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Square Laplacian perturbed by inverse fourth-power potential II. Holomorphic family of type (A) (complex case)

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Abstract. It is proved that $\{\Delta^2 + \kappa|x|^{-4}; \kappa \in \Sigma^c\}$ in $L^2(\mathbb{R}^N)$ forms a holomorphic family of type (A), where Σ is a closed and convex subset of $\mathbb{C}$. In particular, the m-accretivity of $\Delta^2 + \kappa|x|^{-4}$ in $L^2(\mathbb{R}^N)$ is established as an application of the perturbation theorem for linear m-accretive operators. The key lies in two inequalities derived by positive semi-definiteness of Gram matrix.

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§1. Introduction

Let $A := \Delta^2$ with $D(A) := H^4(\mathbb{R}^N)$ and $B := |x|^{-4}$ with $D(B) := D(|x|^{-4}) = \{u \in L^2(\mathbb{R}^N);\ |x|^{-4}u \in L^2(\mathbb{R}^N)\}$ ($N \in \mathbb{N}$), where $\Delta := \sum_{j=1}^N (\partial^2/\partial x_j^2)$ is a usual Laplacian in $\mathbb{R}^N$. This paper is concerned with parameter dependence of the operator sum $A + \kappa B$ ($\kappa \in \mathbb{C}$) in the complex Hilbert space $L^2(\mathbb{R}^N)$:

$$(A + \kappa B)u := \Delta^2 u + \frac{\kappa}{|x|^4}u, \quad u \in D(A) \cap D(B) = H^4(\mathbb{R}^N) \cap D(|x|^{-4}).$$

In the previous paper [**9**] Okazawa, Tamura and Yokota have discussed the selfadjointness of $A+\kappa B$ when "$\kappa \in \mathbb{R}$" in the (complex) Hilbert space $L^2(\mathbb{R}^N)$ ($N \in \mathbb{N}$). Namely, it is proved in [**9**] that $A + \kappa B$ is nonnegative selfadjoint on $D(A) \cap D(B)$ for $\kappa > \kappa_0$, where

$$\kappa_0 = \kappa_0(N) := \begin{cases} k_1 & N \le 8, \\ k_2 & N \ge 9, \end{cases}$$

and k_1, k_2 will be given in Theorem 1.1. In addition we can assert that $A+\kappa_0 B$ is nonnegative and essentially selfadjoint in $L^2(\mathbb{R}^N)$. As a continuation of [**9**] this paper concerns the m-accretivity and the resolvent set of $A+\kappa B$ when "$\kappa \in \mathbb{C}$". First we want to find $\Sigma \subset \mathbb{C}$ such that $\{A+\kappa B; \kappa \in \Sigma^{\mathrm{c}}\}$ is a holomorphic family of type (A) in the sense of Kato [**5**, Chapter VII]. Next we consider the m-accretivity of $A+\kappa B$ for κ in the subset Σ^{c}.

Now we review the notion of holomorphic family in a simple case (the definition of m-accretivity will be given in Section 2).

Definition 1. Let X be a reflexive complex Banach space. Let Ω be a domain in $\mathbb{C}$ and $\{T(\kappa);\ \kappa \in \Omega\}$ a family of linear operators in X. Then $\{T(\kappa);\ \kappa \in \Omega\}$ is said to be a *holomorphic family of type* (A) in X if
(**i**) $T(\kappa)$ is closed in X and $D(T(\kappa)) = D$ independent of κ;
(**ii**) $\kappa \mapsto T(\kappa)u$ is holomorphic in Ω for every $u \in D$.

Kato [**6**] proved that $\{-\Delta + \kappa|x|^{-2};\ \kappa \in \Omega_1\}$ forms a holomorphic family of type (A) in $L^2(\mathbb{R}^N)$, where $\beta := 1-(N-2)^2/4 = -N(N-4)/4$ and

$$\Omega_1 := \{\xi + i\eta \in \mathbb{C};\, \eta^2 > 4(\beta - \xi)\} = \{\xi + i\eta \in \mathbb{C};\, \xi > \gamma(\eta) := \beta - \eta^2/4\}.$$

Borisov-Okazawa [**1**] proved that $\{d/dx + \kappa x^{-1};\ \kappa \in \Omega_2\}$ forms a holomorphic family of type (A) in $L^p(0,\infty)$ $(1 < p < \infty)$, where

$$\Omega_2 := \bigl\{\kappa \in \mathbb{C};\, \operatorname{Re}\kappa > -{p'}^{-1}\bigr\}, \quad p^{-1} + {p'}^{-1} = 1.$$

Concerning fourth order elliptic operators, there seems to be no preceding work on holomorphic family of type (A). So we try to clarify the regions where $A+\kappa B$ forms a holomorphic family of type (A) and where $A+\kappa B$ is m-accretive.

Our result is stated as follows.

Theorem 1.1. *Set $A := \Delta^2$, $B := |x|^{-4}$. Let $k_1 = k_1(N)$ $(N \in \mathbb{N})$ be the constant defined as*

$$k_1 := 112 - 3(N-2)^2. \tag{1.1}$$

Let Σ be the closed convex subset of $\mathbb{C}$ defined as

$$\Sigma := \Bigl\{\xi + i\eta \in \mathbb{C};\ \xi \le k_1, \eta^2 \le 64\bigl[\sqrt{k_1-\xi} + \bigl(10 + N - \frac{N^2}{4}\bigr)\bigr](\sqrt{k_1-\xi}+8)^2\Bigr\}.$$

Then the following (**i**)–(**iii**) *hold.*
(**i**) *B is $(A+\kappa B)$-bounded for $\kappa \in \Sigma^{\mathrm{c}}$, with*

$$\|Bu\| \le \operatorname{dist}(\kappa, \Sigma)^{-1}\|(A+\kappa B)u\|, \quad u \in D(A) \cap D(B),$$

and hence $\{A+\kappa B;\ \kappa\in\Sigma^{\rm c}\}$ *forms a holomorphic family of type* (A) *in* $L^2(\mathbb{R}^N)$. *In particular, if* $N\geq 9$ *then* B *is* A*-bounded, with*

$$\|Bu\|\leq |k_2|^{-1}\|Au\|,\quad u\in D(A)\subset D(B),$$

where $k_2=k_2(N)$ $(N\geq 9)$ *is the negative constant defined as*

$$k_2:=k_1-\Big[\Big(\frac{N-2}{2}\Big)^2-11\Big]^2=-\frac{N}{16}(N-8)(N^2-16). \tag{1.2}$$

In addition, Σ *can be expressed in terms of* k_2 :

$$\Sigma=\Big\{\xi+i\eta\in\mathbb{C};\ \xi\leq k_2,\ \eta^2\leq\frac{64(k_2-\xi)(\sqrt{k_1-\xi}+8)^2}{\sqrt{k_1-\xi}+\big(N^2/4-N-10\big)}\Big\}.$$

(**ii**) $A+\kappa B$ *is m-accretive on* $D(A)\cap D(B)$ *for* $\kappa\in\Sigma^{\rm c}$ *with* $\operatorname{Re}\kappa\geq-\alpha_0$ *and* $A+\kappa B$ *is essentially m-accretive in* $L^2(\mathbb{R}^N)$ *for* $\kappa\in\partial\Sigma$ *with* $\operatorname{Re}\kappa\geq-\alpha_0$, *where* α_0 *is defined as*

$$\alpha_0=\alpha_0(N):=\begin{cases}0, & N\leq 4,\\ \Big[\dfrac{N(N-4)}{4}\Big]^2, & N\geq 5.\end{cases} \tag{1.3}$$

In particular, if $\kappa\in\mathbb{R}$, *then m-accretivity is replaced with nonnegative selfadjointness.*

(**iii**) *Let* $\kappa\in\Sigma^{\rm c}$ *with* $\operatorname{Re}\kappa<-\alpha_0$. *Let* $c_{\alpha_0}(\kappa)$ *and* θ_{α_0} *be defined as*

$$c_{\alpha_0}(\kappa):=\begin{cases}\min\Big\{\dfrac{|-\alpha_0+i\eta-\kappa|}{\operatorname{dist}(-\alpha_0+i\eta,\Sigma)};\ \eta_0<\eta<\infty\Big\}, & \operatorname{Im}\kappa>0,\\ \min\Big\{\dfrac{|-\alpha_0+i\eta-\overline{\kappa}|}{\operatorname{dist}(-\alpha_0+i\eta,\Sigma)};\ \eta_0<\eta<\infty\Big\}, & \operatorname{Im}\kappa<0,\end{cases}$$

$$\theta_{\alpha_0}:=\tan^{-1}\Big(\frac{1-c_{\alpha_0}(\kappa)}{\sqrt{c_{\alpha_0}(\kappa)(2-c_{\alpha_0}(\kappa))}}\Big),$$

where $\eta_0:=\max\{\eta\geq 0;\ -\alpha_0+i\eta\in\Sigma\}$. *Then* $c_{\alpha_0}(\kappa)\in(0,1)$ *and* $\theta_{\alpha_0}\in(0,\pi/2)$.

(a) *If* $\operatorname{Im}\kappa>0$, *then the resolvent set* $\rho(-(A+\kappa B))$ *contains the sector* $S_+(\kappa)$, *where*

$$S_+(\kappa):=\{\lambda\in\mathbb{C};\ -\theta_{\alpha_0}<\arg\lambda<\pi/2\}.$$

(b) *If* $\operatorname{Im}\kappa<0$, *then the resolvent set* $\rho(-(A+\kappa B))$ *contains the sector* $S_-(\kappa)$, *where*

$$S_-(\kappa):=\{\lambda\in\mathbb{C};\ -\pi/2<\arg\lambda<\theta_{\alpha_0}\}.$$

Remark 1.1. When $N \geq 5$, α_0 in (1.3) appears in the Rellich inequality (cf. Davies-Hinz [**3**, Corollary 14], Okazawa [**8**, Lemma 3.8], [**9**, Lemma 3.2]).

Remark 1.2. Theorem 1.1 (**iii**) (and also Theorem 2.1 (**iii**), Theorem 2.7 (**vi**)) can be improved. Actually, the referee[1] informed us that θ_{α_0} in Theorem 1.1 can be replaced with

$$\tan^{-1}\left(\frac{\sqrt{1-c_{\alpha_0}(\kappa)^2}}{c_{\alpha_0}(\kappa)}\right).$$

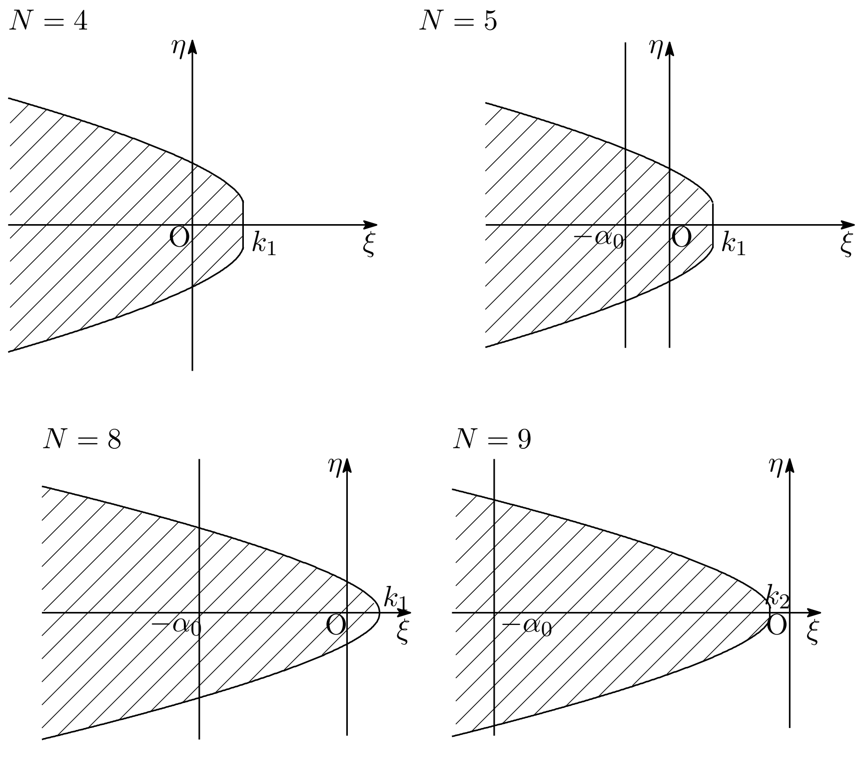

Figure 1: The images of Σ for $N = 4$, 5, 8, 9 and the value of $-\alpha_0$

In Section 2 we propose abstract theorems based on Kato [**6**]. However, the assumption and conclusions are slightly changed. In the proof of Theorem 1.1 we need some generalized forms of the inequalities obtained in [**9**]. Section 3 starts with their proofs depending on the positive semi-definiteness of Gram matrix. At the end of Section 3 we complete the proof of Theorem 1.1 by applying abstract theorems prepared in Section 2.

[1]The author would like to thank the referee for this comment.

§2. Abstract theory toward Theorem 1.1

First we review some definitions required to state Theorems 2.1 and 2.7. Let A be a linear operator with domain $D(A)$ and range $R(A)$ in a (complex) Hilbert space H. Then A is said to be *accretive* if $\operatorname{Re}(Au, u) \geq 0$ for every $u \in D(A)$. An accretive operator A is said to be *m-accretive* if $R(A+1) = H$.

Let A be m-accretive in H. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, $R(A+\lambda) = H$ holds with

$$\|(A+\lambda)^{-1}\| \leq (\operatorname{Re}\lambda)^{-1}.$$

Therefore we can define the Yosida approximation $\{A_\varepsilon;\, \varepsilon > 0\}$ of A:

$$A_\varepsilon := A(1+\varepsilon A)^{-1}, \quad \varepsilon > 0.$$

A nonnegative selfadjoint operator is a typical example of m-accretive operator, while a symmetric m-accretive operator is nonnegative and selfadjoint (see Brézis [**2**, Proposition VII.6] or Kato [**5**, Problem V.3.32]).

Next we consider the m-accretivity of $A + \kappa B$ ($\kappa \in \mathbb{C}$) where A and B are nonnegative selfadjoint operators in H. Since m-accretive operators are closed and densely defined, we will first find $\Omega \subset \mathbb{C}$ where $\{A + \kappa B;\, \kappa \in \Omega\}$ forms a holomorphic family of type (A). Next we will find a set of $\kappa \in \Omega$ where $A+\kappa B$ is m-accretive. We also consider the resolvent set of $A + \kappa B$ for each $\kappa \in \Omega$.

Theorem 2.1. *Let A and B be nonnegative selfadjoint operators in H. Let $\Sigma \subset \mathbb{C}$, and $\gamma : \mathbb{R} \to \mathbb{R}$. Assume that Σ and γ satisfy* $(\boldsymbol{\gamma}\mathbf{1})$–$(\boldsymbol{\gamma}\mathbf{4})$ *and* $(\boldsymbol{\gamma}\mathbf{5})_0$:

$(\boldsymbol{\gamma}\mathbf{1})$ *γ is continuous and concave,*

$(\boldsymbol{\gamma}\mathbf{2})$ *$\gamma(\eta) = \gamma(-\eta)$ for $\eta \in \mathbb{R}$,*

$(\boldsymbol{\gamma}\mathbf{3})$ *$\Sigma = \{\xi + i\eta \in \mathbb{C};\, \xi \leq \gamma(\eta)\}$,*

$(\boldsymbol{\gamma}\mathbf{4})$ *$-(Au, B_\varepsilon u) \in \Sigma$ for $u \in D(A)$ with $\|B_\varepsilon u\| = \|B(1+\varepsilon B)^{-1}u\| = 1$ for any $\varepsilon > 0$,*

$(\boldsymbol{\gamma}\mathbf{5})_0$ *$0 \leq \gamma(0)$ ($\Leftrightarrow 0 \in \Sigma$).*

Then the following (**i**)–(**iii**) *hold.*

(**i**) *B is $(A+\kappa B)$-bounded for $\kappa \in \Sigma^{\mathrm{c}}$, with*

$$\|Bu\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1} \|(A+\kappa B)u\|, \quad u \in D(A) \cap D(B), \tag{2.1}$$

and $\{A + \kappa B;\, \kappa \in \Sigma^{\mathrm{c}}\}$ forms a holomorphic family of type (A).

(**ii**) *$A + \kappa B$ is m-accretive on $D(A) \cap D(B)$ for $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re}\kappa \geq 0$ and $A + \kappa B$ is essentially m-accretive in H for $\kappa \in \partial\Sigma$ with $\operatorname{Re}\kappa \geq 0$.*

(**iii**) *Let* $\kappa \in \Sigma^{\mathrm{c}}$ *with* $\operatorname{Re}\kappa < 0$. *Let* $c_0(\kappa)$ *and* θ_0 *be defined as*

$$c_0(\kappa) := \begin{cases} \min\Big\{\dfrac{|i\eta - \kappa|}{\operatorname{dist}(i\eta, \Sigma)};\ \eta_0 < \eta < \infty\Big\}, & \operatorname{Im}\kappa > 0, \\ \min\Big\{\dfrac{|i\eta - \overline{\kappa}|}{\operatorname{dist}(i\eta, \Sigma)};\ \eta_0 < \eta < \infty\Big\}, & \operatorname{Im}\kappa < 0, \end{cases} \tag{2.2}$$

$$\theta_0 := \tan^{-1}\left(\frac{1 - c_0(\kappa)}{\sqrt{c_0(\kappa)(2 - c_0(\kappa))}}\right), \tag{2.3}$$

where $\eta_0 := \max\{\eta \geq 0;\ i\eta \in \Sigma\}$. *Then* $c_0(\kappa) \in (0,1)$ *and* $\theta_0 \in (0, \pi/2)$, *and the resolvent set is described by* θ_0 *as follows.*

(a) *If* $\operatorname{Im}\kappa > 0$, *then the resolvent set* $\rho(-(A+\kappa B))$ *contains the sector* $S_+(\kappa)$, *where*

$$S_+(\kappa) := \{\mu \in \mathbb{C};\ -\theta_0 < \arg\mu < \pi/2\}.$$

(b) *If* $\operatorname{Im}\kappa < 0$, *then the resolvent set* $\rho(-(A+\kappa B))$ *contains the sector* $S_-(\kappa)$, *where*

$$S_-(\kappa) := \{\mu \in \mathbb{C};\ -\pi/2 < \arg\mu < \theta_0\}.$$

Remark 2.1. Let A and B be as in Theorem 2.1 with $\gamma(0) \geq 0$. Consider the closed interval $(-\infty, \gamma(0)]$ as a subset of $\Sigma \cap \mathbb{R}$ (instead of $\Sigma \subset \mathbb{C}$ itself). Then it is proved in [**8**, Theorem 1.6] that B is $(A+tB)$-bounded for $t > \gamma(0)$ (that is, $t \in (-\infty, \gamma(0)]^c$), with

$$\|Bu\| \leq (t - \gamma(0))^{-1}\|(A+tB)u\|, \quad u \in D(A) \cap D(B),$$

and $A+tB$ is selfadjoint on $D(A)\cap D(B)$ for $t > \gamma(0)$; in particular, if $\gamma(0) > 0$, then $A + \gamma(0)B$ is essentially selfadjoint in H. These facts are regarded as a restriction of Theorem 2.1 (**i**) and (**ii**) to the subset $\Sigma^{\mathrm{c}} \cap \mathbb{R}$.

As stated above Theorem 2.1 is proved along the idea in the proof of [**8**, Theorem 1.6]. We shall divide the proof into several lemmas.

Lemma 2.2. *The assertion* (**i**) *of* Theorem 2.1 *holds.*

Proof. Let $\kappa \in \Sigma^{\mathrm{c}}$ and $\varepsilon > 0$. To prove (2.1) we shall show that

$$\|B_\varepsilon u\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1}\|(A + \kappa B_\varepsilon)u\|, \quad u \in D(A). \tag{2.4}$$

Here we may assume that $B_\varepsilon u = B(1+\varepsilon B)^{-1}u \neq 0$ for $u \in D(A)$. Setting $v := \|B_\varepsilon u\|^{-1}u$, we have $v \in D(A)$ and $\|B_\varepsilon v\| = 1$. It then follows from ($\boldsymbol{\gamma}$**4**) that

$$-(Av, B_\varepsilon v) \in \Sigma.$$

Since Σ is closed and convex by $(\boldsymbol{\gamma}\mathbf{1})$, we have

$$0 < \operatorname{dist}(\kappa, \Sigma) \le |\kappa + (Av, B_\varepsilon v)| = \frac{|(A + \kappa B_\varepsilon)u, B_\varepsilon u)|}{\|B_\varepsilon u\|^2}$$

and hence $\|B_\varepsilon u\|^2 \le \operatorname{dist}(\kappa, \Sigma)^{-1} |((A + \kappa B_\varepsilon)u, B_\varepsilon u)|$. Applying the Cauchy-Schwarz inequality, we have (2.4). Letting $\varepsilon \downarrow 0$ in (2.4) with $u \in D(A) \cap D(B)$ we obtain (2.1). The closedness of $A + \kappa B$ is a consequence of (2.1). This completes the proof of assertion (**i**) in Theorem 2.1. □

Lemma 2.3. *$A + \kappa B$ is m-accretive in H for $\kappa \in \Sigma^{\mathrm{c}}$ with* $\operatorname{Re} \kappa \ge 0$.

Proof. Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa \ge 0$. Then it remains to show that

$$R(A + \kappa B + 1) = H. \tag{2.5}$$

Since $A + \kappa B_\varepsilon$ is also m-accretive (see Pazy [**10**, Corollary 3.3.3]), for $f \in H$ and $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in D(A)$ of the approximate equation

$$Au_\varepsilon + \kappa B_\varepsilon u_\varepsilon + u_\varepsilon = f, \tag{2.6}$$

satisfying $\|u_\varepsilon\| \le \|f\|$ and hence $\|(A + \kappa B_\varepsilon)u_\varepsilon\| = \|f - u_\varepsilon\| \le 2\,\|f\|$. Therefore we see from (2.4) that

$$\|B_\varepsilon u_\varepsilon\| \le 2 \operatorname{dist}(\kappa, \Sigma)^{-1} \|f\|.$$

This implies that $\|B_\varepsilon (A + \kappa B_\varepsilon + 1)^{-1}\|$ is bounded. Thus we obtain (2.5) (see [**7**, Proposition 2.2] or [**4**, Exercise 6.12.7 Chapter 1]). □

Lemma 2.4. *The closure of $A + \kappa B$ (denoted by $(A + \kappa B)^{\sim}$) is m-accretive in H for $\kappa \in \partial\Sigma$ with* $\operatorname{Re} \kappa \ge 0$.

Proof. Let $\kappa \in \partial\Sigma$ with $\operatorname{Re} \kappa \ge 0$. First we note that $A + \kappa B$ is closable and its closure is also accretive (cf. [**10**, Theorem 1.4.5]). Now $(\boldsymbol{\gamma}\mathbf{1})$ means that there exists $\nu \in \mathbb{C}$ satisfying $|\nu| = 1$ and

$$\operatorname{Re}[\nu(\overline{z - \kappa})] \le 0 \quad \forall\, z \in \Sigma. \tag{2.7}$$

(if $\partial\Sigma$ is smooth at a neighborhood of κ, then ν is uniquely defined as a unit outward normal vector of $\partial\Sigma$ at κ). (2.7) implies that the function $\zeta \in \Sigma \mapsto |(\kappa + \nu) - \zeta|$ attains to its minimum at $\zeta = \kappa$ (cf. [**2**, Theorem V.2]). We can show for every $t > 0$ that

$$\operatorname{Re}(\kappa + t\nu) \ge 0, \tag{2.8}$$

$$\operatorname{dist}(\kappa + t\nu, \Sigma) = t. \tag{2.9}$$

In fact, ($\boldsymbol{\gamma}\mathbf{3}$) and $\kappa \in \partial\Sigma$ implies $\kappa - 1 \in \Sigma$. Setting $z = \kappa - 1$ in (2.7), we have $\mathrm{Re}\,\nu \geq 0$ and (2.8). (2.9) is a consequence of (2.7) multiplied by $t > 0$. (2.8) implies that $A + (\kappa + (\nu/n))B$ is m-accretive for each $n \in \mathbb{N}$ (see Lemma 2.3), that is, for every $f \in H$ there is a unique solution $u_n \in D(A) \cap D(B)$ of

$$Au_n + (\kappa + (\nu/n))Bu_n + u_n = f, \tag{2.10}$$

satisfying

$$\|u_n\| \leq \|f\|. \tag{2.11}$$

Now we can prove that $\|(\nu/n)Bu_n\| = n^{-1}\|Bu_n\| \leq 2\,\|f\|$. In fact, we see from (2.1) that

$$\begin{aligned}\|Bu_n\| &\leq \mathrm{dist}\,(\kappa + \nu/n, \Sigma)^{-1}\|(A + (\kappa + \nu/n)B)u_n\| = n\,\|f - u_n\| \\ &\leq 2\,n\,\|f\|.\end{aligned}$$

This yields together with (2.10) that $\|(A + \kappa B)u_n\| \leq 4\,\|f\|$. To finish the proof we show that $(\nu/n)Bu_n$ converges to zero weakly in H. It follows from (2.11) that for every $v \in D(B)$,

$$|((\nu/n)Bu_n, v)| = n^{-1}|(u_n, Bv)| \leq n^{-1}\|f\| \cdot \|Bv\| \to 0 \;\; (n \to \infty).$$

Since $D(B)$ is dense in H and $n^{-1}\|Bu_n\|$ is bounded, we see that $n^{-1}Bu_n \to 0$ $(n \to \infty)$ weakly. (2.11) implies that we can choose a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u := \text{w-lim}_{k\to\infty} u_{n_k}$ exists. Then we have

$$\begin{aligned}(A + \kappa B)u_{n_k} &= f - u_{n_k} - (\nu/n_k)Bu_{n_k} \\ &\to f - u \;\; (k \to \infty) \;\text{ weakly.}\end{aligned}$$

It follows from the (weak) closedness of $(A + \kappa B)^{\sim}$ that $u \in D((A + \kappa B)^{\sim})$ and $(A + \kappa B)^{\sim}u = f - u$. This proves the essential m-accretivity of $A + \kappa B$ for $\kappa \in \partial\Sigma$ with $\mathrm{Re}\,\kappa \geq 0$. □

Lemma 2.5. *Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\mathrm{Re}\,\kappa < 0$. Let $c_0(\kappa)$ be defined in* (2.2).

(a) *If $\mathrm{Im}\,\kappa > 0$, then $\rho(-(A + \kappa B))$ contains the sector $\{\lambda \in \mathbb{C};\ 0 \leq \arg\lambda < \pi/2\}$, with*

$$\|(A + \kappa B + \lambda)^{-1}\| \leq [1 - c_0(\kappa)]^{-1}(\mathrm{Re}\,\lambda)^{-1}, \quad \mathrm{Re}\,\lambda > 0,\ \mathrm{Im}\,\lambda \geq 0. \tag{2.12}$$

(b) *If $\mathrm{Im}\,\kappa < 0$, then $\rho(-(A + \kappa B))$ contains the sector $\{\lambda \in \mathbb{C};\ -\pi/2 < \arg\lambda \leq 0\}$, with*

$$\|(A + \kappa B + \lambda)^{-1}\| \leq [1 - c_0(\kappa)]^{-1}(\mathrm{Re}\,\lambda)^{-1}, \quad \mathrm{Re}\,\lambda > 0,\ \mathrm{Im}\,\lambda \leq 0. \tag{2.13}$$

Proof. Let $\kappa \in \Sigma^{c}$ with $\operatorname{Re}\kappa < 0$. Since Σ is symmetric with respect to the real axis by ($\boldsymbol{\gamma 2}$), it suffices to prove the assertion (a).

(a) Let $\operatorname{Im}\kappa > 0$. Then we shall show that $\lambda \in \rho(-(A+\kappa B))$ for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda \geq 0$. This is equivalent to the unique solvability of the equation for each $f \in H$

$$Au + \kappa Bu + \lambda u = f. \tag{2.14}$$

Let $\zeta \in \Sigma^{c}$ with $\operatorname{Re}\zeta = 0$ and $\operatorname{Im}\zeta > 0$. Then $A+\zeta B$ is m-accretive in H (see Lemma 2.3). Setting $K := (\zeta - \kappa)B(A+\zeta B+\lambda)^{-1}$, (2.14) can be written as

$$(1-K)(A+\zeta B+\lambda)u = f, \tag{2.15}$$

Thus it remains to show the unique solvability of the equation $(1-K)v = f$, since $A+\zeta B+\lambda$ is invertible. To do so it suffices to show that

$$\|K\| = |\zeta - \kappa| \cdot \|B(A+\zeta B+\lambda)^{-1}\| < 1. \tag{2.16}$$

Now let $\kappa \in \Sigma^{c}$ (with $\operatorname{Re}\kappa < 0$ and $\operatorname{Im}\kappa > 0$) satisfy $|\zeta - \kappa| < \operatorname{dist}(\zeta, \Sigma)$ (see Figure 2); in this connection note that if $\operatorname{Im}\zeta < 0$ then we have $|\zeta - \kappa| > \operatorname{dist}(\zeta, \Sigma)$.

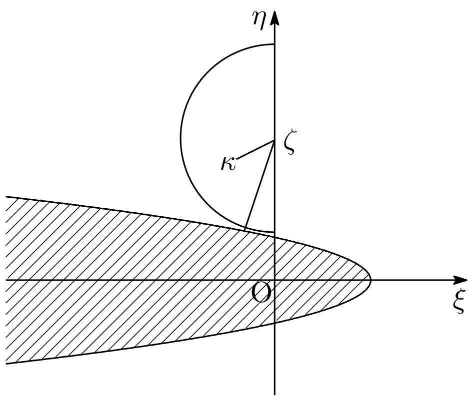

Figure 2: $|\zeta - \kappa| < \operatorname{dist}(\zeta, \Sigma)$

Then we can solve (2.15). It follows from (2.1) that

$$\|Bu\| \leq \operatorname{dist}(\zeta, \Sigma)^{-1}\|(A+\zeta B)u\|. \tag{2.17}$$

On the other hand, we can show that

$$\|(A+\zeta B)u\| \leq \|v\|. \tag{2.18}$$

In fact, making the inner product of $(A+\zeta B+\lambda)u = v$ with $(A+\zeta B)u$ gives

$$\|(A+\zeta B)u\|^2 + (\mathrm{Re}\,\lambda)\|A^{1/2}u\|^2 + \mathrm{Re}\,(\lambda\overline{\zeta})\|B^{1/2}u\|^2 = \mathrm{Re}\,(v, (A+\zeta B)u).$$

Since $\mathrm{Re}\,\zeta = 0$ and $\mathrm{Im}\,\zeta > 0$, we have $\mathrm{Re}\,(\lambda\overline{\zeta}) = (\mathrm{Im}\,\lambda)(\mathrm{Im}\,\zeta) \geq 0$. Hence applying the Cauchy-Schwarz inequality gives (2.18). Combining (2.17) with (2.18), we have

$$\|Bu\| = \|B(A+\zeta B+\lambda)^{-1}v\| \leq \mathrm{dist}\,(\zeta, \Sigma)^{-1}\|v\|.$$

Therefore, since $|\zeta - \kappa| < \mathrm{dist}\,(\zeta, \Sigma)$, we obtain (2.16):

$$\|K\| \leq |\zeta - \kappa|\,\mathrm{dist}\,(\zeta, \Sigma)^{-1} < 1.$$

This completes the proof of $\lambda \in \rho(-(A+\kappa B))$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$ and $\mathrm{Im}\,\lambda \geq 0$.

Now we prove the estimate (2.12). Since $\|v\| = \|(1-K)^{-1}f\| \leq (1 - \|K\|)^{-1}\|f\|$, it follows from (2.15) that

$$\|(A+\kappa B+\lambda)^{-1}f\| = \|(A+\zeta B+\lambda)^{-1}v\| \leq \frac{(\mathrm{Re}\,\lambda)^{-1}\|f\|}{1 - |\zeta-\kappa|\,\mathrm{dist}\,(\zeta, \Sigma)^{-1}}.$$

Here we note that the function $\varphi(\eta) := |i\eta - \kappa|\mathrm{dist}\,(i\eta, \Sigma)^{-1}$ is continuous on the open interval (η_0, ∞), where $\eta_0 := \max\{\eta \geq 0;\, i\eta \in \Sigma\}$. We show that $\inf\{\varphi(\eta); \eta > \eta_0\} = \min\{\varphi(\eta); \eta > \eta_0\} < 1$. Let $P : \mathbb{C} \to \Sigma$ be the projection. Let $\eta_1 \in (\eta_0, \infty)$ satisfy that $P\kappa$, κ and $i\eta_1$ are on the same line. Then we have $\inf\{\varphi(\eta); \eta > \eta_0\} \leq \varphi(\eta_1) < 1$. On the other hand, we have for every $\eta > \eta_0$

$$\begin{aligned}\varphi(\eta) &= \frac{|i\eta - \kappa|}{|i\eta - i\eta_0|}\frac{|i\eta - i\eta_0|}{\mathrm{dist}\,(i\eta, \Sigma)} \\ &\geq \frac{|i\eta - \kappa|}{|i\eta - i\eta_0|},\end{aligned}$$

which implies

$$\liminf_{\eta\to\infty} \varphi(\eta) \geq 1.$$

Thus we can find $\eta_2 \geq \eta_1$ such that $\inf\{\varphi(\eta); \eta > \eta_2\} \geq \varphi(\eta_1)$. Therefore we obtain $\inf\{\varphi(\eta); \eta > \eta_0\} = \min\{\varphi(\eta); \eta > \eta_0\}$. Setting $c_0(\kappa) := \min\{\varphi(\eta); \eta > \eta_0\}$, we obtain (2.12). □

Lemma 2.6. *Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\mathrm{Re}\,\kappa < 0$. Let θ_0 be defined in (2.3). Then*

(a) *If $\mathrm{Im}\,\kappa > 0$, then $\rho(-(A+\kappa B))$ contains $S_+(\kappa) = \{\lambda \in \mathbb{C};\ -\theta_0 < \arg\lambda < \pi/2\}$.*

(b) *If $\mathrm{Im}\,\kappa < 0$, then $\rho(-(A+\kappa B))$ contains $S_-(\kappa) = \{\lambda \in \mathbb{C};\ -\pi/2 < \arg\lambda < \theta_0\}$.*

Proof. We prove only (a) as in the proof of Lemma 2.5.

(a) Let $\operatorname{Im}\kappa > 0$. Then it remains to prove that the sector $\{\lambda \in \mathbb{C};\ -\theta_0 < \arg\lambda < 0\}$ is contained in $\rho(-(A+\kappa B))$ (see Lemma 2.5 (a)). Let $\xi > 0$. Then $\xi \in \rho(-(A+\kappa B))$, with $\|(A+\kappa B+\xi)^{-1}\| \le [1-c_0(\kappa)]^{-1}\,\xi^{-1}$ [see (2.12)]. Now let $f \in H$. Then we want to solve the equation $Au+\kappa Bu+\lambda u = f$, with $\operatorname{Re}\lambda > 0$. Setting $K := (\xi-\lambda)(A+\kappa B+\xi)^{-1}$, we have

$$(1-K)(A+\kappa B+\xi)u = f. \tag{2.19}$$

Noting that if $\operatorname{Im}\lambda > -(\operatorname{Re}\lambda)\tan\theta_0$, then there exists some $\xi > 0$ such that $|\xi-\lambda| < [1-c_0(\kappa)]\,\xi$ (see Figure 2) and hence $\|K\| \le |\xi-\lambda|[1-c_0(\kappa)]^{-1}\,\xi^{-1} < 1$.

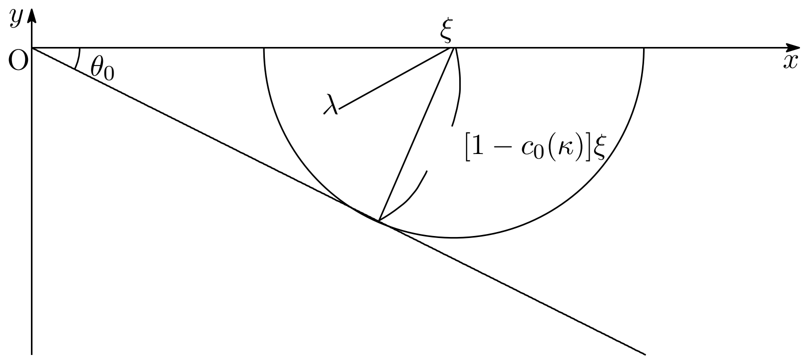

Figure 2: $\tan\theta_0 = (1-c_0(\kappa))/\sqrt{c_0(\kappa)(2-c_0(\kappa))}$

Therefore $u := (A+\kappa B+\xi)^{-1}(1-K)^{-1}f$ is a unique solution of (2.19), with

$$\begin{aligned}\|u\| = \|(A+\kappa B+\xi)^{-1}v\| &\le [1-c_0(\kappa)]^{-1}\xi^{-1}\|v\| \\ &\le \frac{\|f\|}{[1-c_0(\kappa)]\xi - |\xi-\lambda|},\end{aligned}$$

where we have used the inequality

$$\|v\| \le [1-|\xi-\lambda|[1-c_0(\kappa)]^{-1}\xi^{-1}]^{-1}\|f\|$$

derived from (2.19). Therefore we can conclude that $\lambda \in \rho(-(A+\kappa B))$ for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda > -(\operatorname{Re}\lambda)\tan\theta_0$. □

Next, we state two particular cases of Theorem 2.1 in which $B^{1/2}$ is $A^{1/2}$-bounded or B is A-bounded (under the condition $\gamma(0) < 0$).

Theorem 2.7. *Let A, B, Σ and γ be the same as those in* Theorem 2.1 *with* **($\boldsymbol{\gamma}$1)**–**($\boldsymbol{\gamma}$4)**. *Assume that there exists $\alpha_0 > 0$ such that*

$$\alpha_0(B_\varepsilon u, u) \le (Au, u), \quad u \in D(A). \tag{2.20}$$

If $(\boldsymbol{\gamma}\mathbf{5})_0$ *is replaced with*

$(\boldsymbol{\gamma}\mathbf{5})_{\alpha_0}$ $-\alpha_0 \le \gamma(0)$,

then, in addition to (**i**) *of* Theorem 2.1, *the following* (**iv**)–(**vi**) *hold.*

(**iv**) *If* $\gamma(0) < 0$ ($\Leftrightarrow 0 \in \Sigma^{\mathrm{c}}$), *then* B *is* A*-bounded with*

$$\|Bu\| \le |\gamma(0)|^{-1}\|Au\|, \quad u \in D(A) \subset D(B). \tag{2.21}$$

(**v**) $A + \kappa B$ *is m-accretive on* $D(A) \cap D(B)$ *for* $\kappa \in \Sigma^{\mathrm{c}}$ *with* $\operatorname{Re}\kappa \ge -\alpha_0$ *and* $A + \kappa B$ *is essentially m-accretive in* H *for* $\kappa \in \partial\Sigma$ *with* $\operatorname{Re}\kappa \ge -\alpha_0$.

(**vi**) *Let* $\kappa \in \Sigma^{\mathrm{c}}$ *with* $\operatorname{Re}\kappa < -\alpha_0$. *Let* $c_{\alpha_0}(\kappa)$ *and* θ_{α_0} *be defined as*

$$c_{\alpha_0}(\kappa) := \begin{cases} \min\Big\{\dfrac{|-\alpha_0 + i\eta - \kappa|}{\operatorname{dist}(-\alpha_0 + i\eta, \Sigma)};\ \eta_0 < \eta < \infty\Big\}, & \operatorname{Im}\kappa > 0, \\ \min\Big\{\dfrac{|-\alpha_0 + i\eta - \overline{\kappa}|}{\operatorname{dist}(-\alpha_0 + i\eta, \Sigma)};\ \eta_0 < \eta < \infty\Big\}, & \operatorname{Im}\kappa < 0, \end{cases}$$

$$\theta_{\alpha_0} := \tan^{-1}\bigg(\frac{1 - c_{\alpha_0}(\kappa)}{\sqrt{c_{\alpha_0}(\kappa)(2 - c_{\alpha_0}(\kappa))}}\bigg),$$

where $\eta_0 := \max\{\eta \ge 0;\ -\alpha_0 + i\eta \in \Sigma\}$. *Then* $c_{\alpha_0}(\kappa) \in (0,1)$ *and* $\theta_{\alpha_0} \in (0, \pi/2)$.

(a) *If* $\operatorname{Im}\kappa > 0$, *then the resolvent set* $\rho(-(A+\kappa B))$ *contains the sector* $S_+(\kappa)$, *where*

$$S_+(\kappa) := \{\lambda \in \mathbb{C};\ -\theta_{\alpha_0} < \arg\lambda < \pi/2\}.$$

(b) *If* $\operatorname{Im}\kappa < 0$, *then the resolvent set* $\rho(-(A+\kappa B))$ *contains the sector* $S_-(\kappa)$, *where*

$$S_-(\kappa) := \{\lambda \in \mathbb{C};\ -\pi/2 < \arg\lambda < \theta_{\alpha_0}\}.$$

Remark 2.2. Let A and B be as in Theorem 2.7, satisfying (2.20), with $-\alpha_0 \le \gamma(0) < 0$. Then it is proved in [**8**, Theorem 1.7] that B is A-bounded:

$$\|Bu\| \le |\gamma(0)|^{-1}\|Au\|, \quad u \in D(A) \subset D(B),$$

and $A + tB$ is selfadjoint on $D(A)$ for $t > \gamma(0)$; in particular, $A + \gamma(0)B$ is essentially selfadjoint in H. These facts are regarded as a restriction of Theorem 2.7 (**iv**) and (**v**) to the subset $\Sigma^{\mathrm{c}} \cap \mathbb{R}$.

Proof. (**iv**) Let $\gamma(0) < 0$. To prove (2.21) it suffices to show that

$$\|B_\varepsilon u\| \le \operatorname{dist}(0, \Sigma)^{-1}\|Au\| = |\gamma(0)|^{-1}\|Au\|, \quad \varepsilon > 0,\ u \in D(A). \tag{2.22}$$

As in the proof of Lemma 2.2, we see from $(\boldsymbol{\gamma}\mathbf{4})$ that

$$-\operatorname{Re}(Av, B_\varepsilon v) \le \gamma(0) < 0,$$

where $v := \|B_\varepsilon u\|^{-1}u$. So we obtain $\mathrm{Re}\,(Au, B_\varepsilon u) \geq |\gamma(0)| \cdot \|B_\varepsilon u\|^2$ and hence (2.22).

(**v**) Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\alpha_0 + \mathrm{Re}\,\kappa \geq 0$. Then the accretivity of $A + \kappa B_\varepsilon$ (and $A + \kappa B$) is a consequence of (2.20):

$$\mathrm{Re}\,((A + \kappa B_\varepsilon)u, u) \geq (\alpha_0 + \mathrm{Re}\,\kappa)(B_\varepsilon u, u) \geq 0.$$

Now we can consider the unique solvability of the equation for each $f \in H$ and $\lambda > 0$

$$Au_\varepsilon + \kappa B_\varepsilon u_\varepsilon + \lambda u_\varepsilon = f.$$

In order to prove $R(A + \kappa B + \lambda) = H$ we only have to show that $\|u_\varepsilon\|$ and $\|B_\varepsilon u_\varepsilon\|$ are bounded as ε tends to zero. The m-accretivity of $A + \kappa B_\varepsilon$ yields that $\|u_\varepsilon\| \leq \lambda^{-1}\|f\|$ and hence $\|Au_\varepsilon + \kappa B_\varepsilon u_\varepsilon\| \leq 2\|f\|$. In the same way as in the proof of Lemma 2.5 we can show that there exists $c > 0$ such that $\|Au_\varepsilon\| + \|B_\varepsilon u_\varepsilon\| \leq c\|f\|$. This concludes that $R(A + \kappa B + \lambda) = H$. The proof of the essential m-accretivity of $A + \kappa B$ for $\kappa \in \partial\Sigma$ with $\mathrm{Re}\,\kappa \geq -\alpha_0$ is similar to that of Lemma 2.4.

(**vi**) Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\mathrm{Re}\,\kappa < -\alpha_0$ and $\mathrm{Im}\,\kappa > 0$. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$. To show that $\lambda \in \rho(-(A + \kappa B))$ let $f \in H$. Then we want to solve the equation

$$Au + \kappa Bu + \lambda u = f. \tag{2.23}$$

Set $v := (A + \zeta B + \lambda)u$ for $\zeta \in \Sigma^{\mathrm{c}}$ with $\mathrm{Re}\,\zeta = -\alpha_0$. Since $A + \zeta B$ is m-accretive in H [see (**v**)], we can write (2.23) as

$$v - (\zeta - \kappa)B(A + \zeta B + \lambda)^{-1}v = f.$$

Proceeding as in the proof of Lemma 2.5, we can show that $|\zeta - \kappa| \cdot \|B(A + \zeta B + \lambda)^{-1}\| < 1$ if $|\zeta - \kappa| < \mathrm{dist}(\zeta, \Sigma)$. Replacing $c_0(\kappa)$ with $c_{\alpha_0}(\kappa)$, the similar argument to Lemma 2.5 and Lemma 2.6 yields the assertion (a). Considering $\overline{\kappa}$ instead of κ when $\mathrm{Im}\,\kappa < 0$, we can also obtain the assertion (b). □

Remark 2.3. Let $\{\kappa_n = \xi_n + i\eta\} \subset \Sigma^{\mathrm{c}}$ be a sequence satisfying $\xi_n \uparrow -\alpha_0$ $(n \to \infty)$ in assertion (**vi**). Then $c_{\alpha_0}(\kappa_n) \to 0$ and hence the resolvent sets $\rho(-(A + \kappa_n B))$ extend from the sectors to the right half-plane as $n \to \infty$, which suggests the m-accretivity of the limiting operator $A + (-\alpha_0 + i\eta)B$. This is nothing but the conclusion of (**v**).

§3. Proof of Theorem 1.1

In this section we prepare some inequalities to apply Theorems 2.1 and 2.7 to $A := \Delta^2$ and $B := |x|^{-4}$. In [**9**, Lemmas 3.1 and 3.3] we have proved the following

Lemma 3.0. *Let* $v \in C_0^\infty(\mathbb{R}^N)$. *Then*

(**i**) $\mathrm{Re}\,((x\cdot\nabla)v, v) = -\dfrac{N}{2}\|v\|^2$,

(**ii**) $\|(x\cdot\nabla)v\|^2 - (N^2/4)\|v\|^2 \geq 0$,

(**iii**) $\big\||x|^2\Delta v\big\|^2\|v\|^2 + 2N\big\||x|\nabla v\big\|^2\|v\|^2 - \big\||x|\nabla v\big\|^4 - 4\|(x\cdot\nabla)v\|^2\|v\|^2 \geq 0.$

The following lemma is a strict version of Lemma 3.0 (**ii**).

Lemma 3.1. *Let* $v \in C_0^\infty(\mathbb{R}^N)$. *Then*

$$|\mathrm{Im}\,(v, (x\cdot\nabla)v)|^2 \leq \|v\|^2\Big(\|(x\cdot\nabla)v\|^2 - \frac{N^2}{4}\|v\|^2\Big). \tag{3.1}$$

Proof. Let $v \in C_0^\infty(\mathbb{R}^N)$. From the Schwarz inequality we have

$$|\mathrm{Im}\,(v, (x\cdot\nabla)v)|^2 + |\mathrm{Re}\,(v, (x\cdot\nabla)v)|^2 = |(v, (x\cdot\nabla)v)|^2 \leq \|v\|^2\|x\cdot\nabla v\|^2.$$

Combining this with Lemma 3.0 (**i**), we obtain (3.1). □

The following lemma together with Lemma 3.1 give a strict version of Lemma 3.0 (**iii**).

Lemma 3.2. *Let* $v \in C_0^\infty(\mathbb{R}^N)$. *Then*

$$\begin{aligned}&\Big[\|v\|^2\mathrm{Im}\,((x\cdot\nabla)v, |x|^2\Delta v) - \big\||x|\nabla v\big\|^2\mathrm{Im}\,(v, (x\cdot\nabla)v)\Big]^2 \\ &\leq \Big\{\|v\|^2\Big[\|(x\cdot\nabla)v\|^2 - \frac{N^2}{4}\|v\|^2\Big] - |\mathrm{Im}\,(v,(x\cdot\nabla)v)|^2\Big\} \\ &\times\Big[\big\||x|^2\Delta v\big\|^2\|v\|^2 + 2N\big\||x|\nabla v\big\|^2\|v\|^2 - \big\||x|\nabla v\big\|^4 - 4\|(x\cdot\nabla)v\|^2\|v\|^2\Big].\end{aligned} \tag{3.2}$$

Proof. For each $v \in C_0^\infty(\mathbb{R}^N)$ set $v_1 := |x|^2\Delta v$, $v_2 := (x\cdot\nabla)v$, $v_3 := v$. Let $G := ((v_j, v_k))_{jk}$. Let $a, b, c \geq 0$ and $\alpha, \beta, \gamma \in \mathbb{C}$ be defined as

$$\begin{pmatrix} c & \overline{\alpha} & \beta \\ \alpha & b & \overline{\gamma} \\ \overline{\beta} & \gamma & a \end{pmatrix} := \begin{pmatrix} \big\||x|^2\Delta v\big\|^2 & (|x|^2\Delta v, (x\cdot\nabla)v) & (|x|^2\Delta v, v) \\ ((x\cdot\nabla)v, |x|^2\Delta v) & \big\|(x\cdot\nabla)v\big\|^2 & ((x\cdot\nabla)v, v) \\ (v, |x|^2\Delta v) & (v, (x\cdot\nabla)v) & \|v\|^2 \end{pmatrix}.$$

Since G is positive semi-definite, we have $\det G \geq 0$;

$$a|\alpha|^2 + b|\beta|^2 + c|\gamma|^2 \leq abc + 2\mathrm{Re}\,(\alpha\beta\gamma).$$

Setting $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$ with $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}$ $(j = 1, 2)$, we have

$$\begin{aligned}&a\alpha_2^2 + b\beta_2^2 + c\gamma_2^2 + 2(\alpha_1\beta_2\gamma_2 + \alpha_2\beta_1\gamma_2 + \alpha_2\beta_2\gamma_1) \\ &\leq abc + 2\alpha_1\beta_1\gamma_1 - (a\alpha_1^2 + b\beta_1^2 + c\gamma_1^2).\end{aligned} \tag{3.3}$$

Now it is easy to see that

$$\alpha_1 = \operatorname{Re}\alpha = \frac{N}{2}\widetilde{b} - 2b, \tag{3.4}$$

$$\beta_1 = \operatorname{Re}\beta = Na - \widetilde{b}, \tag{3.5}$$

$$\gamma_1 = \operatorname{Re}\gamma = -\frac{N}{2}a, \tag{3.6}$$

where $\widetilde{b} := \big\||x|\nabla v\big\|^2$ (see [**9**, Section 3]). It follows (3.4)–(3.6) that the right-hand side of (3.3) equals

$$(b - (N^2/4)a)(ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab).$$

Multiplying (3.3) by a and using the equality $\beta_2 = 2\gamma_2$, we have

$$\begin{aligned} a^2\alpha_2^2 + 2a(\beta_1 + 2\gamma_1)\alpha_2\gamma_2 + a(4\alpha_1 + 4b + c)\gamma_2^2 \\ \le a(b - (N^2/4)a)(ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab). \end{aligned} \tag{3.7}$$

We see from (3.4)–(3.6) that the left-hand side of (3.7) equals

$$(a\alpha_2 - \widetilde{b}\gamma_2)^2 + (ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab)\gamma_2^2,$$

which implies that

$$(a\alpha_2 - \widetilde{b}\gamma_2)^2 \le (ab - (N^2/4)a^2 - \gamma_2^2)(ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab). \tag{3.8}$$

(3.8) is nothing but (3.2). □

Lemma 3.3. *Let k_1 be the constants defined in* (1.1):

$$k_1 = 112 - 3(N-2)^2.$$

For $u \in H^4(\mathbb{R}^N)$ and $\varepsilon > 0$ put

$$\mathrm{IP} := (\Delta^2 u, (|x|^4 + \varepsilon)^{-1}u),$$

and $a := \|(|x|^4 + \varepsilon)^{-1}u\|^2$. Then $k_1 a + \operatorname{Re}\mathrm{IP} \ge 0$ and

$$\begin{aligned} |\operatorname{Im}\mathrm{IP}|^2 \le\, & 64\sqrt{a}\big(\sqrt{k_1 a + \operatorname{Re}\mathrm{IP}} - ((N^2/4) - N - 10)\sqrt{a}\big) \\ & \times \big(\sqrt{k_1 a + \operatorname{Re}\mathrm{IP}} + 8\sqrt{a}\big)^2. \end{aligned} \tag{3.9}$$

If $N \ge 9$, then $k_2 a + \operatorname{Re}\mathrm{IP} \ge 0$ and

$$|\operatorname{Im}\mathrm{IP}|^2 \le \frac{64\sqrt{a}(k_2 a + \operatorname{Re}\mathrm{IP})(\sqrt{k_1 a + \operatorname{Re}\mathrm{IP}} + 8\sqrt{a})^2}{\sqrt{k_1 a + \operatorname{Re}\mathrm{IP}} + ((N^2/4) - N - 10)\sqrt{a}} \tag{3.10}$$

where $k_2 = k_1 - [(N-2)^2/4 - 11]^2 = -(N/16)(N-8)(N^2-16) < 0$ $(N \ge 9)$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Put $v := (|x|^4 + \varepsilon)^{-1} u$. By using the same notations as in the proof of Lemma 3.2 (3.8) is written as

$$L := \frac{(a\alpha_2 - \widetilde{b}\gamma_2)^2}{ab - (N^2/4)a^2 - \gamma_2^2} \le ac + 2Na\widetilde{b} - \widetilde{b}^2 - 4ab =: R. \tag{3.11}$$

Here we note (see [**9**, Proof of Lemma 3.4]) that

$$\mathrm{IP} = \big\||x|^2 \Delta v\big\|^2 + 8(\,(x\cdot\nabla)v, |x|^2\Delta v) + 4(N+2)(v, |x|^2\Delta v) + \varepsilon\|\Delta v\|^2.$$

It follows that

$$c = \big\||x|^2\Delta v\big\|^2 \le \mathrm{Re}\,\mathrm{IP} + 16b + 8\widetilde{b} - 4N(N+2)a, \tag{3.12}$$

$$\alpha_2 = \mathrm{Im}\,((x\cdot\nabla)v, |x|^2\Delta v) = \frac{1}{8}\mathrm{Im}\,\mathrm{IP} + (N+2)\gamma_2. \tag{3.13}$$

In fact, (3.13) holds as a consequence $\beta_2 = 2\gamma_2$. Applying (3.13) to L yields

$$L = \frac{\Big(\frac{a}{8}\mathrm{Im}\,\mathrm{IP} + ((N+2)a - \widetilde{b})\gamma_2\Big)^2}{a(b-(N^2/4)a) - \gamma_2^2} = \frac{(c_1\gamma_2 + c_2)^2}{c_0 - \gamma_2^2},$$

where

$$c_0 := a(b - (N^2/4)a) \ge \gamma_2^2, \tag{3.14}$$

$$c_1 := (N+2)a - \widetilde{b}, \tag{3.15}$$

$$c_2 := \frac{a}{8}\mathrm{Im}\,\mathrm{IP}; \tag{3.16}$$

note that the inequality in (3.14) is nothing but (3.1). Since the quadratic equation $L(c_0 - t^2) = (c_1 t + c_2)^2$ has a real root $t = \gamma_2$, the discriminant is nonnegative:

$$L(c_0 L + c_0 c_1^2 - c_2^2) \ge 0. \tag{3.17}$$

It is clear that $L \ge 0$. If $L > 0$, then (3.17) yields

$$L \ge (c_2^2/c_0) - c_1^2. \tag{3.18}$$

If $L = 0$, then $\gamma_2 = -c_2/c_1$ and hence (3.14) yields that $0 \ge (c_2^2/c_0) - c_1^2$. This means that (3.18) holds for $L \ge 0$. Hence it follows from (3.14)–(3.16) and (3.18) that

$$L \ge \frac{a|\mathrm{Im}\,\mathrm{IP}|^2}{64(b-(N^2/4)a)} - (\widetilde{b} - (N+2)a)^2. \tag{3.19}$$

On the other hand, since $b \le \widetilde{b}$, (3.11) and (3.12) yields

$$\begin{aligned} R &\le a\mathrm{Re}\,\mathrm{IP} + 12ab + 2(N+4)a\widetilde{b} - \widetilde{b}^2 - 4N(N+2)a^2 \\ &\le a(k_1 a + \mathrm{Re}\,\mathrm{IP}) - (\widetilde{b} - (N+10)a)^2, \end{aligned} \tag{3.20}$$

where $k_1 := (N+10)^2 - 4N(N+2) = 112 - 3(N-2)^2$. Since $L \le R$, it follows from (3.19) and (3.20) that

$$\frac{a|\mathrm{Im}\,\mathrm{IP}|^2}{64(b - N^2a/4)} - (\widetilde{b} - (N+2)a)^2 \le a(k_1 a + \mathrm{Re}\,\mathrm{IP}) - (\widetilde{b} - (N+10)a)^2. \tag{3.21}$$

Therefore we obtain

$$\frac{|\mathrm{Im}\,\mathrm{IP}|^2}{64(b - (N^2/4)a)} - 16(\widetilde{b} - (N+6)a) \le k_1 a + \mathrm{Re}\,\mathrm{IP} =: K. \tag{3.22}$$

Now we see from (3.20) that

$$(\widetilde{b} - (N+10)a)^2 \le R + (\widetilde{b} - (N+10)a)^2 \le aK$$

and hence

$$b \le \widetilde{b} \le \sqrt{aK} + (N+10)a. \tag{3.23}$$

Applying (3.23) to (3.22), we obtain

$$\frac{|\mathrm{Im}\,\mathrm{IP}|^2}{64\sqrt{a}\big[\sqrt{K} - ((N^2/4) - N - 10)\sqrt{a}\big]} \le K + 16(\sqrt{aK} + 4a) = (\sqrt{K} + 8\sqrt{a})^2.$$

This proves (3.9) for $u \in C_0^\infty(\mathbb{R}^N)$. Next note that $N^2/4 - N - 10 \ge 0$ for $N \ge 9$. To obtain (3.10), we have only to use the equality

$$\sqrt{K} - ((N^2/4) - N - 10)\sqrt{a} = \frac{k_2 a + \mathrm{Re}\,\mathrm{IP}}{\sqrt{K} + ((N^2/4) - N - 10)\sqrt{a}}$$

where $k_2 = -N(N-8)(N^2-16)/16$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^4(\mathbb{R}^N)$, we obtain (3.9) for every $u \in H^4(\mathbb{R}^N)$. □

Proof of Theorem 1.1. Let $H := L^2(\mathbb{R}^N)$, $A := \Delta^2$ with $D(A) := H^4(\mathbb{R}^N)$ and $B := |x|^{-4}$ with $D(B) := \{u \in H;\ |x|^{-4}u \in H\}$. For $u \in D(A)$ and $\varepsilon > 0$ take $v := B_\varepsilon u = (|x|^4 + \varepsilon)^{-1}u$ with $\sqrt{a} := \|v\| = 1$. Set $\xi, \eta \in \mathbb{R}$ as

$$\xi + i\eta := -\mathrm{IP} = -(Au, B_\varepsilon u).$$

We shall prove that there exist γ independent of $\varepsilon > 0$ satisfying (γ**1**), (γ**2**), $(\gamma\mathbf{5})_0$ in Theorem 2.1 (or $(\gamma\mathbf{5})_{\alpha_0}$ in Theorem 2.7) and Σ defined in (γ**3**) such that $-\mathrm{IP} \in \Sigma$ for every $u \in D(A)$ and $\varepsilon > 0$, i.e., $(\gamma\mathbf{4})$ holds. First it follows from Lemma 3.3 with $\mathrm{Re}\,\mathrm{IP} = -\xi$, $\mathrm{Im}\,\mathrm{IP} = -\eta$, $a = 1$ that

$$\begin{cases} k_1 a + \mathrm{Re}\,\mathrm{IP} = k_1 - \xi \geq 0, \\ |\eta|^2 \leq \varphi_N(\xi), \end{cases} \tag{3.24}$$

where $\varphi_N : (-\infty, k_1] \to \mathbb{R}$ is given as follows (see (3.9)):

$$\varphi_N(t) := 64\big[\sqrt{k_1 - t} + (10 + N - (N^2/4))\big](\sqrt{k_1 - t} + 8)^2. \tag{3.25}$$

We can easily see that φ_N is monotone decreasing and $\lim_{t\to-\infty} \varphi_N(t) = \infty$. According to the sign of $\varphi_N(k_1)$ we consider two cases $N \leq 8$ and $N \geq 9$.

In the case $N \leq 8$ it holds from $10 + N - (N^2/4) > 0$ that $\varphi_N(k_1) = \min\{\varphi_N(t); t \leq k_1\} > 0$. If $|\eta|^2 \leq \varphi_N(k_1)$, then $|\eta|^2 \leq \varphi_N(\xi)$ holds. If $|\eta|^2 \geq \varphi_N(k_1)$, then $|\eta|^2 \leq \varphi_N(\xi)$ is equivalent to $\xi \leq \varphi_N^{-1}(|\eta|^2)$. Thus we have

$$\begin{cases} \xi \leq k_1 & \text{when } |\eta|^2 \leq \varphi_N(k_1), \\ \xi \leq \varphi_N^{-1}(|\eta|^2) & \text{when } |\eta|^2 \geq \varphi_N(k_1). \end{cases} \tag{3.26}$$

Set

$$\gamma(t) := \begin{cases} k_1 & \text{when } |t|^2 \leq \varphi_N(k_1), \\ \varphi_N^{-1}(|t|^2) & \text{when } |t|^2 \geq \varphi_N(k_1). \end{cases}$$

$(\gamma\mathbf{2})$ is clearly satisfied. Let Σ be defined in $(\gamma\mathbf{3})$. We show that γ is concave. (3.24) implies that

$$\Sigma = \{\xi + i\eta \in \mathbb{C}; \xi \leq k_1, |\eta| \leq \sqrt{\varphi_N(\xi)}\}. \tag{3.27}$$

Since $\sqrt{\varphi_N}$ is concave, (3.27) shows that Σ is convex. Hence γ is concave and $(\gamma\mathbf{1})$ is satisfied. (3.24) and (3.27) imply that $(\gamma\mathbf{4})$ is satisfied. Noting $\gamma(0) = k_1 > 0$, we see that $(\gamma\mathbf{5})_0$ is satisfied. When $N \leq 4$, we apply Theorem 2.1 with A, B, γ and Σ to obtain the assertion of Theorem 1.1 in the case $N \leq 4$. When $N \geq 5$, we have the Rellich inequality

$$\frac{N(N-4)}{4}\|(|x|^2 + \varepsilon)^{-1}u\| \leq \|\Delta u\|, \quad u \in H^2(\mathbb{R}^N),$$

which implies (2.20) with $\alpha_0 := [N(N-4)/4]^2$. Since $\gamma(0) = k_1 > 0 > -\alpha_0$, $(\gamma\mathbf{5})_{\alpha_0}$ is satisfied. Thus we can apply Theorem 2.7 with A, B, γ and Σ to obtain Theorem 2.7 (**v**), (**vi**). Therefore we obtain the assertion of Theorem 1.1 in the case $5 \leq N \leq 8$.

In the case $N \geq 9$ it follows from Lemma 3.3 with $\mathrm{Re}\,\mathrm{IP} = -\xi$, $a = 1$ that

$$\xi \leq k_2 := -(N/16)(N-8)(N^2-16). \tag{3.28}$$

In particular, (3.10) implies that φ_N has another expression:

$$\varphi_N(t) = \frac{64(k_2 - t)(\sqrt{k_1 - t} + 8)}{\sqrt{k_1 - t} + ((N^2/4) - N - 10)}.$$

Then $\varphi_N(k_2) = 0$ and $\sqrt{\varphi_N}$ is concave on $(-\infty, k_2]$. Set

$$\gamma(t) := \varphi_N^{-1}(|t|^2), \quad t \in \mathbb{R}.$$

It is clear that $(\gamma\mathbf{2})$ is satisfied. Let Σ be defined in $(\gamma\mathbf{3})$. Noting $k_2 < k_1$, we see from (3.24) and (3.28) that

$$\Sigma = \{\xi + i\eta \in \mathbb{C};\ \xi \le k_2, |\eta| \le \sqrt{\varphi_N(\xi)}\}. \tag{3.29}$$

Since $\sqrt{\varphi_N}$ is concave, we see from (3.29) that Σ is convex. Hence γ is concave and $(\gamma\mathbf{1})$ is satisfied. (3.24), (3.28) and (3.29) imply that $(\gamma\mathbf{4})$ is satisfied. Applying the Rellich inequality again, we have (2.20) with $\alpha_0 := [N(N-4)/4]^2$. Since $\gamma(0) = k_2 > -\alpha_0$, $(\gamma\mathbf{5})_{\alpha_0}$ is satisfied. Since $\gamma(0) = k_2 < 0$, we obtain Theorem 2.7 (**iv**). Therefore we obtain the assertion of Theorem 1.1 in the case $N \ge 9$. This completes the proof of Theorem 1.1. □

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