n + \nu_0, \mu_2^n - \mu_0^n + \nu_1, \nu_0 + \nu_1 \mid m = 0, 1, \dots, n-1\}$ is a K-basis of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+5}, \Lambda_n)$.*

(g) (1) *If* $n = 0$*, then* $\{\eta\}$ *is a* K*-basis of* $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+6}, \Lambda_n)$.

(2) *If* $n > 0$*, then* $\{\theta_0^m, \theta_0^m - \theta_2^m, \theta_1^m - \theta_0^m, \eta \mid m = 0, 1, \ldots, n-1\}$ *is a* K*-basis of* $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+6}, \Lambda_n)$.

Proof. (a) Let $x = \left(\sum_{k=0}^{2}\sum_{m=0}^{n} p_k^m \alpha_k^m\right) + q\beta \in \operatorname{Hom}_{\Lambda_n^e}(Q^0, \Lambda_n)$, where p_k^m, $q \in K$ for $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$. Suppose that $\operatorname{Hom}_{\Lambda_n^e}(\partial^1, \Lambda_n)(x) = 0$. Then, by (4.1), we have $p_0^0 - q = p_1^0 - q = p_2^0 - q = 0$ if $n = 0$; and $p_0^0 - q = p_1^0 - q = p_2^0 - q = p_0^m - p_2^m = p_1^m - p_2^m = 0$ for $m = 1, \ldots, n$ if $n > 0$. Hence $p_0^0 = p_1^0 = p_2^0 = q$ if $n = 0$; and $p_0^0 = p_1^0 = p_2^0 = q$ and $p_0^m = p_1^m = p_2^m$ for $m = 1, \ldots, n$ if $n > 0$. Therefore we get

$$x = \begin{cases} p_0^0(\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta) & \text{if } n = 0 \\ p_0^0(\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta) + \left(\displaystyle\sum_{m=1}^{n} p_0^m(\alpha_0^m + \alpha_1^m + \alpha_2^m)\right) & \text{if } n > 0. \end{cases}$$

Accordingly it follows that

$$x \in \begin{cases} K(\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta) & \text{if } n = 0 \\ K(\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta) \bigoplus \left(\displaystyle\bigoplus_{m=1}^{n} K(\alpha_0^m + \alpha_1^m + \alpha_2^m)\right) & \text{if } n > 0. \end{cases}$$

Conversely, it is easy to check that the elements $\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta$ and $\alpha_0^m + \alpha_1^m + \alpha_2^m$ are in $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^1, \Lambda_n)$ for $m = 1, \ldots, n$. Thus

$$\begin{aligned} &\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^1, \Lambda_n) \\ &\quad = \begin{cases} K(\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta) & \text{if } n = 0 \\ K(\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta) \bigoplus \left(\displaystyle\bigoplus_{m=1}^{n} K(\alpha_0^m + \alpha_1^m + \alpha_2^m)\right) & \text{if } n > 0. \end{cases} \end{aligned}$$

This gives the required K-basis in (a).

A similar argument provides the required K-basis in (b).

(c) Let $x = \left(\sum_{k=0}^{2}\sum_{m=0}^{n} p_k^m \mu_k^m\right) + q_0\nu_0 + q_1\nu_1 \in \operatorname{Hom}_{\Lambda_n^e}(Q^{6j+1}, \Lambda_n)$, where $p_k^m, q_0, q_1 \in K$ for $k = 0, 1, 2$ and $m = 0, 1, \ldots, n$. Suppose that $\operatorname{Hom}_{\Lambda_n^e}(\partial^{6j+2}, \Lambda_n)(x) = 0$. Then (4.2) yields $(n+1)p_0^0 + (n+1)p_1^0 + np_2^0 - q_0 -$

$q_1 = 0$, so that $q_0 = (n+1)p_0^0 + (n+1)p_1^0 + np_2^0 - q_1$. Thus we have that

$$x = \begin{cases} p_0^0\mu_0^0 + p_1^0\mu_1^0 + p_2^0\mu_2^0 + \left(p_0^0 + p_1^0 - q_1\right)\nu_0 + q_1\nu_1 & \text{if } n = 0 \\ p_0^0\mu_0^0 + p_1^0\mu_1^0 + p_2^0\mu_2^0 \\ \quad + \left((n+1)p_0^0 + (n+1)p_1^0 + np_2^0 - q_1\right)\nu_0 + q_1\nu_1 \\ \quad\quad + \left(\sum_{k=0}^{2}\sum_{m=1}^{n} p_k^m\mu_k^m\right) + q_0\nu_0 + q_1\nu_1 & \text{if } n > 0 \end{cases}$$

$$= \begin{cases} (p_0^0 + p_1^0 + p_2^0)(\mu_0^0 + \nu_0) + (-p_2^0 - q_1)(\mu_0^0 - \mu_2^0 + \nu_0) \\ \quad +(p_1^0 - q_1)(\mu_1^0 - \mu_0^0) - q_1(\mu_2^0 - \mu_1^0 - \nu_1) & \text{if } n = 0 \\ (p_0^0 + p_1^0 + p_2^0)\left(\mu_0^0 + (n+1)\nu_0\right) \\ \quad +(-p_2^0 - q_1)(\mu_0^0 - \mu_2^0 + \nu_0) + (p_1^0 - q_1)(\mu_1^0 - \mu_0^0) \\ \quad\quad -q_1(\mu_2^0 - \mu_1^0 - \nu_1) + \sum_{m=1}^{n}\Big((p_0^m + p_1^m + p_2^m)\mu_0^m \\ \quad\quad\quad -p_2^m(\mu_0^m - \mu_2^m) + p_1^m(\mu_1^m - \mu_0^m)\Big) & \text{if } n > 0. \end{cases}$$

This implies that x belongs to

$$\begin{cases} K(\mu_0^0 + \nu_0) \bigoplus K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \\ \quad \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) & \text{if } n = 0 \\ K\left(\mu_0^0 + (n+1)\nu_0\right) \bigoplus K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \\ \quad \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) \\ \quad\quad \oplus \left(\bigoplus_{m=1}^{n}\Big(K\mu_0^m \bigoplus K(\mu_0^m - \mu_2^m) \bigoplus K(\mu_1^m - \mu_0^m)\Big)\right) & \text{if } n > 0. \end{cases}$$

Conversely it is easy to check by Corollary 4.4 (b) that, for $m = 1, \dots, n$, the elements $\mu_0^0 + \nu_0$, $\mu_0^0 - \mu_2^0 + \nu_0$, $\mu_1^0 - \mu_0^0$, $\mu_2^0 - \mu_1^0 - \nu_1$, $\mu_2^0 - \mu_1^0 - \nu_1$, μ_0^m, $\mu_0^m - \mu_2^m$, and $\mu_1^m - \mu_0^m$ lie in the kernel of $\mathrm{Hom}_{\Lambda_n^e}(\partial^{6j+2}, \Lambda_n)$. This shows that

$$\mathrm{Ker}\,\mathrm{Hom}_{\Lambda_n^e}(\partial^{6j+2}, \Lambda_n)$$

$$= \begin{cases} K(\mu_0^0 + \nu_0) \bigoplus K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \\ \quad \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) & \text{if } n = 0 \\ K\left(\mu_0^0 + (n+1)\nu_0\right) \bigoplus K(\mu_0^0 - \mu_2^0 + \nu_0) \bigoplus K(\mu_1^0 - \mu_0^0) \\ \quad \bigoplus K(\mu_2^0 - \mu_1^0 - \nu_1) \\ \quad\quad \oplus \left(\bigoplus_{m=1}^{n}\Big(K\mu_0^m \bigoplus K(\mu_0^m - \mu_2^m) \bigoplus K(\mu_1^m - \mu_0^m)\Big)\right) & \text{if } n > 0. \end{cases}$$

So we have got the desired K-basis in this case.

Similar arguments provide the required K-basis in the remaining cases. □

As a direct consequence we have the dimensions of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$.

Corollary 4.8. *For $i \geq 1$, the dimension of $\operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n)$ is as follows*:

$$\dim_K \operatorname{Ker}\operatorname{Hom}_{\Lambda_n^e}(\partial^i, \Lambda_n) = \begin{cases} 3n+1 & \textit{if } i \equiv 0 \pmod 6 \\ n+1 & \textit{if } i \equiv 1 \pmod 6 \\ 3n+4 & \textit{if } i \equiv 2 \pmod 6 \\ n+1 & \textit{if } i \equiv 3 \pmod 6 \\ 3n+1 & \textit{if } i \equiv 4 \pmod 6 \textit{ and } \operatorname{char} K \mid 3n+2 \\ 3n & \textit{if } i \equiv 4 \pmod 6 \textit{ and } \operatorname{char} K \nmid 3n+2 \\ n+4 & \textit{if } i \equiv 5 \pmod 6. \end{cases}$$

4.5. The Hochschild cohomology groups of Λ_n

Now, by Lemmas 4.5 and 4.7, we have a K-basis of $\mathrm{HH}^i(\Lambda_n)$ for all $i \geq 0$.

Proposition 4.9. *For $j \geq 0$ we have the following*:

(a) (1) *If $n = 0$, then $\{\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta\}$ is a K-basis of $\mathrm{HH}^0(\Lambda_n)$.*

(2) *If $n > 0$, then $\{\alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta, \alpha_0^m + \alpha_1^m + \alpha_2^m \mid m = 1, \dots, n\}$ is a K-basis of $\mathrm{HH}^0(\Lambda_n)$.*

(b) (1) *If $n = 0$, then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 - \psi\}$ is a K-basis of $\mathrm{HH}^{6j+6}(\Lambda_n)$.*

(2) *If $n > 0$, then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 - \psi, \phi_0^m + \phi_1^m + \phi_2^m \mid m = 1, \dots, n\}$ is a K-basis of $\mathrm{HH}^{6j+6}(\Lambda_n)$.*

(c) (1) *If $n = 0$, then $\{\mu_0^0 + \nu_0\}$ is a K-basis of $\mathrm{HH}^{6j+1}(\Lambda_n)$.*

(2) *If $n > 0$, then $\{\mu_0^0 + (n+1)\nu_0, \mu_0^m \mid m = 1, \dots, n\}$ is a K-basis of $\mathrm{HH}^{6j+1}(\Lambda_n)$.*

(d) (1) *If $n = 0$, then $\mathrm{HH}^{6j+2}(\Lambda_n) = 0$.*

(2) *If $n > 0$, then $\{\theta_0^m + \theta_1^m + \theta_2^m \mid m = 0, 1, \dots, n-1\}$ is a K-basis of $\mathrm{HH}^{6j+2}(\Lambda_n)$.*

(e) (1) *If $\operatorname{char} K \mid 3n+2$ and $n = 0$ (hence $\operatorname{char} K = 2$), then $\{\phi_0^0+\phi_1^0+\phi_2^0+\psi\}$ is a K-basis of $\mathrm{HH}^{6j+3}(\Lambda_n)$.*

(2) *If $\operatorname{char} K \mid 3n+2$ and $n > 0$, then $\{\phi_0^0 + \phi_1^0 + \phi_2^0 + \psi, \phi_0^m + \phi_1^m + \phi_2^m \mid m = 1, \dots, n\}$ is a K-basis of $\mathrm{HH}^{6j+3}(\Lambda_n)$.*

(3) *If* $\operatorname{char} K \nmid 3n+2$ *and* $n = 0$, *then* $\mathrm{HH}^{6j+3}(\Lambda_n) = 0$.

(4) *If* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$, *then* $\{\phi_0^m \mid m = 1, \dots, n\}$ *is a* K*-basis of* $\mathrm{HH}^{6j+3}(\Lambda_n)$.

(f) (1) *If* $\operatorname{char} K \mid 3n+2$, *then* $\{\mu_0^m + \mu_1^m + \mu_2^m \mid m = 0, 1, \dots, n\}$ *is a* K*-basis of* $\mathrm{HH}^{6j+4}(\Lambda_n)$.

(2) *If* $\operatorname{char} K \nmid 3n+2$ *and* $n = 0$, *then* $\mathrm{HH}^{6j+4}(\Lambda_n) = 0$.

(3) *If* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$, *then* $\{\mu_0^m + \mu_1^m + \mu_2^m \mid m = 0, 1, \dots, n-1\}$ *is a* K*-basis of* $\mathrm{HH}^{6j+4}(\Lambda_n)$.

(g) (1) *If* $n = 0$, *then* $\mathrm{HH}^{6j+5}(\Lambda_n) = 0$.

(2) *If* $n > 0$, *then* $\{\theta_0^m \mid m = 0, 1, \dots, n-1\}$ *is a* K*-basis of* $\mathrm{HH}^{6j+5}(\Lambda_n)$.

This proposition provides us with the main result in this paper:

Theorem 4.10. *For* $n \geq 0$ *and* $i \geq 0$ *the dimension of* $\mathrm{HH}^i(\Lambda_n)$ *is given as follows*:

$$\dim_K \mathrm{HH}^i(\Lambda_n) = \begin{cases} n+1 & \text{if } i \equiv 0 \pmod 6 \\ n+1 & \text{if } i \equiv 1 \pmod 6 \\ n & \text{if } i \equiv 2 \pmod 6 \\ n+1 & \text{if } i \equiv 3 \pmod 6 \text{ and } \operatorname{char} K \mid 3n+2 \\ n & \text{if } i \equiv 3 \pmod 6 \text{ and } \operatorname{char} K \nmid 3n+2 \\ n+1 & \text{if } i \equiv 4 \pmod 6 \text{ and } \operatorname{char} K \mid 3n+2 \\ n & \text{if } i \equiv 4 \pmod 6 \text{ and } \operatorname{char} K \nmid 3n+2 \\ n & \text{if } i \equiv 5 \pmod 6. \end{cases}$$

Remark 4.11. Recall that, for any algebra C, the 0th Hochschild cohomology group $\mathrm{HH}^0(C)$ is isomorphic to the centre $Z(C)$ of C. Hence, by Proposition 4.9 (a), the set $\{e_0 + e_1 + e_2 + f_1\}$ is a K-basis of $Z(\Lambda_0)$, and if $n > 0$ the set $\{e_0 + e_1 + e_2 + f_1, \sum_{i=0}^{2}(a_i a_{i+1} a_{i+2})^m \mid m = 1, \dots, n\}$ is a K-basis of $Z(\Lambda_n)$.

We end this paper by giving the dimension of the Hochschild cohomology group of the cluster-tilted algebra Λ_0. By setting $n = 0$ in Theorem 4.10, we have the following corollary:

Corollary 4.12. *For $i \geq 0$ the dimension of* $\mathrm{HH}^i(\Lambda_0)$ *is as follows*:

$$\dim_K \mathrm{HH}^i(\Lambda_0) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 6 \\ 1 & \text{if } i \equiv 1 \pmod 6 \\ 0 & \text{if } i \equiv 2 \pmod 6 \\ 1 & \text{if } i \equiv 3 \pmod 6 \text{ and } \mathrm{char}\, K = 2 \\ 0 & \text{if } i \equiv 3 \pmod 6 \text{ and } \mathrm{char}\, K \neq 2 \\ 1 & \text{if } i \equiv 4 \pmod 6 \text{ and } \mathrm{char}\, K = 2 \\ 0 & \text{if } i \equiv 4 \pmod 6 \text{ and } \mathrm{char}\, K \neq 2 \\ 0 & \text{if } i \equiv 5 \pmod 6. \end{cases}$$

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