SUT Journal of Mathematics
Vol. 46, No. 1 (2010), 35–52

Quarter-symmetric metric connection on 3-dimensional quasi-Sasakian manifolds

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(Received July 17, 2009; Revised January 30, 2010)

Abstract. The object of the present paper is to study a quarter-symmetric metric connection on a 3-dimensional quasi-Sasakian manifold. The existence of the connection is given on a Riemannian manifold. We deduce the relation between the Riemannian connection and the quarter-symmetric metric connection on a 3-dimensional quasi-Sasakian manifold. We investigate the curvature tensor and the Ricci tensor of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection. We study the projective curvature tensor with respect to the quarter-symmetric metric connection and also characterized $\xi-$projectively flat and $\phi-$projectively flat 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection. Finally we study locally $\phi-$symmetric 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection.

AMS 2000 *Mathematics Subject Classification.* 53C15, 53C40.

Key words and phrases. quarter-symmetric metric connection, projective curvature tensor, $\phi-$projectively flat, $\xi-$projectively flat, locally $\phi-$symmetric, $\eta-$Einstein.

§1. Introduction

The notion of quasi-Sasakian structure was introduced by D. E. Blair [7] to unify Sasakian and cosymplectic structures. S. Tanno [28] also added some remarks on quasi-Sasakian structures. The properties of quasi-Sasakian manifolds have been studied by several authors, viz., J. C. Gonzalez and D. Chinea [12], S. Kanemaki [13], [14] and J. A. Oubina [22]. B. H. Kim [15] studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosymplectic structure. Recently, quasi-Sasakian manifolds have been the subject of growing interest in view of finding the significant applications to physics, in particular to super gravity and magnetic theory ([1], [2]). Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory ([3], [10]). Motivated by the roles of curvature tensor and Ricci tensor of quasi-Sasakian manifolds in string theory ([3]) we like to study curvature properties of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Z. Olszak [19] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat ([20]). Next he has proved that if the manifold is additionally conformally flat with $\beta = constant$, then (a) the manifold is locally a product of R and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). This paper is devoted to study quarter-symmetric metric connection in a 3-dimensional quasi-Sasakian manifold.

In 1975, S. Golab [11] defined and studied quarter-symmetric connection in a differentiable manifold with affine connection.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection ([11]) if its torsion tensor T of the connection $\tilde{\nabla}$

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.1}$$

where η is a 1-form and ϕ is a $(1, 1)$ tensor field.

In particular, if $\phi = id$, then the quarter-symmetric connection reduces to the semi-symmetric connection [9]. Thus the notion of quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

for all $X, Y, Z \in T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection.

After S. Golab [11], S. C. Rastogi ([24],[25]) continued the systematic study of quarter-symmetric metric connection.

In 1980, R. S. Mishra and S. N. Pandey [17] studied quarter-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds.

In 1982, K. Yano and T. Imai [30] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifolds.

In 1991, S. Mukhopadhyay, A. K. Roy and B. Barua [18] studied a quarter-symmetric metric connection on a Riemannian manifold (M, g) with an almost complex structure ϕ.

In 1997, U. C. De and S. C. Biswas [4] studied a quarter-symmetric metric connection on an $SP-$Sasakian manifold. Also in 2008, Sular, Ozgur and De [26] studied a quarter-symmetric metric connection in a Kenmotsu manifold.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be an $(2n+1)-$dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $2n+1 \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [16]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.2}$$

for $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively flat (that is, $P = 0$) if and only if the manifold is of constant curvature (pp. 84-85 of [29]). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A 3-dimensional quasi-Sasakian manifold is said to be an $\eta-$Einstein manifold if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold.

The paper is organized as follows:

After preliminaries, we recall the notion of 3-dimensional quasi-Sasakian manifold in section 3. In section 4 we prove the existence of the quarter-symmetric metric connection. In the next section we establish the relation between the Riemannian connection and the quarter-symmetric metric connection on a 3-dimensional quasi-Sasakian manifold. In section 6 we study the curvature tensor, the Ricci tensor, scalar curvature and the first Bianchi identity with respect to the quarter-symmetric metric connection. Section 7 deals with the projective curvature tensor with respect to the quarter-symmetric metric connection and prove that for a 3-dimensional quasi-Sasakian manifold, the Riemannian connection ∇ is $\xi-$projectively flat if and only if the quarter-symmetric metric connection $\tilde{\nabla}$ is so. We also study $\phi-$projectively flat 3-dimensional quasi-Sasakian manifold and prove that a 3-dimensional quasi-Sasakian manifold with constant structure function, is $\phi-$projectively flat with respect to the quarter-symmetric metric connection if and only if the manifold is of constant curvature with respect to the quarter-symmetric metric connection. Finally we characterize locally $\phi-$symmetric 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection.

§2. Preliminaries

Let M be an $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$, $(0,1)$ respectively, such that ([5],[6], [31]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad X, Y \in T(M),$$

where $T(M)$ is the Lie algebra of vector fields of the manifold M.

Then

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).$$

Let Φ be the fundamental 2-form of M defined by

$$\Phi(X,Y) = g(X, \phi Y) \qquad X, Y \in T(M).$$

Then $\Phi(X, \xi) = 0, \quad X \in T(M)$. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed, that is, for every $X, Y \in \mathcal{E}^{(2n+1)}$, where $\mathcal{E}^{(2n+1)}$ denotes the module of vector fields on M,

$$[\phi, \phi](X,Y) + d\eta(X,Y)\xi = 0,$$

$$d\Phi = 0, \quad \Phi(X,Y) = g(X, \phi Y).$$

This was first introduced by Blair [7]. There are many types of quasi-Sasakian structures ranging from the cosymplectic case, $d\eta = 0$ (rank $\eta = 1$), to the Sasakian case, $\eta \wedge (d\eta)^n \neq 0$ (rank $\eta = 2n+1$, $\Phi = d\eta$). The $1-$form η has rank $r' = 2p$ if $d\eta^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has rank $r' = 2p+1$ if $d\eta^p = 0$ and $\eta \wedge (d\eta)^p \neq 0$. We also say that r' is the rank of the quasi-Sasakian structure. Blair [7] also proved that there are no quasi-Sasakian structure of even rank. In order to study the properties of quasi-Sasakian manifolds Blair [7] proved some theorems regarding Kaehlerian manifolds and existence of quasi-Sasakian manifolds. S. Tanno [28] rectified some of these theorems.

However, while Tanno studied locally product quasi-Sasakian manifolds, he mentioned the following:

Let $M_1^{2p+1}(\phi_1, \xi_1, \eta_1, g_1)$ be a Sasakian manifold and let $M_2^{2q}(J_2, G_2)$ be a Kaehlerian manifold. Then $M_1 \times M_2$ has a quasi-Sasakian structure (ϕ, ξ, η, g) of rank $2p+1$ such that

$$\phi X = (\phi_1 X_1, J_2 X_2), \quad \xi = (\xi_1, 0),$$
$$\eta(X) = \eta_1(X_1), \quad g(X, Y) = g_1(X_1, Y_1) + G_2(X_2, Y_2),$$

for the canonical decomposition $X = (X_1, X_2)$ of a vector field X on $M_1 \times M_2$ ([7]).

Theorem [28]: *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold (more generally a normal almost contact Riemannian manifold) of rank $2p+1$. If g^* defined by*

$$2g^*(X, Y) = -d\eta(X, \phi Y),$$

$X, Y \in \mathcal{E}^{2n+1}$, is positive definite on $\mathcal{E}^{2p}$ and $\overline{\nabla}\theta = 0$ with respect to the Riemannian metric $\overline{g}$ defined by

$$\overline{g}(X, Y) = \eta(X)\eta(Y) + g^*(\psi^2 X, \psi^2 Y) + g(\theta^2 X, \theta^2 Y),$$

where the $(1,1)$ tensors ψ and θ are given by

$$\begin{aligned} \psi(X) &= \phi(X) \quad \textit{if } X \in \mathcal{E}^{2p}, \\ &= 0 \quad \textit{if } X \in \mathcal{E}^{2q} \oplus \mathcal{E}^1, \end{aligned}$$

$$\begin{aligned} \theta(X) &= \phi(X) \quad \textit{if } X \in \mathcal{E}^{2q}, \\ &= 0 \quad \textit{if } X \in \mathcal{E}^{2p+1}, \end{aligned}$$

then $(\phi, \xi, \eta, \overline{g})$ is also a quasi-Sasakian structure of rank $2p+1$ and $M(\phi, \xi, \eta, \overline{g})$ is locally the product of a Sasakian manifold and a Kaehler manifold.

It is mentioned that $\mathcal{E}^{2p+1}$, $\mathcal{E}^{2q}$, $\mathcal{E}^1$ are submodules of $\mathcal{E}^{2n+1}$. S. Tanno [28] also gave an example of a 3-dimensional quasi-Sasakian manifold which is not Sasakian. For a quasi-Sasakian manifold we have the relation ([21])

$$(\nabla_X \phi)Y = -g(\nabla_X \xi, \phi Y)\xi - \eta(Y)\phi\nabla_X \xi,$$

which generalizes the well-known conditions $\nabla\phi = 0$ and $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$ characterizing respectively cosymplectic and Sasakian manifolds. The quasi-Sasakian condition also reflects in some properties of curvature and of the vector field ξ. In fact, we have the following results.

Lemma([7], [21]): *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold. Then*

(i) *the vector field ξ is Killing and its integral curves are geodesics*;

(ii) *the Ricci curvature in the direction of ξ is given by $||\nabla\xi||^2$.*

§3. Quasi-Sasakian structure of dimension three

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if ([19])

$$\nabla_X \xi = -\beta\phi X, \quad X \in T(M), \tag{3.1}$$

for a certain function β on M, such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Riemannian connection of M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. Here we have shown that the assumption $\xi\beta = 0$ is not necessary.

As a consequence of (3.1), we have ([19])

$$(\nabla_X\phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \quad X, Y \varepsilon T(M). \tag{3.2}$$

Because of (3.1) and (3.2), we find

$$\nabla_X(\nabla_Y\xi) = -(X\beta)\phi Y - \beta^2\{g(X,Y)\xi - \eta(Y)X\} - \beta\phi\nabla_X Y$$

which implies that

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}. \tag{3.3}$$

Thus we get from (3.3)

$$\begin{aligned} R(X,Y,Z,\xi) \quad &= \quad (X\beta)g(\phi Y, Z) - (Y\beta)g(\phi X, Z) \\ &\quad\; -\beta^2\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}, \end{aligned} \tag{3.4}$$

where $R(X,Y,Z,W) = g(R(X,Y,Z),W)$. Putting $X = \xi$, in (3.4) we obtain

$$R(\xi,Y,Z,\xi) = \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\} + g(\phi Y, Z)\xi\beta. \tag{3.5}$$

Interchanging Y and Z of (3.5) yields

$$R(\xi,Z,Y,\xi) = \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\} + g(\phi Z, Y)\xi\beta. \tag{3.6}$$

Since $R(\xi,Y,Z,\xi) = R(Z,\xi,\xi,Y) = R(\xi,Z,Y,\xi)$, from (3.5) and (3.6) we have

$$\{g(\phi Y, Z) - g(\phi Z, Y)\}\xi\beta = 0.$$

Therefore, we can easily verify that $\xi\beta = 0$.

In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z \quad &= \quad g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad\; -S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{3.7}$$

where Q is the Ricci operator, that is, g(QX,Y)=S(X,Y) and r is the scalar curvature of the manifold.

Let M be a 3-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by ([20])

$$\begin{aligned} S(Y,Z) &= (\frac{r}{2}-\beta^2)g(Y,Z)+(3\beta^2-\frac{r}{2})\eta(Y)\eta(Z) \\ &\quad -\eta(Y)d\beta(\phi Z)-\eta(Z)d\beta(\phi Y), \end{aligned} \tag{3.8}$$

where r is the scalar curvature of M.

As a consequence of (3.8), we get for the Ricci operator Q

$$\begin{aligned} QX &= (\frac{r}{2}-\beta^2)X+(3\beta^2-\frac{r}{2})\eta(X)\xi \\ &\quad +\eta(X)(\phi grad\,\beta)-d\beta(\phi X)\xi, \end{aligned}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X)=g(grad\,f,X)$. From (3.8) we have

$$S(X,\xi)=2\beta^2\eta(X)-d\beta(\phi X). \tag{3.9}$$

As a consequence of (3.1) we also have ([19])

$$(\nabla_X\eta)(Y)=g(\nabla_X\xi,Y)=-\beta g(\phi X,Y). \tag{3.10}$$

Also from (3.8) it follows that

$$S(\phi X,\phi Z)=S(X,Z)-2\beta^2\eta(X)\eta(Z). \tag{3.11}$$

§4. Existence of a quarter-symmetric metric connection

Let X and Y be any two vector fields on (M,g). Let us define a connection $\tilde\nabla_X Y$ by the following equation:

$$\begin{aligned} 2g(\tilde\nabla_X Y,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g([X,Y],Z) \\ &\quad -g([Y,Z],X)+g([Z,X],Y)+g(\eta(Y)\phi X \\ &\quad -\eta(X)\phi Y,Z)+g(\eta(Y)\phi Z-\eta(Z)\phi Y,X) \\ &\quad +g(\eta(X)\phi Z-\eta(Z)\phi X,Y), \end{aligned}$$

which holds for all vector fields $X,Y,Z\in T(M)$.

It can easily be verified that the mapping

$$(X,Y)\longrightarrow\tilde\nabla_X Y$$

satisfies the following equalities:

$$\tilde\nabla_X(Y+Z)=\tilde\nabla_X Y+\tilde\nabla_X Z, \tag{4.1}$$

$$\tilde{\nabla}_{X+Y} Z = \tilde{\nabla}_X Z + \tilde{\nabla}_Y Z, \tag{4.2}$$

$$\tilde{\nabla}_{fX} Y = f \tilde{\nabla}_X Y \tag{4.3}$$

and

$$\tilde{\nabla}_X (fY) = f \tilde{\nabla}_X Y + (Xf) Y \tag{4.4}$$

for all $X, Y, Z \in T(M)$ and $f \in F(M)$, the set of all differentiable mappings over M. From (4.1), (4.2), (4.3) and (4.4) we can conclude that $\tilde{\nabla}$ determines a linear connection on (M, g).

Now we have

$$2g(\tilde{\nabla}_X Y, Z) - 2g(\tilde{\nabla}_Y X, Z) = 2g([X, Y], Z) + 2g(\eta(Y)\phi X - \eta(X)\phi Y, Z). \tag{4.5}$$

Hence,

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y$$

or,

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y.$$

Also we have

$$2g(\tilde{\nabla}_X Y, Z) + 2g(\tilde{\nabla}_X Z, Y) = 2Xg(Y, Z),$$

or,

$$(\tilde{\nabla}_X g)(Y, Z) = 0,$$

that is,

$$\tilde{\nabla} g = 0. \tag{4.6}$$

From (4.5) and (4.6) it follows that $\tilde{\nabla}$ determines a quarter-symmetric metric connection on (M, g). It can be easily verified that $\tilde{\nabla}$ determines a unique quarter-symmetric metric connection on (M, g). Thus we have the following:

Theorem 4.1. *Let M be a Riemannian manifold and η be a 1-form on it. Then there exists a unique linear connection $\tilde{\nabla}$ satisfying* (4.5) *and* (4.6).

Remark: The above theorem proves the existence of a quarter-symmetric metric connection on (M, g).

§5. Relation between the Riemannian connection and the quarter-symmetric metric connection

Let $\tilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an almost contact metric manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y),$$

where U is a tensor of type $(1,1)$. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have ([11])

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{5.1}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{5.2}$$

From (1.1) and (5.2) we get

$$T'(X,Y) = g(\phi Y, X)\xi - \eta(X)\phi Y \tag{5.3}$$

and using (1.1) and (5.3) in (5.2) we obtain

$$U(X,Y) = -\eta(X)\phi Y.$$

Hence a quarter-symmetric metric connection $\tilde{\nabla}$ on a 3-dimensional quasi-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{5.4}$$

Conversely, we show that a linear connection $\tilde{\nabla}$ on a 3-dimensional quasi-Sasakian manifold defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y, \tag{5.5}$$

denotes a quarter-symmetric metric connection.

Using (5.5) the torsion tensor of the connection $\tilde{\nabla}$ is given by

$$\begin{aligned} T(X,Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y. \end{aligned} \tag{5.6}$$

The above equation shows that the connection $\tilde{\nabla}$ is a quarter-symmetric connection ([11]). Also we have

$$\begin{aligned} (\tilde{\nabla}_X g)(Y,Z) &= Xg(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \\ &= \eta(X)[g(\phi Y, Z) + g(\phi Z, Y)] = 0. \end{aligned} \tag{5.7}$$

In virtue of (5.6) and (5.7) we conclude that $\tilde{\nabla}$ is a quarter-symmetric metric connection. Therefore equation (5.4) is the relation between the Riemannian connection and the quarter-symmetric connection on a 3-dimensional quasi-Sasakian manifold.

§6. Curvature tensor of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection

We define the curvature tensor of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \tag{6.1}$$

In view of (6.1) and (5.4) we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z - (\nabla_X\eta)(Y)\phi Z + (\nabla_Y\eta)(X)\phi Z\\ &\quad -\eta(Y)(\nabla_X\phi)Z + \eta(X)(\nabla_Y\phi)Z,\end{aligned}$$

which in view of (3.2) and (3.10) we get

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + 2\beta g(\phi X,Y)\phi Z - \beta\{\eta(Y)g(X,Z)\\ &\quad -\eta(X)g(Y,Z)\}\xi + \beta\{\eta(Y)X - \eta(X)Y\}\eta(Z).\end{aligned} \tag{6.2}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by the relation (6.2). So from (6.2) and (3.3) we have

$$\begin{aligned}\tilde{R}(X,\xi)Y &= R(X,\xi)Y - \beta\{g(X,Y) - \eta(X)\eta(Y)\}\xi\\ &\quad +\beta\{\eta(Y)X - \eta(X)Y\},\end{aligned}$$

and

$$\begin{aligned}\tilde{R}(X,Y)\xi &= \beta(\beta+1)\{\eta(Y)X - \eta(X)Y\}\\ &\quad +d\beta(Y)\phi X - d\beta(X)\phi Y.\end{aligned}$$

Taking inner product of (6.2) with W we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + 2\beta g(\phi X,Y)g(\phi Z,W)\\ &\quad -\beta\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(W)\\ &\quad +\beta\{\eta(Y)g(X,W) - \eta(X)g(Y,W)\}\eta(Z),\end{aligned} \tag{6.3}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y,Z),W)$.

From (6.3) clearly

$$\tilde{R}(X,Y,Z,W) = -\tilde{R}(Y,X,Z,W),$$

$$\tilde{R}(X,Y,Z,W) = -\tilde{R}(X,Y,W,Z).$$

Combining above two relations we have

$$\tilde{R}(X,Y,Z,W) = \tilde{R}(Y,X,W,Z).$$

We also have

$$\begin{aligned}\tilde{R}(X,Y)Z &+ \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y \\ &= 2\beta\{g(\phi X,Y)\phi Z + g(\phi Y,Z)\phi X + g(\phi Z,X)\phi Y\}.\end{aligned} \tag{6.4}$$

This is the first Bianchi identity for $\tilde{\nabla}$.

From (6.4) it is obvious that

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0 \quad if\ \beta = 0.$$

Hence we can state that if the manifold is cosymplectic then the curvature tensor with respect to the quarter-symmetric metric connection satisfies first Bianchi identity.

Contracting (6.3) over X and W, we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) - \beta g(Y,Z) + 3\beta\eta(Y)\eta(Z), \tag{6.5}$$

where $\tilde{S}$ and S are the Ricci tensors of the connection $\tilde{\nabla}$ and ∇ respectively. So in a 3-dimensional quasi-Sasakian manifold the Ricci tensor with respect to the quarter-symmetric metric connection is symmetric. Now, if $\beta = constant$, then using (3.8), the manifold is also an $\eta-$Einstein manifold with respect to the quarter-symmetric metric connection. Also if the manifold is an Einstein manifold then the manifold is an $\eta-$ Einstein manifold with respect to the quarter-symmetric metric connection.

Again contracting (6.5) we have $\tilde{r} = r$, where $\tilde{r}$ and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇ respectively. So we have the following:

Proposition 6.1. *For a* 3*-dimensional quasi-Sasakian manifold M with the quarter-symmetric metric connection* $\tilde{\nabla}$
(a) *The curvature tensor $\tilde{R}$ is given by* (6.3)*,*
(b) *The Ricci tensor $\tilde{S}$ is given by* (6.5)*,*
(c) *The first Bianchi identity is given by* (6.4)*,*
(d) $\tilde{r} = r$,
(e) *The Ricci tensor $\tilde{S}$ is symmetric,*
(f) *If M is Einstein or $\eta-$Einstein with respect to the Riemannian connection, then M is $\eta-$Einstein with respect to the quarter-symmetric metric connection.*

§7. Projective curvature tensor on a 3-dimensional quasi-Sasakian manifold

We define the generalized projective curvature tensor of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$

by ([16])

$$\begin{aligned}\tilde{P}(X,Y)Z &= \tilde{R}(X,Y)Z + \tfrac{1}{4}[\tilde{S}(X,Y)Z - \tilde{S}(Y,X)Z] \\ &\quad + \frac{1}{8}[\{3\tilde{S}(X,Z) + \tilde{S}(Z,X)\}Y \\ &\quad - \{3\tilde{S}(Y,Z) + \tilde{S}(Z,Y)\}X].\end{aligned}$$

Since the Ricci tensor $\tilde{S}$ of the manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ is symmetric, the projective curvature tensor $\tilde{P}$ reduces to

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{7.1}$$

Using (6.2) and (6.5), (7.1) reduces to

$$\begin{aligned}\tilde{P}(X,Y)Z &= P(X,Y)Z + \beta[2g(\phi X,Y)\phi Z - \{\eta(Y)g(X,Z) \\ &\quad - \eta(X)g(Y,Z)\}\xi + \frac{1}{2}\{g(Y,Z)X - g(X,Z)Y \\ &\quad - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\},\end{aligned} \tag{7.2}$$

where P is the projective curvature tensor defined by (1.2). From (7.2) we say that if the manifold is cosymplectic then the projective curvature tensor $\tilde{P}$ and the projective curvature tensor P are coincide.

$\xi-$conformally flat $K-$contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [32]. Analogous to the definition of $\xi-$conformally flat $K-$contact manifold we define the $\xi-$projectively flat 3-dimensional quasi-Sasakian manifold.

Definition 7.1. *A 3-dimensional quasi-Sasakian manifold M is called ξ-projectively flat if the condition $P(X,Y)\xi = 0$ holds on M.*

From (7.2) it is clear that $\tilde{P}(X,Y)\xi = P(X,Y)\xi$. So we have the following:

Theorem 7.1. *For a 3-dimensional quasi-Sasakian manifold, the Riemannian connection ∇ is $\xi-$projectively flat if and only if the quarter-symmetric metric connection $\tilde{\nabla}$ is so.*

Analogous to the definition of $\phi-$conformally flat contact manifold ([8]), we define $\phi-$projectively flat 3-dimensional quasi-Sasakian manifold.

Definition 7.2. *A 3-dimensional quasi-Sasakian manifold satisfying the condition*

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0$$

is called $\phi-$projectively flat ([23]).

Definition 7.3. *A Riemannian manifold M is said to be of constant curvature with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ if*

$$\tilde{R}(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}$$

where k is a constant.

Let us assume that M is a 3-dimensional $\phi-$projectively flat quasi-Sasakian manifold with respect to the quarter-symmetric metric connection. It can be easily seen that $\phi^2\tilde{P}(\phi X,\phi Y)\phi Z = 0$ holds if and only if

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W) = 0, \tag{7.3}$$

for $X,Y,Z,W \in T(M)$.

Using (7.1) and (7.3), $\phi-$projectively flat means

$$\begin{aligned} g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{2}\{\tilde{S}(\phi Y,\phi Z)g(\phi X,\phi W) \\ &\quad -\tilde{S}(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{7.4}$$

Let $\{e_1,e_2,\xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1,\phi e_2,\xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (7.4) and summing up with respect to i, we have

$$\begin{aligned} \textstyle\sum_{i=1}^{2} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \frac{1}{2}\sum_{i=1}^{2}\{\tilde{S}(\phi Y,\phi Z)g(\phi e_i,\phi e_i) \\ &\quad -\tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\}. \end{aligned} \tag{7.5}$$

Using (3.7), (6.2) and (6.5), it can be easily verified that

(7.6)
$$\begin{aligned} \textstyle\sum_{i=1}^{2} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \textstyle\sum_{i=1}^{2} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) - 2\beta g(\phi Y,\phi Z) \\ &= S(\phi Y,\phi Z) - \beta^2 g(\phi Y,\phi Z) - 2\beta g(\phi Y,\phi Z) \\ &= \tilde{S}(\phi Y,\phi Z) - \beta(\beta+1)g(\phi Y,\phi Z), \end{aligned}$$

$$\sum_{i=1}^{2} g(\phi e_i,\phi e_i) = 2, \tag{7.7}$$

$$\sum_{i=1}^{2} \tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = \tilde{S}(\phi Y,\phi Z). \tag{7.8}$$

So using (7.6), (7.7) and (7.8) the equation (7.5) becomes

$$\tilde{S}(\phi Y, \phi Z) = 2\beta(\beta + 1)g(\phi Y, \phi Z). \tag{7.9}$$

Putting $Y = \phi Y$ and $Z = \phi Z$ in (7.9) and using (2.1), (3.9) with $\beta = constant$, we get

$$\tilde{S}(Y, Z) = 2\beta(\beta + 1)g(Y, Z).$$

It is known ([31]) that a 3−dimensional Einstein manifold is a manifold of constant curvature. Also M is projectively flat if and only if it is of constant curvature ([29]). Now trivially, projectively flatness implies $\phi-$projectively flat. Hence we can state the following:

Theorem 7.2. *A 3-dimensional quasi-Sasakian manifold with constant structure function is $\phi-$projectively flat with respect to the quarter-symmetric metric connection if and only if the manifold is of constant curvature with respect to the quarter-symmetric metric connection.*

§8. Locally $\phi-$symmetric 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection

Definition 8.1 *A quasi-Sasakian manifold is said to be locally $\phi-$symmetric if*

$$\phi^2(\nabla_W R)(X, Y)Z = 0,$$

for all vector fields W, X, Y, Z orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [27].

Analogous to the definition of $\phi-$symmetric 3-dimensional quasi-Sasakian manifold with respect to the Riemannian connection, we define locally $\phi-$ symmetric 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric metric connection by

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X, Y)Z = 0,$$

for all vector fields W, X, Y, Z orthogonal to ξ. Using (5.5) we can write

$$(\tilde{\nabla}_W \tilde{R})(X, Y)Z = (\nabla_W \tilde{R})(X, Y)Z - \eta(W)\phi\tilde{R}(X, Y)Z. \tag{8.1}$$

Now differentiating (6.2) with respect to W, we obtain

$$\begin{aligned}(\nabla_W\tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z+\beta[2g(\phi X,Y)(\nabla_W\phi)Z\\ &-\{(\nabla_W\eta)(Y)g(X,Z)-(\nabla_W\eta)(X)g(Y,Z)\}\xi\\ &-\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\}(\nabla_W\xi)+\\ &(\nabla_W\eta)(Y)\eta(Z)X+(\nabla_W\eta)(Z)\eta(Y)X\\ &-(\nabla_W\eta)(X)\eta(Z)Y-(\nabla_W\eta)(Z)\eta(X)Y]\\ &+(W\beta)[2g(\phi X,Y)\phi Z-\eta(Y)g(X,Z)\xi\\ &+\eta(X)g(Y,Z)\xi+\eta(Y)\eta(Z)X\\ &-\eta(X)\eta(Z)Y].\end{aligned}$$

Using (3.1),(3.2) and (3.11) we have

$$\begin{aligned}(\nabla_W\tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z+\beta^2[2g(\phi X,Y)g(Z,W)\xi\\ &-2g(\phi X,Y)\eta(Z)W+g(\phi W,Y)g(X,Z)\xi\\ &-g(\phi W,X)g(Y,Z)\xi+\eta(Y)g(X,Z)\phi W\\ &-\eta(X)g(Y,Z)\phi W-g(\phi W,Y)\eta(Z)X\\ &-g(\phi W,Z)\eta(Y)X+g(\phi W,X)\eta(Z)Y\\ &+g(\phi W,Z)\eta(X)Y]+(W\beta)[2g(\phi X,Y)\phi Z\\ &-\eta(Y)g(X,Z)\xi+\eta(X)g(Y,Z)\xi\\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y].\end{aligned} \tag{8.2}$$

Using (8.2) and (2.1) from (8.1) we get

$$\begin{aligned}\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= \phi^2(\nabla_W R)(X,Y)Z+\beta^2[2g(\phi X,Y)\eta(Z)W\\ &-2g(\phi X,Y)\eta(Z)\eta(W)\xi-\eta(Y)g(X,Z)\phi W\\ &+\eta(X)g(Y,Z)\phi W+g(\phi W,Y)\eta(Z)X\\ &-g(\phi W,Y)\eta(Z)\eta(X)\xi+g(\phi W,Z)\eta(Y)X\\ &-g(\phi W,X)\eta(Z)Y+g(\phi W,X)\eta(Z)\eta(Y)\xi\\ &-g(\phi W,Z)\eta(X)Y]-(W\beta)\{2g(\phi X,Y)\phi Z\\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &-\eta(W)\phi^2(\phi\tilde{R})(X,Y)Z.\end{aligned} \tag{8.3}$$

If $\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z=\phi^2(\nabla_W R)(X,Y)Z$, then

$$\begin{aligned}\beta^2[2g(\phi X,Y)\eta(Z)W &- 2g(\phi X,Y)\eta(Z)\eta(W)\xi-\eta(Y)g(X,Z)\phi W\\ &+\eta(X)g(Y,Z)\phi W+g(\phi W,Y)\eta(Z)X\\ &-g(\phi W,Y)\eta(Z)\eta(X)\xi+g(\phi W,Z)\eta(Y)X\\ &-g(\phi W,X)\eta(Z)Y+g(\phi W,X)\eta(Z)\eta(Y)\xi\\ &-g(\phi W,Z)\eta(X)Y]-(W\beta)\{2g(\phi X,Y)\phi Z\\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &-\eta(W)\phi^2(\phi\tilde{R})(X,Y)Z\\ &= 0.\end{aligned} \tag{8.4}$$

Taking W, X, Y, Z orthogonal to ξ, (8.4) reduces to

$$(W\beta)g(\phi X, Y)\phi Z = 0,$$

which implies that

$$W\beta = 0 \ \ for\ all\ W.$$

Hence $\beta = constant$.

Conversely, if $\beta = constant$ and X, Y, Z, W orthogonal to ξ, then in view of (8.3) we obtain

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X, Y)Z = \phi^2(\nabla_W R)(X, Y)Z.$$

Hence we can state the following:

Theorem 8.1. *For a 3-dimensional non-cosymplectic quasi-Sasakian manifold, locally $\phi-$symmetry for the Riemannian connection ∇ and the quarter-symmetric metric connection $\tilde{\nabla}$ are coincide if and only if the structure function β=constant.*

Acknowledgement.

The authors are thankful to the referee for his comments and valuable suggestions towards the improvement of this paper.

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