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Holomorphic families of Schrödinger operators in L^p

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Abstract. Given two m-sectorial operators in a reflexive Banach space X, a sufficient condition is presented for the family $\{T+\kappa A;\ \kappa \in \Sigma^c\}$ to be holomorphic of type (A), where Σ is a closed convex region in the κ-plane such that $\partial\Sigma$ is the left branch of a hyperbola. The abstract formulation is almost identical with Kato's when X is a Hilbert space. However, $\partial\Sigma$ is typically reduced to a parabola when X is a Hilbert space. The m-sectoriality is a generalized notion of the nonnegative selfadjointness so that Schrödinger operators with singular potentials in the L^p-spaces ($1 < p < \infty$) are typical examples of the abstract theory. In this connection a detailed analysis is given in the case of $-\Delta+\kappa|x|^{-2}$. This application is an L^p-generalization of Kato's.

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§1. Introduction

This paper is concerned with holomorphic families of the Schrödinger operator $-\Delta+\kappa V(x)$ in $L^p = L^p(\mathbb{R}^N)$, $1 < p < \infty$, $N \in \mathbb{N}$. Here $V(x)$ is a nonnegative potential with singularity at the origin or at infinity, and κ is a complex parameter running outside a closed convex region Σ in the κ-plane. Namely, we ask when $\{-\Delta + \kappa V(x);\ \kappa \in \Sigma^c\}$, $\Sigma^c := \mathbb{C} \setminus \Sigma$, forms a family of closed linear operators with constant domain in L^p. A solution of this problem with $p = 2$ had already been given by Kato [**9**] using the abstract theory in the Hilbert space setting. For example, put

$$\Sigma_2 := \{\xi + i\eta \in \mathbb{C};\ \eta^2 \le 4(\beta_0(2) - \xi)\} = \{\xi + i\eta \in \mathbb{C};\ \xi \le \beta_0(2) - \eta^2/4\},$$

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where

$$\beta_0(2) := 1 - \frac{(N-2)^2}{4} = -\frac{(N-4)N}{4} \quad \forall\, N \in \mathbb{N}. \tag{1.1}$$

Then Kato proved that, in his terms, $\{-\Delta + \kappa|x|^{-2};\ \kappa \in \Sigma_2^c\}$ forms a holomorphic family of type (A) in L^2. His result means that this family consists of closed operators with constant domain in L^2. Later, Borisov-Okazawa [**3**] proved that $\{d/dx + \kappa\, x^{-1};\ \kappa \in \Sigma_1^c\}$ forms a holomorphic family of type (A) in $L^p(0,\infty)$ $(1 < p < \infty)$, where

$$\Sigma_1 := \Big\{\kappa \in \mathbb{C};\ \operatorname{Re}\kappa \le -\frac{1}{p'}\Big\} \quad \Big(\frac{1}{p} + \frac{1}{p'} = 1\Big).$$

Recently, Tamura [**21**] proved that $\{\Delta^2 + \kappa|x|^{-4};\ \kappa \in \Sigma_4^c\}$ forms a holomorphic family of type (A) in L^2, where

$$\Sigma_4 := \{\xi + i\eta \in \mathbb{C};\ \eta^2 \le f(\xi),\ \xi \le 112 - 3(N-2)^2\};$$

however, f is too complicated to be stated here.

In this connection we note that the practical use of the holomorphic families of type (A) and other types is exemplified in Kato [**8**].

Let T and A be two linear m-accretive operators in a Hilbert space H; the domain and range of T are denoted by $D(T)$ and $R(T)$, respectively. Then Kato gave two abstract theorems [**9**, Theorems 2.1 and 2.2] on holomorphic families of the form $\{T + \kappa A;\ \kappa \in \Sigma^c\}$. For a general definition of holomorphic families of type (A) see Definition 1 in Section 2. It seems from the viewpoint of operator semigroups that [**9**, Theorem 2.1] deals with generators of "general" contraction semigroups on H, while [**9**, Theorem 2.2] deals with generators of "analytic" contraction semigroups. In [**9**, Theorem 2.1] he assumes that A^{-1} exists (but not necessarily bounded) and there is a constant $a \in \mathbb{R}$ such that

$$\limsup_{\substack{\varepsilon \to 0 \\ (|\varepsilon|^{-1}\operatorname{Re}\varepsilon \ge \delta > 0)}} \operatorname{Re}\,((A+\varepsilon)^{-1}v, T^*v) \ge -a\|v\|^2 \quad \forall\, v \in D(T^*), \tag{1.2}$$

where T^* is the adjoint of T (and δ may depend on v). Under these conditions he proves among others that $\{T + \kappa A;\ \operatorname{Re}\kappa > a\}$ forms a holomorphic family of type (A), that is, a typical case of Σ is the left half-plane:

$$\Sigma_{acc} = \{\kappa \in \mathbb{C};\ \operatorname{Re}\kappa \le a\}. \tag{1.3}$$

In this case the family maintains the m-accretivity for only real $\kappa > a_+ := \max\{a, 0\}$. Kato remarks that if A^{-1} is bounded, then (1.2) is equivalent to

$$\operatorname{Re}\,(A^{-1}v, T^*v) \ge -a\|v\|^2 \quad \forall\, v \in D(T^*) \tag{1.4}$$

which is identical with the condition introduced by Sohr [**19**] and [**20**]. For an interesting characterization of condition (1.4) with $a = 0$ is presented by Miyajima [**10**]. In his proof the notion of Yosida approximation is employed.

Now let $\{A_\varepsilon;\, \varepsilon > 0\}$ be the family of Yosida approximation of A:

$$A_\varepsilon := A(1+\varepsilon A)^{-1} = \varepsilon^{-1}[1-(1+\varepsilon A)^{-1}], \quad \varepsilon > 0.$$

Then the second author of the present paper introduced the following condition for $T + A$ (or its closure) to be m-accretive in H: there are constants $a \le 1$ and b, $c \ge 0$ such that

$$\mathrm{Re}\,(Tu, A_\varepsilon u) \ge -a\|A_\varepsilon u\|^2 - b\|A_\varepsilon u\| \cdot \|u\| - c\|u\|^2 \quad \forall\, u \in D(T) \tag{1.5}$$

(see [**12**, Theorem 4.2 and Corollary 5.5]). As stated in [**3**, Introduction], we can show that (1.5) is also a generalization of (1.4). It should be noted further that A need not be invertible in condition (1.5). Inequalities of the form (1.5) makes sense even in a (reflexive) Banach space X if we replace the inner product $(Tu, A_\varepsilon u)$ with the semi-inner product $(Tu, F(A_\varepsilon u))$:

$$\mathrm{Re}\,(Tu, F(A_\varepsilon u)) \ge -a_1\|A_\varepsilon u\|^2 - b_1\|A_\varepsilon u\| \cdot \|u\| - c_1\|u\|^2 \tag{1.6}$$

for all $u \in D(T)$, where F is the duality map on X to its adjoint X^*, that is, (f, g) denotes the pairing between $f \in X$ and $g \in X^*$. By virtue of (1.6) we could generalize [**9**, Theorem 2.1] from the Hilbert space case to the (reflexive) Banach space case (see [**3**, Theorems 2.2 and 2.4]). As an application we considered the first order singular differential operator $d/dx + \kappa\, x^{-1}$ in $L^p(0,\infty)$, $1 < p < \infty$. However, a detailed analysis of the second order singular differential operator such as $d^2/dx^2 + \kappa\, x^{-2}$ in $L^p(0,\infty)$ has been left open over ten years.

In order to generalize [**9**, Theorem 2.2] let T and A be two linear m-sectorial operators in a (reflexive) Banach space X (in applications in [**9**, Section 7] Kato assumed in addition that T and A are also selfadjoint in L^2). As introduced by Goldstein [**4**, Definition 1.5.8] (see also Ouhabaz [**17**, p. 97]), a linear operator T in X is said to be *sectorial of type* $S(\tan\theta)$ if $(Tu, F(u)) \in S(\tan\theta)$ for $u \in D(T)$, where $S(\tan\theta)$ is the sector defined by

$$S(\tan\theta) := \begin{cases} [\,0,\infty) & (\theta = 0), \\ \{z \in \mathbb{C};\ |\mathrm{Im}\, z| \le (\tan\theta)\,\mathrm{Re}\, z\} & \left(0 < \theta < \dfrac{\pi}{2}\right), \\ \{z \in \mathbb{C};\ \mathrm{Re}\, z \ge 0\} & \left(\theta = \dfrac{\pi}{2}\right); \end{cases} \tag{1.7}$$

additionally, an accretive operator is sectorial of type $S(\tan\pi/2)$ and a sectorial operator T is *m-sectorial* if $R(1+T) = X$. It should be noted that sectoriality (or more precisely, sectorial-valuedness) was first introduced in a Hilbert space (see Kato [**7**, Section V.3.10]) as an extension of the notion of nonnegative selfadjointness. In fact, a linear operator T is the generator of an analytic contraction semigroup on X if and only if $-T$ is an m-sectorial operator in (reflexive) X (see [**4**, Theorem 1.5.9 and Proposition 1.3.9]).

Given two linear m-sectorial operators T and A, we introduce, in addition to (1.6), the bound on $\mathrm{Im}\,(Tu, F(A_\varepsilon u))$:

$$\begin{aligned}(1.8)\qquad &|\mathrm{Im}\,(Tu, F(A_\varepsilon u))|^2\\ &\le d_1\{\mathrm{Re}(Tu, F(A_\varepsilon u)) + a_1\|A_\varepsilon u\|^2 + b_1\|A_\varepsilon u\|\cdot\|u\| + c_1\|u\|^2\}\\ &\quad\times\{d_2\mathrm{Re}\,(Tu, F(A_\varepsilon u)) + a_2\|A_\varepsilon u\|^2 + b_2\|A_\varepsilon u\|\cdot\|u\| + c_2\|u\|^2\}\end{aligned}$$

for $u \in D(T)$, with $a_1 d_2 \le a_2$. Letting $\varepsilon \downarrow 0$ in (1.8) with $u \in D(T)\cap D(A)$, we have

$$\begin{aligned}&|\mathrm{Im}\,(Tu, F(Au))|^2\\ &\le d_1\{\mathrm{Re}(Tu, F(Au)) + a_1\|Au\|^2 + b_1\|Au\|\cdot\|u\| + c_1\|u\|^2\}\\ &\quad\times\{d_2\mathrm{Re}\,(Tu, F(Au)) + a_2\|Au\|^2 + b_2\|Au\|\cdot\|u\| + c_2\|u\|^2\}.\end{aligned}$$

The closedness and m-accretivity of $T+\kappa A$ for a fixed κ is equivalent to that of $t(T+\kappa A) = (tT)+\kappa(tA)$ for every $t > 0$. Therefore the essential parameters in (1.6) and (1.8), with A_ε replaced with A, seems to be a_j and d_j ($j = 1$, 2). In fact, (1.6) and (1.8) yield the "hyperbolic" region (bounded by the left branch of a hyperbola) in terms of a_j and d_j ($j = 1$, 2):

$$(1.9)\qquad \Sigma_{sec} = \{x+iy \in \mathbb{C};\ y^2 \le d_1(x-a_1)(d_2x-a_2),\ \ -\infty < x \le a_1\}$$

such that $\{T+\kappa A;\ \kappa = x+iy \in \Sigma_{sec}^c\}$ forms a holomorphic family of type (A) (see Theorems 3.1 and 3.3 below); note that $\Sigma_{acc}^c \subset \Sigma_{sec}^c$ (if $a_1 = a$) as a consequence of the difference between accretivity and sectoriality.

Here we want to explain that our result (Theorems 3.1 and 3.3) below is regarded as a generalization of [**9**, Theorem 2.2] to the (reflexive) Banach space case. To this end we give a brief description of two examples of Σ_{sec} given by (1.9). Let $T := -\Delta$ be the minus Laplacian in $L^p = L^p(\mathbb{R}^N)$, with domain $D(T) := W^{2,p}(\mathbb{R}^N)$. Then it is well-known recently that T is m-sectorial of type $S(c_p)$ in L^p:

$$|\mathrm{Im}\,(-\Delta u, F(u))| \le c_p\mathrm{Re}\,(-\Delta u, F(u)) \quad \forall\, u \in W^{2,p}(\mathbb{R}^N),$$

where the best constant $c_p := |p-2|/(2\sqrt{p-1})$ was found by Henry [**6**, p. 32] (see also Okazawa [**14**] and Voigt [**23**]). Next, let $A := |x|^{-2}$ be the maximal multiplication operator in L^p. Then $-\Delta + \kappa|x|^{-2}$ is an example of the

Schrödinger operator in L^p for which the hyperbolic region is given by

$$\Sigma_p = \Big\{x + iy \in \mathbb{C};\ y^2 \le (x - \beta_0(p))\Big[\frac{|\,p-2\,|^2}{p-1}x - a_2\Big],\ -\infty < x \le \beta_0(p)\Big\},$$

with

$$a_1 = \beta_0(p) < \frac{a_2}{d_2} = \frac{p-1}{|\,p-2\,|^2}a_2$$

(for the details see Theorem 5.1 below). Letting $p \to 2$, we see that $a_2 \to 4$ and the "hyperbolic" region Σ_p is reduced to the "parabolic" region (bounded by a parabola)

$$\Sigma_2 = \{x + iy \in \mathbb{C};\ y^2 \le 4(\beta_0(2) - x),\ \ -\infty < x \le \beta_0(2)\} \tag{1.10}$$

as described in [**9**, Theorem 7.1 and Example 7.4 (a)] (for the constant $\beta_0(2)$ see (1.1)); note that

$$\frac{a_2}{d_2} = \frac{p-1}{|\,p-2\,|^2}a_2 \to \infty\ \ (p \to 2),$$

while $a_1 = \beta_0(p)$ remains bounded. This explains the restriction $a_1 d_2 \le a_2$ in (1.8). In this connection, if $N \ge 5$ then $-\beta_0(2) = (N-4)N/4 > 0$ is well-known as the constant in the Rellich inequality

$$\frac{(N-4)N}{4}\big\||x|^{-2}u\big\|_{L^2} \le \|(-\Delta)u\|_{L^2}, \quad u \in H^2(\mathbb{R}^N) \subset D(|x|^{-2})$$

(for a simple proof see, e.g., [**16**, Lemma 3.2]).

In the next example the potential is of harmonic oscillator type: $T + \kappa A = -\Delta + \kappa|x|^2$. In this case, computing for $V(x) + \alpha = |x|^2 + \alpha$ instead of $V(x) = |x|^2$ itself, we have the "hyperbolic" region $\Sigma_p(\alpha)$ (depending on $\alpha > 0$) defined by the following inequalities:

$$y^2 \le \frac{p^2}{4(p-1)}\Big(x - \frac{p-1}{4}\rho(\alpha)\Big)\Big(\frac{|\,p-2\,|^2}{p-1}x - \frac{p^2}{4}\rho(\alpha)\Big),\ -\infty < x \le \frac{p-1}{4}\rho(\alpha),$$

where $\rho(\alpha) := 16/(27\alpha^2)$. Letting $\alpha \to \infty$, we obtain the sector on the left half-plane:

$$\Sigma_p(\infty) = \Big\{x + iy \in \mathbb{C};\ |y| \le -\frac{p\,|\,p-2\,|}{2\,(p-1)}\,x,\ \ -\infty < x \le 0\Big\}.$$

This is a special case of Example 1 in Section 4. In this limiting process it should be noted that the closedness of $-\Delta + \kappa V(x)$ is equivalent to that of $-\Delta + \kappa[V(x) + \alpha]$, while the maximality $R(1 - \Delta + \kappa V(x)) = L^p$ is equivalent to $R(1 - \Delta + \kappa[V(x) + \alpha]) = L^p$ if $\operatorname{Re}\kappa \ge 0$. Thus we see that if $p \to 2$ then the sector $\Sigma_p(\infty)$ degenerates into the ray $\Sigma_2(\infty) = (-\infty, 0]$, as described in [**9**, Section 7, Remark (b)].

Actually, Σ_2 given by (1.10) and $\Sigma_2(\infty)$ are typical examples of the closed convex subset S which was employed by Kato in the second abstract theorem [**9**, Theorem 2.2]. That is, our result forms a generalization of Kato's to the (reflexive) Banach space case.

This paper is divided into six sections. Section 2 is concerned with the holomorphic family of type (A) of closed linear operators in a Banach space. In Section 3 we prepare an abstract result (Theorems 3.1 and 3.3) for holomorphic family of linear m-sectorial operators in a reflexive Banach space, satisfying (1.6) and (1.8). We consider in Section 4 the general Schrödinger operator $-\Delta + \kappa V(x)$ in L^p. Detailed analysis of the inverse square potential case is carried out in Section 5. The final Section 6 is concerned with the parabolic Cauchy problem for the Schrödinger operator with inverse square potential.

§2. Preliminaries

Let T and A be two closed linear operators from a Banach space X to another Y. Then we consider the operator

$$T + \kappa A \quad \text{with domain} \quad D_0 := D(T) \cap D(A), \tag{2.1}$$

where κ is a complex parameter and D_0 is assumed to be non-trivial. We want to ask if $T + \kappa A$ forms a holomorphic family of type (A) on a non-empty complex region in the sense of Kato. To this end let us recall the general definition of holomorphic family of type (A).

Definition 1 ([**7**, Section VII.2])**.** Let G_0 be a domain in $\mathbb{C}$. Then a family $\{T(\kappa);\ \kappa \in G_0\}$ of linear operators (from X to Y) is said to be *holomorphic of type* (A) if

(i) $T(\kappa)$ is closed for every $\kappa \in G_0$, with $D(T(\kappa)) = D_0$ independent of κ;
(ii) $T(\kappa)u$ is holomorphic with respect to $\kappa \in G_0$ for every $u \in D_0$.

In particular, if $T(\kappa)$ is a linear function of κ as in (2.1) then only (i) is required. In this case $\kappa = 0$ may be an *exceptional point*; in fact, $D(T) = D(T(0))$ may differ from $D(T) \cap D(A) = D(T(\kappa))$ with $\kappa \neq 0$.

Now we introduce the duality map F on X to X^*, the adjoint of X:

$$F(v) := \{f \in X^*;\ (v, f) = \|v\|^2 = \|f\|^2\} \quad \forall\, v \in X. \tag{2.2}$$

Then we have the homogeneity of F: $F(rv) = rF(v)$, $r \geq 0$, and an answer to the above-mentioned question is given in terms of (Tu, f), $f \in F(Au)$.

Proposition 2.1. *Let T and A be closed linear operators from X to Y. Assume that for every $u \in D_0$ there is $f \in F(Au)$ such that*

$$\text{(2.3)} \qquad \mathrm{Re}\,(Tu, f) \geq -\, a_1\|Au\|^2 - b_1\|Au\| \cdot \|u\| - c_1\|u\|^2,$$

$$\text{(2.4)} \quad |\mathrm{Im}\,(Tu, f)|^2 \leq d_1\{\mathrm{Re}\,(Tu, f) + a_1\|Au\|^2 + b_1\|Au\| \cdot \|u\| + c_1\|u\|^2\} \times \{d_2\mathrm{Re}\,(Tu, f) + a_2\|Au\|^2 + b_2\|Au\| \cdot \|u\| + c_2\|u\|^2\},$$

where $a_j \in \mathbb{R}$, b_j, c_j, $d_j \geq 0$ are constants $(j = 1, 2)$, with $a_1 d_2 \leq a_2$. Let Σ be the closed convex subset of $\mathbb{C}$ given by

$$\text{(2.5)} \qquad \Sigma := \{x + iy \in \mathbb{C};\ y^2 \leq d_1(x - a_1)(d_2 x - a_2),\ -\infty < x \leq a_1\}.$$

Then
(**i**) *$T + \kappa A$ is closed for $\kappa \in \Sigma^c$. In particular, if $b_j = c_j = 0$ $(j = 1, 2)$, then*

$$\text{(2.6)} \qquad \|Au\| \leq \mathrm{dist}(\kappa, \Sigma)^{-1}\|(T + \kappa A)u\|, \quad u \in D_0.$$

(**ii**) *$\{T + \kappa A;\ \kappa \in \Sigma^c, \kappa \neq 0\}$ forms a holomorphic family of type* (A); *$\kappa = 0$ is an exceptional point even if $a_1 < 0$.*

In the next lemma we prove an inequality which yields the $(T + \kappa A)$-boundedness of A (and hence of T).

Lemma 2.2. *Let T, A be closed operators from X to Y satisfying* (2.3) *and* (2.4). *Define the following four functions of θ with $|\theta| < \pi/2$:*

$$\text{(2.7)} \qquad \delta(\theta) := 1 + \frac{1}{4}\, d_1 d_2 \tan^2\theta + i\tan\theta, \quad \alpha(\theta) := a_1 + \frac{1}{4}\, a_2\, d_1 \tan^2\theta,$$

$$\text{(2.8)} \qquad \beta(\theta) := b_1 + \frac{1}{4}\, b_2\, d_1 \tan^2\theta, \quad \gamma(\theta) := c_1 + \frac{1}{4}\, c_2\, d_1 \tan^2\theta.$$

Then, the following assertions hold.

(**i**) *Let θ be an arbitrary angle with $|\theta| < \pi/2$ and put $\delta = \delta(\theta)$, $\alpha = \alpha(\theta)$, $\beta = \beta(\theta)$ and $\gamma = \gamma(\theta)$ for simplicity. Then for every $u \in D_0$ there is $f \in F(Au)$ such that*

$$\text{(2.9)} \qquad \mathrm{Re}\,(Tu, \delta f) \geq -\alpha\|Au\|^2 - \beta\|Au\| \cdot \|u\| - \gamma\|u\|^2.$$

(**ii**) *If κ is a complex number with $\mathrm{Re}\,\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$ $(\Leftrightarrow \kappa \in \Sigma^c)$, then A is $(T + \kappa A)$-bounded: for $u \in D_0$,*

$$\text{(2.10)} \qquad \|Au\| \leq |\delta|\,(\mathrm{Re}\,\{\overline{\delta}\,\kappa\} - \alpha)^{-1}\|(T + \kappa A)u\| + \left[\beta\,(\mathrm{Re}\,\{\overline{\delta}\,\kappa\} - \alpha)^{-1} + \sqrt{\gamma\,(\mathrm{Re}\,\{\overline{\delta}\,\kappa\} - \alpha)^{-1}}\right]\|u\|,$$

and hence T is also $(T+\kappa A)$-bounded: *for $u \in D_0$*,

$$\begin{aligned}\|Tu\| \le {} & [1+|\kappa|\cdot|\delta|\,(\mathrm{Re}\,\{\overline{\delta}\,\kappa\}-\alpha)^{-1}]\|(T+\kappa A)u\| \\ & + |\kappa|\left[\beta\,(\mathrm{Re}\,\{\overline{\delta}\,\kappa\}-\alpha)^{-1}+\sqrt{\gamma(\mathrm{Re}\,\{\overline{\delta}\,\kappa\}-\alpha)^{-1}}\right]\|u\|.\end{aligned} \tag{2.11}$$

Proof. Fix θ with $|\theta| < \pi/2$. Then it follows from (2.4) that

$$\begin{aligned}&\mathrm{Re}\,(Tu, e^{i\theta}f) - (\cos\theta)\mathrm{Re}\,(Tu,f) = (\sin\theta)\mathrm{Im}\,(Tu,f) \\ &\ge -\,d_1^{1/2}|\sin\theta|\{\mathrm{Re}\,(Tu,f)+a_1\|Au\|^2+b_1\|Au\|\cdot\|u\|+c_1\|u\|^2\}^{1/2} \\ &\quad\times\{d_2\mathrm{Re}\,(Tu,f)+a_2\|Au\|^2+b_2\|Au\|\cdot\|u\|+c_2\|u\|^2\}^{1/2} \\ &= -\,2(\cos\theta)^{1/2}\{\mathrm{Re}\,(Tu,f)+a_1\|Au\|^2+b_1\|Au\|\cdot\|u\|+c_1\|u\|^2\}^{1/2} \\ &\quad\times\frac{d_1^{1/2}|\sin\theta|}{2(\cos\theta)^{1/2}}\{d_2\mathrm{Re}\,(Tu,f)+a_2\|Au\|^2+b_2\|Au\|\cdot\|u\|+c_2\|u\|^2\}^{1/2}.\end{aligned}$$

Applying the inequality $2ab \le a^2+b^2$, we have

$$\begin{aligned}\mathrm{Re}\,(Tu,e^{i\theta}f) \ge {} & -\frac{(\sin\theta)^2}{4\cos\theta}d_1d_2\mathrm{Re}\,(Tu,f)-\alpha(\cos\theta)\|Au\|^2 \\ & -\beta(\cos\theta)\|Au\|\cdot\|u\|-\gamma(\cos\theta)\|u\|^2.\end{aligned}$$

Thus we obtain (2.9) with α, β and γ given by (2.7) and (2.8).

Now let κ be a complex number with $\mathrm{Re}\,\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$. Then it follows from (2.9) that

$$\begin{aligned}\mathrm{Re}\,((T+\kappa A)u,\delta f) &= \mathrm{Re}\,(Tu,\delta f)+\mathrm{Re}\,\{\overline{\delta}\,\kappa\}\|Au\|^2 \\ &\ge (\mathrm{Re}\,\{\overline{\delta}\,\kappa\}-\alpha)\|Au\|^2-\beta\,\|Au\|\cdot\|u\|-\gamma\,\|u\|^2\end{aligned}$$

and hence

$$(\mathrm{Re}\,\{\overline{\delta}\,\kappa\}-\alpha)\|Au\|^2 \le (|\delta|\cdot\|(T+\kappa A)u\|+\beta\,\|u\|)\|Au\|+\gamma\,\|u\|^2$$

which implies (2.10) and (2.11). (We shall prove the equivalence $\kappa \in \Sigma^c \Leftrightarrow \mathrm{Re}\,\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$ in the proof of Proposition 2.1.) □

Proof of Proposition 2.1. Obviously, (2.10) and (2.11) imply that if κ is a complex number with $\mathrm{Re}\,\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta)$, then $T+\kappa A$ is closed in X. Thus it remains to prove the equivalence:

$$\kappa \in \Sigma^c \;\Leftrightarrow\; \mathrm{Re}\,\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta) \text{ for some } \theta \text{ with } |\theta| < \pi/2. \tag{2.12}$$

First we have to determine the shape of Σ or its boundary $\partial\Sigma$. The key lies in the equation $\mathrm{Re}\,\{\overline{\delta}\,\kappa\} = \alpha$. In fact, we see from (2.7) that the equation depends on the parameter $\tan\theta$ ($|\theta| < \pi/2$). Thus we have a one-parameter family of lines on the (x, y)-plane:

$$\begin{aligned} (2.13) \qquad & \operatorname{Re}\{\overline{\delta}\,\kappa\} - \alpha = \operatorname{Re}\{\overline{\delta}\,(x+iy)\} - \alpha \\ &= \Bigl(1 + \frac{1}{4}d_1d_2\tan^2\theta\Bigr)x + (\tan\theta)y - \Bigl(a_1 + \frac{1}{4}a_2d_1\tan^2\theta\Bigr) = 0. \end{aligned}$$

Setting $\lambda := \tan\theta \in \mathbb{R}$, these lines are rewritten as

$$(2.14) \qquad F(x, y, \lambda) := \Bigl(1 + \frac{1}{4}d_1d_2\lambda^2\Bigr)x + \lambda y - \Bigl(a_1 + \frac{1}{4}a_2d_1\lambda^2\Bigr) = 0.$$

Now it is easy to understand that $\partial\Sigma$ is nothing but the envelope E of the family (2.14). In fact, E is determined as

$$E = \{x + iy \in \mathbb{C};\ F(x, y, \lambda) = 0 = F_\lambda(x, y, \lambda)\},$$

where $F_\lambda(x, y, \lambda) = (1/2)d_1d_2\lambda x + y - (1/2)a_2d_1\lambda$. By a simple computation we obtain

$$\partial\Sigma = \{x + iy \in \mathbb{C};\ y^2 = d_1(x - a_1)(d_2x - a_2),\ \ -\infty < x \le a_1\}$$

as required. Incidentally, for a fixed λ, (2.14) is a tangential line of $\partial\Sigma$ at (x_0, y_0) with

$$x_0 = \frac{a_1 - (1/4)a_2d_1\lambda^2}{1 - (1/4)d_1d_2\lambda^2}, \quad y_0 = \frac{(1/2)(a_2 - a_1d_2)d_1\lambda}{1 - (1/4)d_1d_2\lambda^2}.$$

In particular, we have $(x_0, y_0) = (a_1, 0)$ when $\lambda = 0$. Therefore we can conclude that $T + \kappa A$ is closed for $\kappa \in \Sigma^c$ so that $\{T + \kappa A;\ \kappa \in \Sigma^c, \kappa \ne 0\}$ forms a holomorphic family of type (A).

Finally, we show that if $b_j = c_j = 0$ ($j = 1, 2$), then (2.10) is simplified as (2.6). Given $\kappa \in \Sigma^c$, we can find a supporting line ℓ to Σ such that

$$(2.15) \qquad \operatorname{dist}(\kappa, \Sigma) = \operatorname{dist}(\kappa, \ell),$$

where ℓ is one of the lines in the family (2.13). Therefore it suffices to remind of the well-known formula

$$(2.16) \qquad \operatorname{dist}(\kappa, \ell) = |\delta|^{-1}(\operatorname{Re}\{\overline{\delta}\,\kappa\} - \alpha).$$

It then follows from (2.10), (2.15) and (2.16) that

$$\|Au\| \le \operatorname{dist}(\kappa, \Sigma)^{-1}\|(T + \kappa A)u\|, \quad u \in D_0;$$

note that $\beta = 0 = \gamma$ is a consequence of $b_j = 0 = c_j$ ($j = 1, 2$). This completes the proof of Proposition 2.1. $\square$

Remark 1. In the perturbation theory of (unbounded) linear operators the relative boundedness is a very popular condition (see [**7**, Section IV.1.1]). Now suppose that A is T-bounded:

$$\|Au\| \le a_0\|u\| + b_0\|Tu\|, \quad u \in D(T) \subset D(A).$$

Then for every $g \in F(Au)$ we have

$$\mathrm{Re}\,(Tu, g) \ge -\|Tu\| \cdot \|Au\| \ge -b_0\|Tu\|^2 - a_0\|Tu\| \cdot \|u\|. \tag{2.17}$$

This is obviously different from (2.3). We note that (2.17) is used by Borisov [**2**] when X is a Hilbert space.

§3. Holomorphic families of linear m-sectorial operators

Let F be the duality map on a Banach space X to its adjoint X^* (see (2.2)). Then we begin with the definition of a sectorial operator in X.

Definition 2. A linear operator T in X is said to be *sectorial of type* $S(\tan\theta_T)$ if there is $\theta_T \in [0, \pi/2]$ such that for all $u \in D(T)$ there is a $f \in F(u)$ satisfying

$$(Tu, f) \in S(\tan\theta_T), \tag{3.1}$$

where $S(\tan\theta_T)$ is the sector defined as (1.7) with $\theta = \theta_T$. In other words, a linear operator T in X is sectorial of type $S(\tan\theta_T)$ if $e^{i\,\theta}T$ is accretive for every θ with $|\theta| \le \pi/2 - \theta_T$. In particular, T is *m-sectorial* in X if T is sectorial of type $S(\tan\theta_T)$ and $R(1+T) = X$.

We say that an operator T in X is simply *sectorial* if T is sectorial of type $S(\tan\theta)$ for some $\theta \in [0, \pi/2]$.

Let T be m-sectorial in a reflexive space X. Then, since $e^{i\,\theta}T$ is m-accretive, we have

$$\mathrm{Re}\,(e^{i\,\theta}Tu, f) \ge 0 \quad \forall\, f \in F(u), \ \ \forall\, u \in D(T), \ \ |\theta| \le \pi/2 - \theta_T$$

(see Pazy [**18**, Theorem 1.4.6] and Tanabe [**22**, Theorem 2.1.5]). Therefore we obtain (3.1) for *all* $f \in F(u)$ and $u \in D(T)$. Thus we see that as for m-sectorial operators in a reflexive space Definition 2 coincides with that given by Goldstein [**4**, Definition 1.5.8]. Consequently, if T and A are two m-sectorial operators in a reflexive space, then $T + A$ is also sectorial.

Now let T and A be linear m-sectorial operators in a *reflexive* Banach space X. In this section we consider the m-sectoriality of $T + \kappa A$ with domain $D_0 = D(T) \cap D(A)$. Our basic assumption (**A1**) is stated in terms of the Yosida approximation $\{A_\varepsilon\}$ of A:

(**A1**) For every $u \in D(T)$ and $\varepsilon > 0$ there is $f_\varepsilon \in F(A_\varepsilon u)$ such that

$$\operatorname{Re}(Tu, f_\varepsilon) \ge -a_1\|A_\varepsilon u\|^2 - b_1\|A_\varepsilon u\| \cdot \|u\| - c_1\|u\|^2, \tag{3.2}$$

$$\begin{aligned} &|\operatorname{Im}(Tu, f_\varepsilon)|^2 \\ &\le d_1\{\operatorname{Re}(Tu, f_\varepsilon) + a_1\|A_\varepsilon u\|^2 + b_1\|A_\varepsilon u\| \cdot \|u\| + c_1\|u\|^2\} \\ &\quad \times \{d_2\operatorname{Re}(Tu, f_\varepsilon) + a_2\|A_\varepsilon u\|^2 + b_2\|A_\varepsilon u\| \cdot \|u\| + c_2\|u\|^2\}, \end{aligned} \tag{3.3}$$

where $a_j \in \mathbb{R}$, b_j, c_j, $d_j \ge 0$ $(j = 1, 2)$ are constants, with $a_1 d_2 \le a_2$.

Then we have the first half of the main theorem in this section.

Theorem 3.1. *Let T and A be linear m-sectorial operators in a reflexive Banach space X. Assume that condition* (**A1**) *is satisfied. Let Σ be the closed convex subset of $\mathbb{C}$ given by*

$$\Sigma = \{x + iy \in \mathbb{C};\ y^2 \le d_1(x - a_1)(d_2 x - a_2),\ \ -\infty < x \le a_1\}. \tag{3.4}$$

Then the following assertions hold:

(**i**) *Let $a_1 \ge 0$. Then $T + \kappa A$ is closed for $\kappa \in \Sigma^c$ and hence $\{T + \kappa A;\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type* (A). *In particular, if $b_j = c_j = 0$ $(j = 1, 2)$, then for $u \in D_0$,*

$$\|Au\| \le \operatorname{dist}(\kappa, \Sigma)^{-1}\|(T + \kappa A)u\|.$$

(**ii**) *Let $a_1 < 0$. Then $0 \in \Sigma^c$ and A is T-bounded with T-bound less than or equal to $(-a_1)^{-1} = \operatorname{dist}(0, \Sigma)^{-1}$: for $u \in D_0 = D(T)$,*

$$\|Au\| \le (-a_1)^{-1}\|Tu\| + \big[b_1(-a_1)^{-1} + \sqrt{c_1(-a_1)^{-1}}\big]\|u\|. \tag{3.5}$$

Consequently, $\{T + \kappa A;\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A).

Proof. First we note that the constants in (3.2) and (3.3) are identical to those in (2.3) and (2.4).

(**i**) In the same way as in the proof of Lemma 2.2 we can obtain

$$\operatorname{Re}(Tu, \delta f_\varepsilon) \ge -\alpha\|A_\varepsilon u\|^2 - \beta\|A_\varepsilon u\| \cdot \|u\| - \gamma\|u\|^2, \tag{3.6}$$

where α, β, γ and δ are the same as in (2.9). Hence we can obtain (2.10) and (2.11) with A replaced with A_ε. Since $A_\varepsilon u \to Au$ for $u \in D_0$, we have (2.10) and (2.11) themselves. Therefore it follows from Proposition 2.1 that $\{T + \kappa A;\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A).

(**ii**) Since $-a_1 > 0$, it follows from (3.2) that

$$(-a_1)\|A_\varepsilon u\|^2 \le (\|Tu\| + b_1\|u\|)\|A_\varepsilon u\| + c_1\|u\|^2,$$

which implies

$$\|A_\varepsilon u\| \le (-a_1)^{-1}\|Tu\| + \big[b_1(-a_1)^{-1} + \sqrt{c_1(-a_1)^{-1}}\big]\|u\|, \quad u \in D(T).$$

Therefore we conclude that $u \in D(A)$ and $A_\varepsilon u \rightharpoonup Au$ as $\varepsilon \downarrow 0$, with (3.5). $\square$

Remark 2. The m-sectoriality of T and A was not used in the proof of Theorem 3.1. However, the notion is essential in applications in Sections 4 and 5.

Lemma 3.2. *Assume that* (**A1**) *above and* (**A2**) *below are satisfied*:

(**A2**) *Let* $v \in X$ *and* $\varepsilon > 0$. *Then* $\mathrm{Im}\,(A_\varepsilon v, g) = 0$ *for every* $g \in F(v)$, *and*

$$(v, g_\varepsilon) \geq 0 \text{ so that } \mathrm{Im}\,(v, g_\varepsilon) = 0 \quad \forall\, g_\varepsilon \in F(A_\varepsilon v).$$

Then (3.2) *and* (3.3) *hold, with* T *replaced with* $T + \lambda$ $(\lambda > 0)$.

In fact, we have $\mathrm{Re}\,(Tu, f_\varepsilon) \leq \mathrm{Re}\,((T+\lambda)u, f_\varepsilon)$ for $f_\varepsilon \in F(A_\varepsilon u)$.

Here we note that condition (**A2**) means that A is symmetric and nonnegative if X is a Hilbert space.

Theorem 3.3. *Let* T *be a linear* m*-sectorial operator of type* $S(\tan\theta_T)$ *in a reflexive Banach space* X *and let* A *be a linear* m*-sectorial operator in* X. *Assume that conditions* (**A1**) *and* (**A2**) *are satisfied. Let* Σ *be the region as in* (3.4). *Then in addition to* (**i**), (**ii**) *in* Theorem 3.1 *one has*

(**iii**) $T + \kappa A$ *is* m*-accretive in* X *for* $\kappa \in \Sigma^c$ *with* $\mathrm{Re}\,\kappa \geq 0$.

(**iv**) $T + \kappa A$ *is* m*-sectorial of type* $S(\tan\theta)$ *in* X *for* $\theta \in [\theta_T, \pi/2)$ *and* $\kappa \in \Sigma^c$ *with* $|\arg\kappa| \leq \theta$.

Assume further that $b_j = c_j = 0$ $(j = 1, 2)$ *in* (**A1**). *Then one has*

(**v**) $\mathbb{R}_- := (-\infty, 0)$ *belongs to the resolvent set of* $T + \kappa A$ *for* $\kappa \in \Sigma^c$.

(**vi**) *In particular, if there exists a constant* $\gamma \geq 0$ *such that*

$$\mathrm{Re}\,(Tu, f) \geq \gamma\,(A_\varepsilon u, f), \quad u \in D(T), \quad f \in F(u), \tag{3.7}$$

then $T + \kappa A$ *is* m*-accretive in* X *for* $\kappa \in \Sigma^c$ *with* $\mathrm{Re}\,\kappa \geq -\gamma$.

To prove Theorem 3.3 we need the following

Lemma 3.4. *Let* A *be a linear* m*-accretive operator in a reflexive Banach space* X.

(i) *Let* $\{u_\varepsilon\}$ *be a family in* $D(A)$ *such that* $u_\varepsilon \rightharpoonup u \in X$ $(\varepsilon \downarrow 0)$. *If* $\|Au_\varepsilon\|$ *are bounded as* ε *tends to* 0, *then* $u \in D(A)$ *and* $Au_\varepsilon \rightharpoonup Au$ $(\varepsilon \downarrow 0)$.

(ii) *Let* $\{v_\varepsilon\}$ *be a family in* X *such that* $v_\varepsilon \rightharpoonup u \in X$ $(\varepsilon \downarrow 0)$. *If* $\|A_\varepsilon v_\varepsilon\|$ *are bounded as* ε *tends to* 0, *then* $u \in D(A)$ *and* $A_\varepsilon v_\varepsilon \rightharpoonup Au$ $(\varepsilon \downarrow 0)$.

Note that (ii) is reduced to (i) in Lemma 3.4. In fact, put $u_\varepsilon := (1+\varepsilon A)^{-1} v_\varepsilon$. Then $Au_\varepsilon = A_\varepsilon v_\varepsilon$ is bounded and $u_\varepsilon = v_\varepsilon - \varepsilon A_\varepsilon v_\varepsilon \rightharpoonup u$ $(\varepsilon \downarrow 0)$.

Proof of Theorem 3.3. (**iii**) Since $(A_\varepsilon u, f)$ is real-valued (see (**A2**)), we have

$$\operatorname{Re}((T+\kappa A_\varepsilon)u, f) = \operatorname{Re}(Tu, f) + (A_\varepsilon u, f)\operatorname{Re}\kappa, \quad u \in D(T), \quad f \in F(u).$$

Therefore $T + \kappa A_\varepsilon$ (and hence $T + \kappa A$) is accretive in X if $\kappa \in \Sigma^c$ with $\operatorname{Re}\kappa \geq 0$. To prove the assertion it remains to show that

$$R(1 + T + \kappa A) = X \quad \text{for } \kappa \in \Sigma^c \text{ with } \operatorname{Re}\kappa \geq 0.$$

Let $\kappa \in \Sigma^c$ with $\operatorname{Re}\kappa \geq 0$. Since κA_ε is bounded and accretive in X, $T + \kappa A_\varepsilon$ is m-accretive in X, that is, for every $v \in X$ and $\varepsilon > 0$ there is $u_\varepsilon \in D(T)$ such that

$$u_\varepsilon + Tu_\varepsilon + \kappa A_\varepsilon u_\varepsilon = v,$$

with $\|u_\varepsilon\| \leq \|v\|$. Thus we see from (2.10), (2.15) and (2.16) that

$$\|A_\varepsilon u_\varepsilon\| \leq \Big[(2 + \beta|\delta|^{-1})\operatorname{dist}(\kappa, \Sigma)^{-1} + \sqrt{\gamma|\delta|^{-1}\operatorname{dist}(\kappa, \Sigma)^{-1}}\Big]\|v\|,$$

where δ is given by (2.7). This implies that $v \in R(1 + T + \kappa A)$ (cf. [**11**, Proposition 2.2]; see [**4**, Exercises 1.6.12 (7)]).

(**iv**) Let θ be an angle with $\theta_T \leq \theta < \pi/2$ and $\kappa \in \mathbb{C}$ with $|\arg\kappa| \leq \theta$. Here we may assume that $\kappa \neq 0$. Then condition (**A2**) yields that for every $u \in D_0$ and $f \in F(u)$,

$$\begin{aligned}
|\operatorname{Im}\big((T + \kappa A)u, f\big)| &\leq |\operatorname{Im}\big(Tu, f)| + (Au, f)\,|\operatorname{Im}\kappa| \\
&\leq (\tan\theta_T)\operatorname{Re}(Tu, f) + \frac{|\operatorname{Im}\kappa|}{\operatorname{Re}\kappa}\operatorname{Re}\big(\kappa Au, f\big) \\
&= (\tan\theta_T)\operatorname{Re}(Tu, f) + \big|\tan(\arg\kappa)\big|\operatorname{Re}\big(\kappa Au, f\big) \\
&\leq (\tan\theta)\operatorname{Re}\big((T + \kappa A)u, f\big).
\end{aligned}$$

Since $T + \kappa A$ is m-accretive (see (**iii**) above), we see that $T + \kappa A$ is m-sectorial of type $S(\tan\theta)$ in X.

(**v**) Let $\kappa \in \Sigma^c$. Then $\operatorname{Re}\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$ (see (2.12)), where $\delta = \delta(\theta)$ and $\alpha = \alpha(\theta)$ are given by (2.7). Next let

$$t > \alpha_+ = \alpha_+(\theta) := \max\{\alpha, 0\}.$$

Then we see that $t/\overline{\delta} \in \Sigma^c$ with $\operatorname{Re}\{t/\overline{\delta}\} > 0$ and hence $(T + (t/\overline{\delta})A + \lambda)^{-1}$ exists for $\lambda > 0$ as a consequence of the m-accretivity of $T + (t/\overline{\delta})A$. Therefore $(T + \kappa A + \lambda)^{-1}$ also exists for $\lambda > 0$. In fact, we can show that if $\lambda > 0$, then the resolvent of $T + \kappa A$ is given by the Neumann series:

$$\begin{aligned}
&(T + \kappa A + \lambda)^{-1} \\
&\quad = (T + (t/\overline{\delta})A + \lambda)^{-1}\sum_{n=0}^{\infty}((t/\overline{\delta}) - \kappa)^n[A(T + (t/\overline{\delta})A + \lambda)^{-1}]^n.
\end{aligned} \tag{3.8}$$

To this end we have to show that

$$\|A(T+(t/\overline{\delta})A+\lambda)^{-1}\| \le |\delta|(t-\alpha)^{-1}, \quad \lambda > 0. \tag{3.9}$$

Now, it follows from (2.10) with T replaced with $T+\lambda$ that

$$\|Au\| \le |\delta|(t-\alpha)^{-1}\|(T+(t/\overline{\delta})A+\lambda)u\|, \quad u \in D_0,\ \lambda > 0$$

(see Lemma 3.2); note that $\beta = \gamma = 0$. Since $T+(t/\overline{\delta})A$ is m-accretive, we obtain (3.9). Hence the resolvent (3.8) exists if κ satisfies

$$|(t/\overline{\delta})-\kappa|\cdot|\delta|(t-\alpha)^{-1} < 1 \quad \Leftrightarrow \quad |t-\overline{\delta}\,\kappa| < t-\alpha.$$

Noting that

$$\begin{aligned}&\{\kappa \in \mathbb{C};\ \exists\,\theta \text{ such that } \mathrm{Re}\,\{\overline{\delta(\theta)}\,\kappa\} > \alpha(\theta)\}\\ =& \bigcup_{t>\alpha_+(\theta)} \{\kappa \in \mathbb{C};\ \exists\,\theta \text{ such that } |t-\overline{\delta(\theta)}\,\kappa| < t-\alpha(\theta)\},\end{aligned}$$

we can obtain the conclusion.

(**vi**) Let $\kappa \in \Sigma^c$. Then, since (**v**) yields $R(1+T+\kappa A) = X$, it suffices to note that if $\mathrm{Re}\,\kappa+\gamma \ge 0$,

$$\mathrm{Re}\,((T+\kappa A)u, f) = \lim_{\varepsilon\downarrow 0} \mathrm{Re}\,((T+\kappa A_\varepsilon)u, f) \ge 0, \quad u \in D_0.$$

This completes the proof of Theorem 3.3. □

Remark 3. Borisov-Okazawa [**3**, Theorem 2.4 (ii)] is false unless $b = 0 = c$ in (2.2).

§4. Application to Schrödinger operators in $L^p(\mathbb{R}^N)$

Let $V \ge 0$ be a function in $C^1(\mathbb{R}^N)$ (or in $C^1(\mathbb{R}^N \setminus \{0\})$), $N \in \mathbb{N}$. Then we consider the Schrödinger operator $-\Delta + \kappa V(x)$ in $L^p = L^p(\mathbb{R}^N)$, $1 < p < \infty$, with domain

$$W^{2,p}(\mathbb{R}^N) \cap \{u \in L^p;\ Vu \in L^p\}$$

(for the domain characterization of $-\Delta$ see, e.g., Hempel-Voigt [**5**]). We shall figure out the region Σ for which $\{-\Delta + \kappa V(x);\ \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A). To do so we shall use Theorem 3.3 with $T := -\Delta$ and $A := A(\alpha) := V(x) + \alpha$ ($\alpha > 0$) instead of $V(x)$ itself. As stated in Introduction, T is m-sectorial of type $S(c_p)$:

$$|\mathrm{Im}\,(-\Delta u, F(u))| \le c_p \mathrm{Re}\,(-\Delta u, F(u)) \quad \forall\, u \in W^{2,p}(\mathbb{R}^N).$$

Here (f,g) denotes the pairing between $f \in L^p$ and $g \in L^{p'}$ $(p^{-1}+p'^{-1}=1)$, and (f,g) is linear in f and conjugate linear in g. Thus the computation depends on the single-valued duality map on L^p: $F(0)=0$ and

$$F(v)(x) := \|v\|^{2-p}|v(x)|^{p-2}v(x), \quad 0 \neq v \in L^p.$$

As for $A(\alpha)$ its m-sectoriality is a simple consequence of the positivity of $V(x)+\alpha$. In fact, it is easy to see that $R(1+A(\alpha)) = L^p$, with

$$|\mathrm{Im}\,(A(\alpha)u, F(u))| = 0 \le \mathrm{Re}\,(A(\alpha)u, F(u)) = \|u\|^{2-p}\int_{\mathbb{R}^N}[V(x)+\alpha]|u(x)|^p\,dx.$$

Next we introduce the Yosida approximation $A(\alpha)_\varepsilon$ of $A(\alpha)$: for $v \in L^p$ and $\varepsilon > 0$,

$$\begin{aligned} A(\alpha)_\varepsilon v(x) &:= (V+\alpha)_\varepsilon(x)v(x) \\ &= (V(x)+\alpha)[1+\varepsilon(V(x)+\alpha)]^{-1}v(x). \end{aligned} \tag{4.1}$$

Then we see that $A(\alpha)_\varepsilon$ satisfies condition (**A2**):

$$\mathrm{Im}\,(A(\alpha)_\varepsilon v, F(v)) = 0, \quad (v, F(A(\alpha)_\varepsilon v)) \ge 0 \quad \forall\, v \in L^p.$$

In this section we do not use the Yosida approximation of $V(x)$ itself.

Now we assume that $A(\alpha) = V(x)+\alpha$ satisfies

$(\mathbf{V})_\varepsilon$ $(V+\alpha)_\varepsilon \in C^1(\mathbb{R}^N)$ and for every $\alpha > 0$ and $\varepsilon > 0$ there is a constant $\rho(\alpha) \ge 0$ (independent of $\varepsilon > 0$) such that

$$|\nabla(V+\alpha)_\varepsilon(x)|^2 \le \rho(\alpha)[(V+\alpha)_\varepsilon(x)]^3 \quad \forall\, x \in \mathbb{R}^N. \tag{4.2}$$

Before stating the main theorem we want to give a sufficient condition for $(\mathbf{V})_\varepsilon$.

Proposition 4.1. *Assume that $V \in C^1(\mathbb{R}^N)$ and for every $\alpha > 0$ there is a constant $\rho(\alpha) \ge 0$ such that*

$$|\nabla V(x)|^2 \le \rho(\alpha)[V(x)+\alpha]^3 \quad \forall\, x \in \mathbb{R}^N. \tag{4.3}$$

Then $V(x)$ satisfies condition $(\mathbf{V})_\varepsilon$.

Proof. Since $V \in C^1(\mathbb{R}^N)$, we see from (4.1) that

$$\nabla(V+\alpha)_\varepsilon(x) = \frac{1}{\varepsilon}\nabla\Big[1 - \frac{1}{1+\varepsilon(V(x)+\alpha)}\Big] = \frac{\nabla V(x)}{[1+\varepsilon(V(x)+\alpha)]^2}.$$

Then it follows from (4.3) that

$$|\nabla(V+\alpha)_\varepsilon(x)|^2 \le \rho(\alpha)\frac{[(V+\alpha)_\varepsilon(x)]^3}{1+\varepsilon(V(x)+\alpha)} \le \rho(\alpha)[(V+\alpha)_\varepsilon(x)]^3.$$

This completes the proof. □

Let $\rho(\alpha)$ be as introduced in condition $(\mathbf{V})_\varepsilon$. Then we assume further that

$$\rho := \lim_{\alpha\to\infty} \rho(\alpha) < \infty \tag{4.4}$$

(for this class of potentials cf. Kato [**9**, Section 6]).

In terms of ρ we can obtain a generalization of Kato [**9**, Theorem 7.1].

Theorem 4.2. *For the potential $V(x)$ assume that condition $(\mathbf{V})_\varepsilon$ is satisfied, together with (4.4). Define the hyperbolic region $\Sigma_p(\infty)$ as*

$$\Sigma_p(\infty) := \big\{x+iy \in \mathbb{C};\ y^2 \le f(x),\ \ -\infty < x \le \frac{p-1}{4}\rho\big\}, \tag{4.5}$$

where f is defined as

$$f(x) := \frac{p^2}{4(p-1)}\Big(x - \frac{p-1}{4}\rho\Big)\Big(\frac{|\,p-2\,|^2}{p-1}x - \frac{p^2}{4}\rho\Big).$$

Then

(**a**) *$\{-\Delta + \kappa V(x);\ \kappa \in \Sigma_p(\infty)^{\,c}\}$ forms a holomorphic family of type* (A).

(**b**) *$-\Delta + \kappa V(x)$ is m-accretive in L^p for $\kappa \in \Sigma_p(\infty)^{\,c}$ with* $\operatorname{Re}\kappa \ge 0$.

(**c**) *$-\Delta + \kappa V(x)$ is m-sectorial of type $S(\tan\theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and $\kappa \in \Sigma_p(\infty)^{\,c}$ with $|\arg\kappa| \le \theta$.*

Proof. Let $\Sigma_p(\infty)$ be the region defined by (4.5). Then for $\kappa \in \Sigma_p(\infty)^{\,c}$ we prove the assertions (**a**), (**b**) and (**c**). Since $\Sigma_p(\infty)^{\,c}$ is open, we see from (4.4) that $\kappa \in \Sigma_p(\alpha)^{\,c}$ for sufficiently large $\alpha > 0$, where $\Sigma_p(\alpha)$ is defined by (4.5) with f and ρ replaced with f_α and $\rho(\alpha)$, respectively:

$$\Sigma_p(\alpha) := \big\{x+iy \in \mathbb{C};\ y^2 \le f_\alpha(x),\ \ -\infty < x \le \frac{p-1}{4}\rho(\alpha)\big\},$$
$$f_\alpha(x) := \frac{p^2}{4(p-1)}\Big(x - \frac{p-1}{4}\rho(\alpha)\Big)\Big(\frac{|\,p-2\,|^2}{p-1}x - \frac{p^2}{4}\rho(\alpha)\Big).$$

It is expected that $\Sigma_p(\alpha)$ is the region associated with $T = -\Delta$ and $A(\alpha) = V + \alpha$. Therefore it suffices to prove the closedness, m-accretivity and m-sectoriality of $-\Delta + \kappa[V(x)+\alpha]$ for $\kappa \in \Sigma_p(\alpha)^{\,c}$ instead of (**a**), (**b**) and (**c**). Since (**A2**) is already verified above, we are going to show that $T = -\Delta$ and $A(\alpha)_\varepsilon = (V+\alpha)_\varepsilon$ satisfy condition (**A1**): for $u \in C_0^\infty(\mathbb{R}^N)$,

$$\operatorname{Re}(Tu, F(A(\alpha)_\varepsilon u)) \ge -\frac{p-1}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^2, \tag{4.6}$$

$$\begin{aligned}
&|\operatorname{Im}(Tu, F(A(\alpha)_\varepsilon u))|^2 \\
&\le \frac{p^2}{4(p-1)}\Big\{\operatorname{Re}(Tu, F(A(\alpha)_\varepsilon u)) + \frac{p-1}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^2\Big\} \\
&\quad\times \Big\{\frac{|\,p-2\,|^2}{p-1}\operatorname{Re}(Tu, F(A(\alpha)_\varepsilon u)) + \frac{p^2}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^2\Big\};
\end{aligned} \tag{4.7}$$

note that $C_0^\infty(\mathbb{R}^N)$ is a core for T. The proof is divided into three steps.

Step 1 [Proof of (4.6)]. It suffices to show that for $u \in C_0^\infty(\mathbb{R}^N)$,

$$\operatorname{Re}(Tu, |A(\alpha)_\varepsilon u|^{p-2}A(\alpha)_\varepsilon u)) \ge -\frac{p-1}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^p. \tag{4.8}$$

First we note that if $1 < p < 2$, then

$$\operatorname{Re}(Tu, |A(\alpha)_\varepsilon u|^{p-2}A(\alpha)_\varepsilon u)) = \lim_{\delta\downarrow 0}\operatorname{Re} I(\delta),$$

where

$$I(\delta) := -\int_{\mathbb{R}^N}[(V+\alpha)_\varepsilon(x)]^{p-1}(|u(x)|^2+\delta)^{(p-2)/2}\overline{u(x)}\Delta u(x)\,dx;$$

if $p \ge 2$ then we may take $\delta = 0$ so that the computation will be simpler. Thus we restrict ourselves to the case of $1 < p < 2$. Put $Q(x) := [(V+\alpha)_\varepsilon(x)]^{p-1}$. Then integration by parts gives

$$I(\delta) = \int_{\mathbb{R}^N} f(x,\delta)\,dx + \int_{\mathbb{R}^N} g(x,\delta)\,dx, \tag{4.9}$$

where $f(x,\delta) := Q(x)(|u(x)|^2+\delta)^{(p-2)/2}|\nabla u(x)|^2$ and

$$\begin{aligned} g(x,\delta) := \Big\{\frac{p-2}{2}\nabla|u(x)|^2 + \frac{\nabla Q(x)}{Q(x)}(|u(x)|^2+\delta)\Big\}& \\ \cdot\, Q(x)(|u(x)|^2+\delta)^{(p-4)/2}\overline{u(x)}\nabla u(x).& \end{aligned}$$

This yields that

$$\begin{aligned} \operatorname{Re} I(\delta) - \int_{\mathbb{R}^N} f(x,\delta)\,dx &= \operatorname{Re}\int_{\mathbb{R}^N} g(x,\delta)\,dx \\ = (p-2)\int_{\mathbb{R}^N} &Q(x)|u(x)|^2(|u(x)|^2+\delta)^{(p-4)/2}|\nabla|u(x)||^2\,dx \\ + \int_{\mathbb{R}^N} &|u(x)|(|u(x)|^2+\delta)^{(p-2)/2}\nabla Q(x)\cdot\nabla|u(x)|\,dx. \end{aligned} \tag{4.10}$$

Now suppose that $\operatorname{supp} u \subset B = B(0,R)$ for some $R > 0$. Then, applying the Cauchy-Schwarz and geometric-arithmetic mean inequalities to

$$(|u(x)|^2+\delta)^{(p-2)/2}|u(x)|\nabla Q(x)\cdot\nabla|u(x)||,$$

we can obtain

$$\begin{aligned} \operatorname{Re} I(\delta) + \frac{1}{p^2}J(\delta) - \delta\int_{\mathbb{R}^N} Q(x)(|u(x)|^2+\delta)^{(p-4)/2}|\nabla u(x)|^2\,dx& \\ \ge \int_{\mathbb{R}^N} Q(x)|u(x)|^2(|u(x)|^2+\delta)^{(p-4)/2}\Big(|\nabla u(x)|^2 - \big|\nabla|u(x)|\big|^2\Big)\,dx \ge 0& \end{aligned} \tag{4.11}$$

(note that $|\nabla|u(x)|| \le |\nabla u(x)|$), where

$$J(\delta) := \frac{p^2}{4(p-1)} \int_{B(0,R)} \frac{|\nabla Q(x)|^2}{Q(x)} (|u(x)|^2 + \delta)^{p/2}\, dx. \tag{4.12}$$

Here (4.2) in condition $(\mathbf{V})_\varepsilon$ applies to give

$$\frac{|\nabla Q(x)|^2}{Q(x)} \le (p-1)^2 \rho(\alpha)[(V+\alpha)_\varepsilon(x)]^p.$$

Therefore we have

$$-\mathrm{Re}\, I(\delta) \le \frac{1}{p^2} J(\delta) \le \frac{p-1}{4} \rho(\alpha) \|A(\alpha)_\varepsilon (|u|^2 + \delta)^{1/2}\|^p_{L^p(B)}. \tag{4.13}$$

Letting $\delta \downarrow 0$, we obtain (4.8); the computation stated above is essentially the same as in [**13**, Proof of Theorem 2.1].

Step 2. Before computing the $\mathrm{Im}\, I(\delta)$, we note that (4.11) is written as

$$\begin{aligned} &|\,p-2\,|^2 \Big[\mathrm{Re}\, I(\delta) - \int_{\mathbb{R}^N} f(x,\delta)\, dx\Big] + \frac{|\,p-2\,|^2}{p^2} J(\delta) \\ &\ge -\,|\,p-2\,|^2 \int_{\mathbb{R}^N} Q(x)|u(x)|^2(|u(x)|^2+\delta)^{(p-4)/2} \big|\nabla|u(x)|\big|^2\, dx. \end{aligned} \tag{4.14}$$

Setting

$$\begin{aligned} h(x,\delta) := &\left|\frac{p-2}{2} \nabla |u(x)|^2 + \frac{\nabla Q(x)}{Q(x)} (|u(x)|^2 + \delta)\right|^2 \\ &\times Q(x)(|u(x)|^2+\delta)^{(p-4)/2}, \end{aligned}$$

we have $|g(x,\delta)| \le \sqrt{f(x,\delta)}\sqrt{h(x,\delta)}$; note that

$$f(x,\delta) \ge Q(x)|u(x)|^2(|u(x)|^2+\delta)^{(p-4)/2}|\nabla u(x)|^2.$$

Therefore the Cauchy-Schwarz inequality yields that

$$\left|\int_{\mathbb{R}^N} g(x,\delta)\, dx\right|^2 \le \Big(\int_{\mathbb{R}^N} f(x,\delta)\, dx\Big)\Big(\int_{B(0,R)} h(x,\delta)\, dx\Big). \tag{4.15}$$

Now we see that (4.10) can be written in terms of $J(\delta)$ and $h(x,\delta)$:

$$\begin{aligned} &2(p-2)\Big[\mathrm{Re}\, I(\delta) - \int_{\mathbb{R}^N} f(x,\delta)\, dx\Big] + \frac{4(p-1)}{p^2} J(\delta) \\ &= \int_{B(0,R)} h(x,\delta)\, dx \\ &\quad + |\,p-2\,|^2 \int_{\mathbb{R}^N} Q(x)|u(x)|^2(|u(x)|^2+\delta)^{(p-4)/2} \big|\nabla|u(x)|\big|^2\, dx. \end{aligned}$$

Combining this equality with (4.14), we obtain

$$\int_{B(0,R)} h(x,\delta)\,dx \le J(\delta) + p(p-2)[\operatorname{Re} I(\delta) - K(\delta)], \tag{4.16}$$

where we have set

$$K(\delta) := \int_{\mathbb{R}^N} f(x,\delta)\,dx.$$

Step 3 [Proof of (4.7)]. First by virtue of (4.9) and (4.15) we have that

$$\begin{aligned}|\operatorname{Im} I(\delta)|^2 &= \Big|\operatorname{Im}\int_{\mathbb{R}^N} g(x,\delta)\,dx\Big|^2 = \Big|\int_{\mathbb{R}^N} g(x,\delta)\,dx\Big|^2 - \Big|\operatorname{Re}\int_{\mathbb{R}^N} g(x,\delta)\,dx\Big|^2 \\ &\le \Big(\int_{B(0,R)} h(x,\delta)\,dx\Big)K(\delta) - |\operatorname{Re} I(\delta) - K(\delta)|^2.\end{aligned}$$

Then we see from (4.16) that

$$\begin{aligned}|\operatorname{Im} I(\delta)|^2 &\le \big\{J(\delta) + p(p-2)[\operatorname{Re} I(\delta) - K(\delta)]\big\}K(\delta) - |\operatorname{Re} I(\delta) - K(\delta)|^2 \\ &= -(p-1)^2[K(\delta)]^2 \\ &\quad + \big\{J(\delta) + (p^2-2p+2)\operatorname{Re} I(\delta)\big\}K(\delta) - |\operatorname{Re} I(\delta)|^2.\end{aligned}$$

By completing the square we have

$$\begin{aligned}|\operatorname{Im} I(\delta)|^2 &\le \frac{1}{4(p-1)^2}\big\{J(\delta) + (p^2-2p+2)\operatorname{Re} I(\delta)\big\}^2 - |\operatorname{Re} I(\delta)|^2 \\ &= \frac{p^2|\,p-2\,|^2}{4(p-1)^2}\Big\{\operatorname{Re} I(\delta) + \frac{1}{p^2}J(\delta)\Big\}\Big\{\operatorname{Re} I(\delta) + \frac{1}{|\,p-2\,|^2}J(\delta)\Big\}.\end{aligned}$$

It then follows from (4.13) that

$$\begin{aligned}|\operatorname{Im} I(\delta)|^2 &\le \frac{p^2}{4(p-1)}\Big\{\operatorname{Re} I(\delta) + \frac{p-1}{4}\rho(\alpha)\|A(\alpha)_\varepsilon(|u|^2+\delta)^{1/2}\|^p_{L^p(B)}\Big\} \\ &\quad \times \Big\{\frac{|\,p-2\,|^2}{p-1}\operatorname{Re} I(\delta) + \frac{p^2}{4}\rho(\alpha)\|A(\alpha)_\varepsilon(|u|^2+\delta)^{1/2}\|^p_{L^p(B)}\Big\}.\end{aligned}$$

Letting $\delta \downarrow 0$, we obtain (4.7). Thus for T and $A(\alpha)_\varepsilon$, (**A1**) is satisfied. □

Example 1. $V(x) := |x|^{2\,l}$ ($l \in (-\infty,-1) \cup [1,\infty)$). Then Theorem 4.2 yields that

(**a**) $\{-\Delta + \kappa|x|^{2\,l};\ \kappa \in \Sigma_p(\infty)^{\,c}\}$ forms a holomorphic family of type (A), where $\Sigma_p(\infty)$ is given by the following sector of the κ-plane:

$$\Sigma_p(\infty) = \Big\{x + iy \in \mathbb{C};\ |y| \le -\frac{p\,|\,p-2\,|}{2\,(p-1)}x,\ \ -\infty < x \le 0\Big\};$$

(**b**) $-\Delta + \kappa|x|^{2l}$ is m-accretive in L^p for κ with $\operatorname{Re}\kappa \ge 0$;

(**c**) $-\Delta + \kappa|x|^{2l}$ is m-sectorial of type $S(\tan\theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and κ with $|\arg\kappa| \le \theta$.

The proof is divided into two cases.

Case 1: $l \ge 1$ (**Harmonic oscillator type potential**). Put

$$\rho(\alpha) := \frac{4}{27l}(l+1)^{1+1/l}(2l-1)^{2-1/l}\frac{1}{\alpha^{1+1/l}}.$$

Then we have

$$\frac{|\nabla V(x)|^2}{[V(x)+\alpha]^3} = \frac{4l^2|x|^{2(2l-1)}}{(|x|^{2l}+\alpha)^3} \le \rho(\alpha).$$

In fact, we can obtain $\rho(\alpha)$ as the maximum of $g(t) := 4l^2t^{2l-1}/(t^l+\alpha)^3$ ($t := |x|^2 \ge 0$). Therefore $\rho(\alpha) \to 0$ ($\alpha \to \infty$).

Case 2: $l < -1$. Put $k := -l > 1$. Then the Yosida approximation of $V(x)+\alpha$ is given by

$$(V+\alpha)_\varepsilon(x) = \frac{1+\alpha|x|^{2k}}{(1+\alpha\varepsilon)|x|^{2k}+\varepsilon} \quad (\varepsilon > 0).$$

Put

$$\rho(\alpha) := \frac{4}{27k}(k-1)^{1-1/k}(2k+1)^{2+1/k}\frac{1}{\alpha^{1-1/k}}.$$

Then we have

$$\frac{|\nabla(V+\alpha)_\varepsilon(x)|^2}{[(V+\alpha)_\varepsilon(x)]^3} \le \frac{4k^2|x|^{2(k-1)}}{(1+\alpha|x|^{2k})^3} \le \rho(\alpha).$$

In fact, we can obtain $\rho(\alpha)$ as the maximum of $g(t) := 4k^2t^{k-1}/(1+\alpha t^k)^3$ ($t := |x|^2 \ge 0$). Therefore $\rho(\alpha) \to 0$ ($\alpha \to \infty$). Thus we obtain the same conclusion both in Case 1 and in Case 2.

Example 2 (**Inverse square potential**). $V(x) := |x|^{-2}$. Then we have

$$\frac{|\nabla(V+\alpha)_\varepsilon(x)|^2}{[(V+\alpha)_\varepsilon(x)]^3} \le \frac{4}{(1+\alpha|x|^2)^3} \le \rho(\alpha) := 4 \to 4 \ (\alpha \to \infty).$$

Hence we see from Theorem 4.2 that

(**a**) $\{-\Delta+\kappa|x|^{-2};\ \kappa \in \Sigma_p(\infty)^c\}$ forms a holomorphic family of type (A), where $\Sigma_p(\infty)$ is given by the following hyperbolic region of the κ-plane:

$$\Sigma_p(\infty) = \big\{x+iy \in \mathbb{C};\ y^2 \le f(x),\ \ -\infty < x \le p-1\big\},$$

and f is defined as

$$f(x) := \frac{p^2}{4(p-1)}\Big(x-(p-1)\Big)\Big(\frac{|\,p-2\,|^2}{p-1}x-p^2\Big);$$

(**b**) $-\Delta+\kappa|x|^{-2}$ is m-accretive in L^p for $\kappa\in\Sigma_p(\infty)^c$ with $\operatorname{Re}\kappa\geq 0$;

(**c**) $-\Delta+\kappa|x|^{-2}$ is m-sectorial of type $S(\tan\theta)$ in L^p for $\theta\in[\tan^{-1}c_p,\pi/2)$ and $\kappa\in\Sigma_p(\infty)^c$ with $|\arg\kappa|\leq\theta$.

§5. More about the case of $-\Delta+\kappa|x|^{-2}$ in L^p

In this section we shed new light on the Schrödinger operator $-\Delta+\kappa|x|^{-2}$ in $L^p = L^p(\mathbb{R}^N)$ through detailed computation ($1<p<\infty$, $N\in\mathbb{N}$). As stated above, $T=-\Delta$ with domain $D(T)=W^{2,p}(\mathbb{R}^N)$ is m-sectorial in L^p, while $A=|x|^{-2}$ is also m-sectorial as a maximal multiplication operator in L^p and its Yosida approximation is given by

$$A_\varepsilon = A(1+\varepsilon A)^{-1} = (|x|^2+\varepsilon)^{-1}, \quad \varepsilon>0.$$

It is easy to see that condition (**A2**) is satisfied (see the previous section). On the other hand, to derive (**A1**) we have utilized only the quotient

$$|\nabla(|x|^{-2}+\alpha)_\varepsilon|^2/[(|x|^{-2}+\alpha)_\varepsilon]^3.$$

As its consequence the boundary of $\Sigma_p(\infty)$ was given by the function with constants depending only on p. However, some of those constants are already known to depend also on N. Thus, we shall derive (**A1**) and (3.7) with sharp constants: for $u\in C_0^\infty(\mathbb{R}^N)$,

$$\operatorname{Re}(Tu,F(A_\varepsilon u)) \geq -\,\beta_0(p)\|A_\varepsilon u\|^2, \tag{5.1}$$

$$|\operatorname{Im}(Tu,F(A_\varepsilon u))|^2 \leq \big\{\operatorname{Re}(Tu,F(A_\varepsilon u))+\beta_0(p)\|A_\varepsilon u\|^2\big\} \times\Big\{\frac{|\,p-2\,|^2}{p-1}\operatorname{Re}(Tu,F(A_\varepsilon u))+\beta_1(p)\|A_\varepsilon u\|^2\Big\}, \tag{5.2}$$

$$\operatorname{Re}(Tu,F(u)) \geq \gamma_0(p)(Au,F(u)) \geq \gamma_0(p)(A_\varepsilon u,F(u)), \tag{5.3}$$

where $\beta_0(p)$, $\beta_1(p)$ and $\gamma_0(p)$ depend not only on p but also on N:

$$\beta_0(p) := p^{-2}(p-1)(2p-N)N \quad (1<p<\infty),$$

$$\beta_1(p) := \begin{cases} 4+2N(p-2) & (2\leq p<\infty),\\ \dfrac{4}{p-1}+\dfrac{2N}{p}(2-p)^2 & (1<p<2),\end{cases}$$

$$\gamma_0(p) := p^{-2}(p-1)(N-2)^2 \qquad (N\geq 3).$$

Note that $\beta_0(p)$ and $\gamma_0(p)$ are optimal (see [**15**, Section 5.2]).

Then as a consequence of Theorem 3.3 we can state the following

Theorem 5.1. *Let $T = -\Delta$, $A = |x|^{-2}$ and*

$$\Sigma_p = \Big\{x + iy \in \mathbb{C}; y^2 \le (x - \beta_0(p))\Big[\frac{|\,p-2\,|^2}{p-1}x - \beta_1(p)\Big], -\infty < x \le \beta_0(p)\Big\}.$$

(**i**) *Let $2p \ge N$. Then $T + \kappa A = -\Delta + \kappa|x|^{-2}$ is closed for $\kappa \in \Sigma_p^c$, with*

$$\big\||x|^{-2}u\big\| \le \operatorname{dist}(\kappa, \Sigma_p)^{-1}\big\|(-\Delta + \kappa|x|^{-2})u\big\|, \quad u \in W^{2,p}(\mathbb{R}^N) \cap D(|x|^{-2}),$$

and hence $\{T + \kappa A; \kappa \in \Sigma_p^c\}$ forms a holomorphic family of type (A).

(**ii**) *Let $2p < N$. Then $-\beta_0(p) > 0$ so that A is T-bounded with T-bound $[-\beta_0(p)]^{-1}$:*

$$\big\||x|^{-2}u\big\| \le \frac{p^2}{(p-1)(N-2p)N}\|(-\Delta)u\|, \quad u \in W^{2,p}(\mathbb{R}^N),$$

and hence $\{T + \kappa A; \kappa \in \Sigma_p^c\}$ forms a holomorphic family of type (A).

(**iii**) *If $N = 1, 2$, then $T + \kappa A$ is m-accretive in L^p for $\kappa \in \Sigma_p^c$ with* $\operatorname{Re}\kappa \ge 0$.

(**iv**) *$T + \kappa A$ is m-sectorial of type $S(\tan\theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and $\kappa \in \Sigma_p^c$ with $|\arg\kappa| \le \theta$.*

(**v**) $\mathbb{R}_- = (-\infty, 0)$ *belongs to the resolvent set of $T + \kappa A$ for $\kappa \in \Sigma_p^c$.*

(**vi**) *If $N \ge 3$, then $T + \kappa A$ is m-accretive in L^p for $\kappa \in \Sigma_p^c$ with* $\operatorname{Re}\kappa \ge -\gamma_0(p)$.

The conclusion of Theorem 5.1 is a considerable improvement of Example 2. In fact, let Σ_p and $\Sigma_p(\infty)$ be as in Theorem 5.1 and Example 2, respectively. Then, since $\beta_0(p) - (p-1) = -p^{-2}(p-1)(p-N)^2 \le 0$, we see from the computation of asymptotes that

$$\Sigma_p(\infty)^c \subset \Sigma_p^c.$$

Actually, both (5.1) and (5.3) had already been proved in [**15**, Lemma 3.5 and Theorem 2.4]. As for (5.1) we shall give a simplified proof in Lemma 5.3 below. In this connection we note that for $N \ge 3$,

$$-\gamma_0(p) \le \beta_0(p) \quad \text{when } 2 - \frac{2}{N} \le p < \infty,$$
$$-\gamma_0(p) \ge \beta_0(p) \quad \text{when } 1 < p \le 2 - \frac{2}{N}.$$

Thus, in order to prove Theorem 5.1 it remains to prove the most complicated inequality (5.2). This will be done in Lemma 5.6 below.

The next lemma is a new version of Hardy's inequality.

Lemma 5.2. *Let $v \in C_0^1(\mathbb{R}^N)$ and $\delta > 0$. Suppose that* $\operatorname{supp} v \subset B(0,R)$ *for some $R > 0$. Then one has*

$$\text{(5.4)}\quad \frac{N^2}{p^2}\int_{B(0,R)} (|v(x)|^2+\delta)^{p/2}\,dx + \frac{2N^2\delta}{p(2-p)}\int_{B(0,R)} \big(|v(x)|^2+\delta\big)^{(p-2)/2}\,dx \le \int_{\mathbb{R}^N} |x|^2|v(x)|^2(|v(x)|^2+\delta)^{(p-4)/2}|\nabla v(x)|^2\,dx.$$

Consequently, if $1 < p < 2$ then

$$\text{(5.5)}\quad \int_{\mathbb{R}^N} |x|^2|v(x)|^2(|v(x)|^2+\delta)^{(p-4)/2}|\nabla v(x)|^2\,dx \ge \frac{N^2}{p^2}\int_{B(0,R)} (|v(x)|^2+\delta)^{p/2}\,dx.$$

On the other hand, if $p \ge 2$ then one can set $\delta = 0$ in (5.4):

$$\text{(5.6)}\quad \int_{\mathbb{R}^N} |x|^2|v(x)|^{p-2}|\nabla v(x)|^2\,dx \ge \frac{N^2}{p^2}\int_{\mathbb{R}^N} |v(x)|^p\,dx.$$

These inequalities remain true even if v is replaced with $|v| \in W^{1,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

Remark 4. The right-hand side of (5.5) can be replaced with $(N/p)^2\|v\|^p$. However, we need (5.5) itself at the end of the proof of Lemma 5.3.

Proof of Lemma 5.2. Let $v \in C_0^1(\mathbb{R}^N)$ and $\delta > 0$. Then we have

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} |x|^2|v(x)|^2\big(|v(x)|^2+\delta\big)^{(p-4)/2}\left|\nabla v(x) + \frac{N}{p}\frac{x}{|x|^2}v(x)\right|^2 dx \\ &\le \int_{\mathbb{R}^N} |x|^2|v(x)|^2\big(|v(x)|^2+\delta\big)^{(p-4)/2}|\nabla v(x)|^2\,dx \\ &\quad + \frac{N^2}{p^2}\int_{B(0,R)} \big(|v(x)|^2+\delta\big)^{p/2}\,dx \\ &\quad + \frac{N}{p}\int_{\mathbb{R}^N} |v(x)|^2\big(|v(x)|^2+\delta\big)^{(p-4)/2}(x\cdot\nabla)|v(x)|^2\,dx. \end{aligned}$$

The integrand of the third term on the right-hand side is rewritten as

$$\begin{aligned} &|v(x)|^2\big(|v(x)|^2+\delta\big)^{(p-4)/2}(x\cdot\nabla)|v(x)|^2 \\ &= \big(|v(x)|^2+\delta\big)^{(p-2)/2}(x\cdot\nabla)|v(x)|^2 \\ &\quad - \delta\big(|v(x)|^2+\delta\big)^{(p-4)/2}(x\cdot\nabla)|v(x)|^2 \\ &= \frac{2}{p}(x\cdot\nabla)\big(|v(x)|^2+\delta\big)^{p/2} - \frac{2\,\delta}{p-2}(x\cdot\nabla)\big(|v(x)|^2+\delta\big)^{(p-2)/2}. \end{aligned}$$

Therefore integration by parts gives

$$\begin{aligned}
&\frac{N}{p}\int_{\mathbb{R}^N}|v(x)|^2\big(|v(x)|^2+\delta\big)^{(p-4)/2}(x\cdot\nabla)|v(x)|^2\,dx\\
&= -\frac{2N^2}{p^2}\int_{B(0,R)}\big(|v(x)|^2+\delta\big)^{p/2}\,dx\\
&\quad -\frac{2N^2\delta}{p(2-p)}\int_{B(0,R)}\big(|v(x)|^2+\delta\big)^{(p-2)/2}\,dx.
\end{aligned}$$

Thus we obtain (5.4). □

Lemma 5.3. *Let* $T=-\Delta$ *and* $A=|x|^{-2}$. *Then* (5.1) *holds*:

$$\mathrm{Re}\,(Tu, F(A_\varepsilon u)) \ge -\beta_0(p)\|A_\varepsilon u\|^2, \quad u\in C_0^\infty(\mathbb{R}^N),\ \ \varepsilon>0.$$

Proof. For $u\in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon>0$ let $v:=(|x|^2+\varepsilon)^{-1}u=A_\varepsilon u\in C_0^\infty(\mathbb{R}^N)$. Then we can express Tu in terms of v as follows:

$$Tu = TA_\varepsilon^{-1}v = -2Nv(x) - 2\,(x\cdot\nabla)v(x) - \mathrm{div}[(|x|^2+\varepsilon)\nabla v(x)].$$

Therefore, instead of (5.1), we shall show that

$$\mathrm{Re}\,I_0(\delta) \ge -\beta_0(p)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}, \tag{5.7}$$

where we assume that $\mathrm{supp}\,v\subset B=B(0,R)$ for some $R>0$ and

$$I_0(\delta) := (TA_\varepsilon^{-1}v, (|v|^2+\delta)^{(p-2)/2}v). \tag{5.8}$$

In addition $\delta>0$ if $1<p<2$, while $\delta=0$ if $p\ge 2$. First we have

$$\begin{aligned}
&I_0(\delta) + 2N\int_{\mathbb{R}^N}|v(x)|^2(|v(x)|^2+\delta)^{(p-2)/2}\,dx\\
&= \int_{\mathbb{R}^N}\big(f_{01}(x,\delta)+g_0(x,\delta)\big)\,dx\\
&\quad + \delta\int_{\mathbb{R}^N}(|x|^2+\varepsilon)(|v(x)|^2+\delta)^{(p-4)/2}|\nabla v(x)|^2\,dx,
\end{aligned}\tag{5.9}$$

where $f_{0j}(x,\delta)$ $(j=1,2)$ and $g_0(x,\delta)$ are defined as follows:

$$\begin{aligned}
f_{01}(x,\delta) &:= (|x|^2+\varepsilon)|v(x)|^2(|v(x)|^2+\delta)^{(p-4)/2}|\nabla v(x)|^2\\
&\ge (|x|^2+\varepsilon)|v(x)|^2(|v(x)|^2+\delta)^{(p-4)/2}\big|\nabla|v(x)|\big|^2 =: f_{02}(x,\delta),\\
g_0(x,\delta) &:= \Big\{-2(|v(x)|^2+\delta)x + \frac{p-2}{2}(|x|^2+\varepsilon)\nabla|v(x)|^2\Big\}\\
&\qquad\cdot(|v(x)|^2+\delta)^{(p-4)/2}\overline{v(x)}\nabla v(x).
\end{aligned}$$

The definition of $g_0(x,\delta)$ and (5.9) respectively yield

$$\begin{aligned}(5.10)\qquad \operatorname{Re}\int_{\mathbb{R}^N} g_0(x,\delta)\,dx &= \frac{2N}{p}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \\ &\quad + (p-2)\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx,\end{aligned}$$

$$(5.11)\qquad \operatorname{Im}\int_{\mathbb{R}^N} g_0(x,\delta)\,dx = \operatorname{Im} I_0(\delta).$$

Now it follows from (5.9) and (5.10) that

$$\begin{aligned}&\operatorname{Re} I_0(\delta) + \frac{2\,(p-1)}{p} N\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \\ &\ge \int_{\mathbb{R}^N}\big(f_{01}(x,\delta)+(p-2)f_{02}(x,\delta)\big)\,dx \\ &= (p-1)\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx - (p-2)\int_{\mathbb{R}^N}\big(f_{01}(x,\delta)-f_{02}(x,\delta)\big)\,dx \\ &= (p-1)\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx + \int_{\mathbb{R}^N}\big(f_{01}(x,\delta)-f_{02}(x,\delta)\big)\,dx.\end{aligned}$$

Thus (5.5) and (5.6) with v replaced with $|v|$ apply to give

$$\begin{aligned}&\operatorname{Re} I_0(\delta) + \frac{2\,(p-1)}{p} N\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \\ &-\int_{\mathbb{R}^N}\big(f_{01}(x,\delta)-f_{02}(x,\delta)\big)\,dx \\ &\ge (p-1)\int_{\mathbb{R}^N}|x|^2|v(x)|^2(|v(x)|^2+\delta)^{(p-4)/2}\big|\nabla|v(x)|\big|^2\,dx \\ &\ge \frac{p-1}{p^2}N^2\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}.\end{aligned}$$

Therefore we obtain a useful inequality

$$\operatorname{Re} I_0(\delta) + \beta_0(p)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \ge \int_{\mathbb{R}^N}\big(f_{01}(x,\delta)-f_{02}(x,\delta)\big)\,dx \ge 0,$$

which is more precise than (5.7). Letting $\delta\downarrow 0$, we have

$$\operatorname{Re}\,(TA_\varepsilon^{-1}v, |v|^{p-2}v) = \lim_{\delta\downarrow 0}\operatorname{Re} I_0(\delta) \ge -\beta_0(p)\|v\|^p.$$

This is equivalent to (5.1). □

Now set

$$
\begin{aligned}
J_0(\delta) &:= \operatorname{Re} I_0(\delta) + \beta_0(p)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)},\\
K_0(\delta) &:= \int_{\mathbb{R}^N} \big(f_{01}(x,\delta) - f_{02}(x,\delta)\big)\,dx\\
&= \int_{\mathbb{R}^N} (|x|^2+\varepsilon)|v(x)|^2(|v(x)|^2+\delta)^{(p-4)/2}\big(|\nabla v(x)|^2 - \big|\nabla|v(x)|\big|^2\big)\,dx.
\end{aligned}
$$

In the proof of the previous lemma we have established the following

Lemma 5.4. *For $v \in C_0^\infty(\mathbb{R}^N)$ let $f_{0j}(x,\delta)$ be as in the proof of* Lemma 5.3 *$(j = 1, 2)$. Then one has*

$$
\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx - \Big(\frac{N}{p}\Big)^2\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \le \frac{J_0(\delta)}{p-1} + \frac{p-2}{p-1}K_0(\delta); \tag{5.12}
$$

$$
\frac{K_0(\delta)}{p-1} + \int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx \le \frac{\operatorname{Re} I_0(\delta)}{p-1} + \frac{2N}{p}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}; \tag{5.13}
$$

$$
0 \le K_0(\delta) \le J_0(\delta). \tag{5.14}
$$

Next we can prepare a lemma in terms of $f_{0j}(x,\delta)$ and $K_0(\delta)$ that will be basic for the precise estimate of $\operatorname{Im}(TA_\varepsilon^{-1}v, |v|^{p-2}v)$.

Lemma 5.5. *For $v \in C_0^\infty(\mathbb{R}^N)$ let $f_{0j}(x,\delta)$, $g_0(x,\delta)$ be as in the proof of* Lemma 5.3 *$(j = 1, 2)$. Set*

$$
\begin{aligned}
h_0(x,\delta) := &\left|\frac{p-2}{2}(|x|^2+\varepsilon)^{1/2}\nabla|v(x)|^2 - 2\frac{|v(x)|^2+\delta}{(|x|^2+\varepsilon)^{1/2}}\,x\right|^2\\
&\times (|v(x)|^2+\delta)^{(p-4)/2}.
\end{aligned}
$$

Assume that $\operatorname{supp} v \subset B(0,R)$ *for some $R > 0$. Then one has*

$$
\begin{aligned}
&\int_{B(0,R)} h_0(x,\delta)\,dx\\
&\quad\le \Big[4 + \frac{4N}{p}(p-2)\Big]\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} + |\,p-2\,|^2\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx;
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
&(F_1H)(\delta) - \Big|\operatorname{Re}\int_{\mathbb{R}^N} g_0(x,\delta)\,dx\Big|^2\\
&\quad\le 4\,\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}\Big[\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx - \Big(\frac{N}{p}\Big)^2\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}\Big]\\
&\qquad + \Big[\frac{4N}{p}(p-2)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} + |\,p-2\,|^2\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx\Big]K_0(\delta),
\end{aligned}
\tag{5.16}
$$

where we have set

$$(F_1H)(\delta) := \Big(\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx\Big)\Big(\int_{B(0,R)} h_0(x,\delta)\,dx\Big).$$

Proof. It follows from the definition of $h_0(x,\delta)$ that

$$\begin{aligned} &h_0(x,\delta) - |\,p-2\,|^2 f_{02}(x,\delta)\\ &= \frac{4\,|x|^2}{|x|^2+\varepsilon}\,(|v(x)|^2+\delta)^{p/2} - \frac{4}{p}(p-2)(x\cdot\nabla)(|v(x)|^2+\delta)^{p/2}. \end{aligned}$$

Integrating this equality over the ball $B(0,R)$, we obtain (5.15). Then it follows from (5.15) and the definition of $K_0(\delta)$ that

$$\begin{aligned} &(F_1H)(\delta)\\ &\le \Big[4+\frac{4N}{p}(p-2)\Big]\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^p\Big(\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx\Big)\\ &\quad + |\,p-2\,|^2\Big(\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx\Big)\Big(\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx\Big)\\ &= 4\,\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^p\Big(\int_{\mathbb{R}^N} f_{01}(x,\delta)\,dx\Big) - \Big(\frac{2N}{p}\Big)^2\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^{2p}\\ &\quad + \Big[\frac{4N}{p}(p-2)\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^p + |\,p-2\,|^2\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx\Big]K_0(\delta)\\ &\quad + \Big|\frac{2N}{p}\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^p + (p-2)\Big(\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx\Big)\Big|^2. \end{aligned}$$

Therefore (5.16) is a consequence of (5.10). □

Now we are in a position to derive the estimate of $\mathrm{Im}\,(TA_\varepsilon^{-1}v, |v|^{p-2}v)$ in terms of $\mathrm{Re}\,(TA_\varepsilon^{-1}v, |v|^{p-2}v)$.

Lemma 5.6. *Let $T = -\Delta$ and $A = |x|^{-2}$. Then the inequality* (5.2) *holds: for all $u \in C_0^\infty(\mathbb{R}^N)$, $\varepsilon > 0$,*

$$\begin{aligned} |\mathrm{Im}\,(Tu, F(A_\varepsilon u))|^2 &\le \big\{\mathrm{Re}\,(Tu, F(A_\varepsilon u)) + \beta_0(p)\|A_\varepsilon u\|^2\big\}\\ &\quad\times\Big\{\frac{|\,p-2\,|^2}{p-1}\mathrm{Re}\,(Tu, F(A_\varepsilon u)) + \beta_1(p)\|A_\varepsilon u\|^2\Big\}. \end{aligned}$$

Proof. As in the proof of Lemma 5.3 we set $v := (|x|^2+\varepsilon)^{-1}u = A_\varepsilon u \in C_0^\infty(\mathbb{R}^N)$ for $u \in C_0^\infty(\mathbb{R}^N)$. Let $I_0(\delta)$ be as defined in (5.8). Then we shall show that for v with $\mathrm{supp}\, v \subset B = B(0,R)$ for some $R > 0$,

$$\begin{aligned} (5.17)\qquad |\mathrm{Im}\,I_0(\delta)|^2 &\le \big\{\mathrm{Re}\,I_0(\delta) + \beta_0(p)\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^p\big\}\\ &\quad\times\Big\{\frac{|\,p-2\,|^2}{p-1}\mathrm{Re}\,I_0(\delta) + \beta_1(p)\|(|v|^2+\delta)^{1/2}\|_{L^p(B)}^p\Big\}. \end{aligned}$$

Since $|g_0(x,\delta)| \le \sqrt{f_{01}(x,\delta)}\sqrt{h_0(x,\delta)}$, the Cauchy-Schwarz inequality yields

$$\Big|\int_{\mathbb{R}^N} g_0(x,\delta)\,dx\Big|^2 \le (F_1H)(\delta). \tag{5.18}$$

It then follows from (5.12) and (5.16) that

$$\begin{aligned}(F_1H)(\delta) - &\Big|\mathrm{Re}\int_{\mathbb{R}^N} g_0(x,\delta)\,dx\Big|^2 \\ \le{}& \frac{4}{p-1}\,\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}J_0(\delta) \\ &+ \Big[4\Big(\frac{N}{p}+\frac{1}{p-1}\Big)(p-2)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \\ &+ |\,p-2\,|^2\int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx\Big]K_0(\delta).\end{aligned} \tag{5.19}$$

In this way we can compute $|\mathrm{Im}\,I_0(\delta)|$. In fact, (5.11) and (5.18) yield that

$$\begin{aligned}|\mathrm{Im}\,I_0(\delta)|^2 &= \Big|\int_{\mathbb{R}^N} g_0(x,\delta)\,dx\Big|^2 - \Big|\mathrm{Re}\int_{\mathbb{R}^N} g_0(x,\delta)\,dx\Big|^2 \\ &\le (F_1H)(\delta) - \Big|\mathrm{Re}\int_{\mathbb{R}^N} g_0(x,\delta)\,dx\Big|^2.\end{aligned}$$

Therefore we see from (5.19) that

$$\begin{aligned}|\mathrm{Im}\,I_0(\delta)|^2 \le{}& \frac{4}{p-1}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}J_0(\delta) \\ &+ 4\Big(\frac{N}{p}+\frac{1}{p-1}\Big)(p-2)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}K_0(\delta) \\ &+ |\,p-2\,|^2\Big\{\frac{K_0(\delta)}{p-1} + \int_{\mathbb{R}^N} f_{02}(x,\delta)\,dx\Big\}K_0(\delta) \\ &- \frac{|\,p-2\,|^2}{p-1}[K_0(\delta)]^2.\end{aligned} \tag{5.20}$$

Applying (5.13) to the third term (and dropping the fourth term) on the right-hand side of (5.20), we have

$$\begin{aligned}|\mathrm{Im}\,I_0(\delta)|^2 \le{}& \frac{4}{p-1}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}J_0(\delta) \\ &+ 4\Big(\frac{N}{p}+\frac{1}{p-1}\Big)(p-2)\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}K_0(\delta) \\ &+ |\,p-2\,|^2\Big\{\frac{\mathrm{Re}\,I_0(\delta)}{p-1} + \frac{2N}{p}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}\Big\}K_0(\delta).\end{aligned} \tag{5.21}$$

Now in order to use (5.14) we have to divide our computation into two cases. In fact, the second term on the right-hand side of (5.21) contains the factor $(p-2)$ so that the computation depends on its sign.

Case 1: $2 \le p < \infty$. Here we can set $\delta = 0$ and apply (5.14) to (or replace $K_0(0)$ with $J_0(0)$ in) the second and third terms on the right-hand side of (5.21). Then $|\mathrm{Im}\, I_0(0)|^2$ is less than or equal to $J_0(0)$ multiplied by

$$\begin{aligned} &\frac{4}{p-1}\|v\|^p + \frac{|\,p-2\,|^2}{p-1}\mathrm{Re}\, I_0(0) \\ &+ 2\Big(\frac{2N}{p} + \frac{2}{p-1} + \frac{N}{p}(p-2)\Big)(p-2)\|v\|^p \\ = &\frac{(p-2)^2}{p-1}\mathrm{Re}\, I_0(0) + [4 + 2N(p-2)]\|v\|^p. \end{aligned}$$

Thus we obtain (5.17) (with $p \ge 2$ and $\delta = 0$) which is equivalent to (5.2).

Case 2: $1 < p < 2$. In this case we note that the second term on the right-hand side of (5.21) is nonpositive. So we have

$$\begin{aligned} |\mathrm{Im}\, I_0(\delta)|^2 \le &\frac{4}{p-1}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} J_0(\delta) \\ &+ |\,p-2\,|^2\Big\{\frac{\mathrm{Re}\, I_0(\delta)}{p-1} + \frac{2N}{p}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}\Big\} K_0(\delta). \end{aligned}$$

Now (5.14) yields that $|\mathrm{Im}\, I_0(\delta)|^2$ is less than or equal to $J_0(\delta)$ multiplied by

$$\begin{aligned} &\frac{4}{p-1}\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} + \frac{|\,p-2\,|^2}{p-1}\mathrm{Re}\, I_0(\delta) \\ &+ \frac{2N}{p}|\,p-2\,|^2\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)} \\ = &\frac{(2-p)^2}{p-1}\mathrm{Re}\, I_0(\delta) + \Big[\frac{4}{p-1} + \frac{2N}{p}(2-p)^2\Big]\|(|v|^2+\delta)^{1/2}\|^p_{L^p(B)}. \end{aligned}$$

Letting $\delta \downarrow 0$, we can obtain the inequality which is equivalent to (5.2) with $p \in (1, 2)$. □

§6. Concluding remark

Here we want to give a rough description of the solvability of the Cauchy problem for $N \ge 3$:

$$\text{(6.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{\kappa}{|x|^2}u = 0, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases}$$

together with a conjecture. Let Σ_p be the region as in Section 5. Then, since $T+\kappa A = -\Delta + \kappa|x|^{-2}$ is m-accretive in L^p for $\kappa \in \Sigma_p^c$ with $\operatorname{Re}\kappa \geq -\gamma_0(p)$, $-(T+\kappa A)$ generates a contraction semigroup $\{\exp(-t(T+\kappa A));\, t \geq 0\}$ on L^p. Therefore for every $u_0 \in W^{2,p}(\mathbb{R}^N) \cap D(|x|^{-2})$ a unique solution to (6.1) is given by $\exp(-t(T+\kappa A))u_0$. Here we note that

$$-\gamma_0(2) = -\frac{(N-2)^2}{4} \leq -\frac{(N-2)^2}{p\,p'} = -\gamma_0(p) \quad \forall\, p \in [1,\infty).$$

In fact, we have

$$\frac{1}{p\,p'} \leq \frac{p-1}{p^2} + \Big(\frac{1}{p} - \frac{1}{2}\Big)^2 = \frac{1}{4}.$$

Thus the lower bound of $\operatorname{Re}\kappa$ for which (6.1) is solvable may be given by $-\gamma_0(2)$. Therefore we may restrict ourselves to the case of $p = 2$.

Let $\kappa \in \Sigma_2$ with $\operatorname{Re}\kappa \geq -\gamma_0(2)$ and assume that there exists θ with $|\theta| < \pi/2$ such that

$$|\operatorname{Im}\kappa| \leq (\tan\theta)(\operatorname{Re}\kappa + \gamma_0(2)).$$

Then we can show that $T+\kappa A$ is sectorial in L^2. In fact, it follows from (5.3) that

$$\begin{aligned} &|\operatorname{Im}((T+\kappa A)u,u)| = |\operatorname{Im}\kappa|(Au,u)\\ &\leq (\tan\theta)(\operatorname{Re}\kappa + \gamma_0(2))(Au,u)\\ &\leq (\tan\theta)\big((Au,u)\operatorname{Re}\kappa + (Tu,u)\big)\\ &= (\tan\theta)\operatorname{Re}((T+\kappa A)u,u). \end{aligned}$$

Now let F be the Friedrichs extension of $T+\kappa A$ (see, e.g., [**7**] or [**17**]). Then F is m-sectorial in L^2 and hence $-F$ generates an analytic contraction semigroup $\{\exp(-tF);\, t \geq 0\}$ on L^2. Therefore a unique solution to (6.1) is given by $\exp(-tF)u_0$ for every $u_0 \in L^2$. Though the argument is not perfect, the above observation roughly explains the solvability of (6.1) for κ with $\operatorname{Re}\kappa \geq -\gamma_0(2)$.

On the other hand, Baras-Goldstein [**1**] have shown that there exists no solution to (6.1) with $u_0 \geq 0$ ($u_0 \not\equiv 0$) for $\kappa \in \mathbb{R}$ with $\kappa < -\gamma_0(2)$. Thus we may conjecture that there exists no solution to (6.1) with $u_0 \geq 0$ ($u_0 \not\equiv 0$) for $\kappa \in \mathbb{C}$ with $\operatorname{Re}\kappa < -\gamma_0(2)$.

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