

3-dimensional quasi-Sasakian manifolds and Ricci solitons

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(Received November 20, 2011; Revised June 5, 2012)

Abstract. The object of the present paper is to obtain a necessary and sufficient condition for a 3-dimensional quasi-Sasakian manifold to be an η -Einstein manifold. An example is given to verify the theorem. Finally Ricci solitons and gradient Ricci solitons have been studied.

AMS 2010 Mathematics Subject Classification. 53c15, 53c40.

Key words and phrases. quasi-Sasakian manifold, structure function, Ricci soliton, gradient Ricci soliton.

§1. Introduction

The notion of quasi-Sasakian structure was introduced by D. E. Blair [4] to unify Sasakian and cosymplectic structures. S. Tanno [24] also added some remarks on quasi-Sasakian structures. The properties of quasi-Sasakian manifolds have been studied by several authors, viz., J. C. Gonzalez and D. Chinea [13], S. Kanemaki [16], [17] and J. A. Oubina [22], De and Sarkar [9], De and Mondal [8]. B. H. Kim [18] studied quasi-Sasakian manifolds and proved that every fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosymplectic structure. Recently, quasi-Sasakian manifolds have been the subject of growing interest in view of finding the significant applications to physics, in particular to super gravity and magnetic theory [1], [2]. Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory [3], [12]. Motivated by the roles of curvature tensor and Ricci tensor of quasi-Sasakian manifolds in string theory [3] we would like to study some curvature properties and Ricci soliton in a 3-dimensional quasi-Sasakian manifold. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Z. Olszak [20] and with the help of this function

he has obtained necessary and sufficient conditions for the manifold to be conformally flat [21]. Next he has proved that if the manifold is additionally conformally flat with $\beta = \text{constant}$, then (a) the manifold is locally a product of R and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). It is also known that D -homothetic and homothetic deformations of (quasi-) Sasakian structures lead to quasi-Sasakian structures [19]. In dimension 3, certain D -conformal deformations of Sasakian structures yield quasi-Sasakian and non-Sasakian structures of rank 3 [24].

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [6]. On the manifold M , a Ricci soliton is a triple (g, V, λ) with g , a Riemannian metric, V a vector field and λ a real scalar such that

$$(1.1) \quad \mathcal{L}_V g + 2S + 2\lambda g = 0$$

where \mathcal{L} is the Lie derivative. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.1) takes the form

$$(1.2) \quad \nabla \nabla f = S + \lambda g,$$

where ∇ denotes the Riemannian connection.

A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [14]), and also in dimension 3 (Ivey [15]). For details we refer to Chow and Knoff [7] and Derdzinski [11]. Recently in [6] C. Calin and M. Crasmareanu have studied Ricci solitons in f -Kenmotsu manifolds. We also recall the following significant result of Perelman [23]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optic, phase space of dynamical system, thermodynamics and control theory.

The paper is organized as follows: After preliminaries in section 3 we obtain a necessary and sufficient condition for a 3-dimensional quasi-Sasakian manifold to be an η -Einstein manifold and also verify the result by a concrete example. In the last section we study Ricci solitons and gradient Ricci solitons in 3-dimensional quasi-Sasakian manifold and prove that in a 3-dimensional non-cosymplectic quasi-Sasakian manifold, the Ricci soliton (g, ξ, λ) is expanding provided β is constant and the manifold is of constant curvature. Finally

we prove that if the metric g of a 3-dimensional quasi-Sasakian manifold with constant structure function β is a gradient Ricci soliton, then the manifold is an Einstein manifold.

§2. Preliminaries

Let M be a $(2n+1)$ -dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) , where ϕ, ξ, η are tensor fields on M of types $(1, 1), (1, 0), (0, 1)$ respectively, such that [5]

$$(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Then also

$$(2.2) \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).$$

Let Φ be fundamental 2-form of M defined by

$$\Phi(X, Y) = g(X, \phi Y) \quad X, Y \in T(M).$$

M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair [4]. The normality condition gives that the induced almost complex structure of $M \times \mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M . The rank of a quasi-Sasakian structure is always an odd integer [4] which is equal to 1 if the structure is cosymplectic and it is equal to $(2n + 1)$ if the structure is Sasakian.

A Riemannian manifold M is said to be an η -Einstein manifold if it satisfies the condition

$$(2.3) \quad S(X, Y) = \lambda g(X, Y) + \delta \eta(X)\eta(Y),$$

where λ and δ are smooth functions on the manifold.

In [10] De and Sengupta prove the following:

Lemma 1. *A parallel symmetric $(0, 2)$ tensor field in a 3-dimensional non-cosymplectic quasi-Sasakian manifold is a constant multiple of the associated metric tensor.*

§3. 3-dimensional quasi-Sasakian manifold

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if [20]

$$(3.1) \quad \nabla_X \xi = -\beta \phi X, \quad X \in T(M),$$

for a certain function β on M , such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M . Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. If $\beta = \text{constant}$, then the manifold reduces to a β -Sasakian manifold and if in particular $\beta = 1$, the manifold becomes a Sasakian manifold. Here we have shown that the assumption $\xi\beta = 0$ is not necessary.

As a consequence of (3.1), we have [20]

$$(3.2) \quad (\nabla_X \phi)(Y) = \beta(g(X, Y)\xi - \eta(Y)X), \quad X, Y \in T(M).$$

Because of (3.1) and (3.2), we find

$$\nabla_X(\nabla_Y \xi) = -(X\beta)\phi Y - \beta^2\{g(X, Y)\xi - \eta(Y)X\} - \beta\phi(\nabla_X Y)$$

which implies that

$$(3.3) \quad R(X, Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}.$$

Thus we get from (3.3)

$$(3.4) \quad \begin{aligned} R(X, Y, Z, \xi) &= (X\beta)g(\phi Y, Z) - (Y\beta)g(\phi X, Z) \\ &- \beta^2\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}, \end{aligned}$$

where $R(X, Y, Z, W) = g(R(X, Y, Z), W)$.

Putting $X = \xi$, in (3.4) we obtain

$$(3.5) \quad R(\xi, Y, Z, \xi) = \beta^2\{g(Y, Z) - \eta(Y)\eta(Z)\} + g(\phi Y, Z)\xi\beta.$$

Interchanging Y and Z of (3.5) yields

$$(3.6) \quad R(\xi, Z, Y, \xi) = \beta^2\{g(Y, Z) - \eta(Y)\eta(Z)\} + g(\phi Z, Y)\xi\beta.$$

Since $R(\xi, Y, Z, \xi) = R(Z, \xi, \xi, Y) = R(\xi, Z, Y, \xi)$, from (3.5) and (3.6) we have

$$\{g(\phi Y, Z) - g(\phi Z, Y)\}\xi\beta = 0.$$

Therefore, we can easily verify that $\xi\beta = 0$. So we have the following:

Proposition 1. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold the structure function β satisfies the condition $\xi\beta = 0$.*

Let M be a three-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [21]

$$(3.7) \quad \begin{aligned} S(Y, Z) &= \left(\frac{r}{2} - \beta^2\right)g(Y, Z) + \left(3\beta^2 - \frac{r}{2}\right)\eta(Y)\eta(Z) \\ &- \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y), \end{aligned}$$

where r is the scalar curvature of M .

From (3.7), the Ricci operator Q can be written as

$$(3.8) \quad QX = \left(\frac{r}{2} - \beta^2\right)X + \left(3\beta^2 - \frac{r}{2}\right)\eta(X)\xi - \eta(X)(\phi \text{grad } \beta) - d\beta(\phi X)\xi,$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(\text{grad } f, X)$.

From (3.7) it is clear that if $\beta = \text{constant}$ then M is an η -Einstein manifold. Conversely if we consider that M is an η -Einstein manifold, then from (3.7) and (2.3), we have

$$\eta(X)d\beta(\phi Y) + \eta(Y)d\beta(\phi X) = \left(-\lambda + \frac{r}{2} - \beta^2\right)g(X, Y) + \left(-\delta - \frac{r}{2} + 3\beta^2\right)\eta(X)\eta(Y).$$

Taking $Y = \xi$ in the last equation we get

$$d\beta(\phi X) = (-\lambda - \delta + 2\beta^2)\eta(X).$$

Now taking ϕX instead of X in the above equation and using Proposition 1 we obtain that β is a constant. Hence the Ricci tensor S of an η -Einstein quasi-Sasakian manifold is of the form

$$S(Y, Z) = \left(\frac{r}{2} - \beta^2\right)g(Y, Z) + \left(3\beta^2 - \frac{r}{2}\right)\eta(Y)\eta(Z).$$

Hence we can state the following:

Theorem 1. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold is an η -Einstein manifold if and only if β is constant.*

We verify the above theorem by an example.

Example: We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq 0\}$, where (x, y, z) are standard co-ordinate of \mathbb{R}^3 .

The vector fields

$$e_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M . Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_1, e_2) = g(e_2, e_3) = 0 \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in T(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g , we have

$$\begin{aligned} \eta(e_3) &= 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \\ g(\phi Z, \phi W) &= g(Z, W) - \eta(Z)\eta(W), \end{aligned}$$

for any $Z, W \in T(M)$.

Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M .

Let ∇ be the Levi-Civita connection with respect to metric g . Then we have

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0 \quad \text{and} \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ (3.9) \quad &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned}$$

which is known as Koszul's formula.

Using (3.9) we obtain $2g(\nabla_{e_1} e_3, X) = 2g(-\frac{1}{2}e_2, X)$ for all $X \in T(M)$.

Thus, $\nabla_{e_1} e_3 = -\frac{1}{2}e_2$.

(3.9) further yields

$$\begin{aligned} \nabla_{e_1} e_3 &= -\frac{1}{2}e_2, \quad \nabla_{e_1} e_2 = \frac{1}{2}e_3, \quad \nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_3 = \frac{1}{2}e_1, \quad \nabla_{e_2} e_2 = 0, \\ (3.10) \quad \nabla_{e_2} e_1 &= -\frac{1}{2}e_3, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = \frac{1}{2}e_1, \quad \nabla_{e_3} e_1 = -\frac{1}{2}e_2. \end{aligned}$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula $\nabla_X \xi = -\beta \phi X$ for $\beta = -\frac{1}{2}$. Hence the manifold is a three-dimensional quasi-Sasakian manifold with the constant structure function β .

It is known that

$$(3.11) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

With the help of the above results and using (3.11) it can be easily verified that

$$\begin{aligned} R(e_1, e_2)e_3 &= 0, & R(e_2, e_3)e_3 &= \frac{1}{4}e_2, & R(e_1, e_3)e_3 &= \frac{1}{4}e_1, \\ R(e_1, e_2)e_2 &= -\frac{3}{4}e_1, & R(e_2, e_3)e_2 &= -\frac{1}{4}e_3, & R(e_1, e_3)e_2 &= 0, \\ R(e_1, e_2)e_1 &= \frac{3}{4}e_2, & R(e_2, e_3)e_1 &= 0, & R(e_1, e_3)e_1 &= -\frac{1}{4}e_3. \end{aligned}$$

From the above expression of the curvature tensor we obtain

$$S(e_1, e_1) = -\frac{1}{2}, \quad S(e_2, e_2) = -\frac{1}{2} \quad \text{and} \quad S(e_3, e_3) = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} r &= S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) \\ &= -\frac{1}{2}. \end{aligned}$$

Here we note that the scalar curvature r is a constant.

With the help of the above expressions of the Ricci tensor it can be easily verified that the manifold satisfies (2.3) for $\lambda = -\frac{1}{2}$ and $\delta = 1$. Hence the manifold is an η -Einstein manifold. Therefore Theorem 1 is verified.

§4. Ricci solitons and Gradient Ricci solitons

Suppose a 3-dimensional quasi-Sasakian manifold admits a Ricci soliton defined by (1.1). It is well known that $\nabla g = 0$. Since λ in the Ricci soliton equation (1.1) is a constant, so $\nabla \lambda g = 0$. Thus $\mathcal{L}_V g + 2S$ is parallel. Hence using Lemma 1 we can say that $\mathcal{L}_V g + 2S$ is a constant multiple of metric tensors g , that is, $\mathcal{L}_V g + 2S = ag$, where a is constant. Hence $\mathcal{L}_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$. Using (1.1) we get $\lambda = -a/2$. So we have the following:

Proposition 2. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold, the Ricci soliton (g, V, λ) is shrinking or expanding according as a is positive or negative.*

Now in particular we investigate the case $V = \xi$. Then (1.1) reduces to

$$(4.1) \quad \mathcal{L}_\xi g + 2S + 2\lambda g = 0.$$

It is known that [4] in a 3-dimensional quasi-Sasakian manifold ξ is Killing, that is, $\mathcal{L}_\xi g = 0$. Then from (4.1) $\lambda = -S(\xi, \xi) = -2\beta^2$, provided β is constant. Also from (4.1) it follows that the manifold is an Einstein manifold. But it is known [25] that a 3-dimensional Einstein manifold is a manifold of constant curvature.

Thus we have

Corollary 1. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold, the Ricci soliton (g, ξ, λ) is shrinking provided β is constant and the manifold is of constant curvature.*

Let M be a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β and g a gradient Ricci soliton. Then the equation (1.2) can be written as

$$(4.2) \quad \nabla_Y Df = QY + \lambda Y$$

for all vector fields Y in M , where D denotes the gradient operator of g . From (4.2) it follows that

$$(4.3) \quad R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \quad X, Y \in TM.$$

Using (3.3) we have

$$(4.4) \quad g(R(\xi, Y)Df, \xi) = g(\beta^2(Df - (\xi f)\xi), Y).$$

Also in a 3-dimensional quasi-Sasakian manifold, it follows that

$$(4.5) \quad g((\nabla_\xi Q)Y - (\nabla_Y Q)\xi, \xi) = 0.$$

From (4.3), (4.4) and (4.5) we get

$$\beta^2(Df - (\xi f)\xi) = 0$$

that is,

$$(4.6) \quad Df = (\xi f)\xi,$$

since M is non-cosymplectic. Using (4.6) in (4.2) we obtain

$$(4.7) \quad S(X, Y) + \lambda g(X, Y) = -\beta(\xi f)g(\phi Y, X) + Y(\xi f)\eta(X).$$

Putting $X = \xi$ in (4.7) we get

$$(4.8) \quad Y(\xi f) = (2\beta^2 + \lambda)\eta(Y).$$

From (4.7) and (4.8) we get

$$(4.9) \quad S(X, Y) + \lambda g(X, Y) = -\beta(\xi f)g(\phi Y, X) + (2\beta^2 + \lambda)\eta(X)\eta(Y).$$

Using (4.9) in (4.2), we have

$$(4.10) \quad \nabla_Y Df = (2\beta^2 + \lambda)\eta(Y)\xi - \beta(\xi f)\phi Y.$$

Using (4.10) we compute $R(X, Y)Df$ and obtain

$$(4.11) \quad g(R(X, Y)(\xi f)\xi, \xi) = (2\beta^2 + \lambda)d\eta(X, Y).$$

Thus we get

$$(4.12) \quad 2\beta^2 + \lambda = 0.$$

Therefore from equation (4.8) we have $Y(\xi f) = 0$ that is $\xi f = c$, where c is a constant. Thus the equation (4.6) gives $df = c\eta$. Its exterior derivative implies that $cd\eta = 0$, which implies $c = 0$. Hence f is a constant. Consequently (4.2) reduces to $S(X, Y) = 2\beta^2g(X, Y)$. Hence M is Einstein. So we have the following:

Theorem 2. *If the metric g of a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β is a gradient Ricci soliton, then the manifold is an Einstein manifold.*

Since a 3-dimensional Einstein manifold is a manifold of constant curvature, hence we get the following:

Corollary 2. *If the metric g of a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β is a gradient Ricci soliton, then the manifold is a manifold of constant curvature.*

Also using the result of Perelman [23], we can state the following:

Corollary 3. *If the metric g of a 3-dimensional non-cosymplectic compact quasi-Sasakian manifold with constant structure function β is a Ricci soliton, then the manifold is an Einstein manifold.*

Acknowledgement: The authors are thankful to the referee for his/her comments and valuable suggestions towards the improvement of this paper.

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