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# On $\phi$-Ricci symmetric $N(k)$-contact metric manifolds

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**Abstract.** The object of the present paper is to study $\phi$-Ricci symmetric $N(k)$-contact metric manifolds and $\phi$-Ricci symmetric $N(k)$-contact metric manifolds of dimension three. It is proved that a $\phi$-Ricci symmetric $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is Sasakian. Next we prove that a three dimensional $N(k)$-contact metric manifold is locally $\phi$-Ricci symmetric if and only if the scalar curvature is constant. Finally we give examples of $\phi$-Ricci symmetric $N(k)$-contact metric manifolds.

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## §1. Introduction

In modern mathematics geometry of contact manifolds have become a matter of growing interest. An important class of contact manifolds consists of Sasakian manifolds. Again, through the works of Ch. Baikoussis, D. E. Blair and Th. Koufogiorgos [2] a new class of non-Sasakian contact manifolds has evolved. Such manifolds are known as $N(k)$-contact metric manifolds. However, among the geometric properties of contact metric manifold symmetry is an important one. Symmetry of a contact metric manifold has been studied in several ways by several authors. For instance, the notion of locally $\phi$-symmetric Sasakian manifolds has been introduced by T. Takahashi [12]. He studied several interesting properties of such a manifold in the context of Sasakian geometry. Recently U. C. De, A. A. Shaikh and Sudipta Biswas [8] introduced the notion of $\phi$-recurrent Sasakian manifolds which generalizes the notion of $\phi$-symmetric Sasakian manifolds. Also in another paper U. C. De and Aboul Kalam Gazi [7] introduced the notion of $\phi$-recurrent $N(k)$-contact metric manifolds. In a recent paper U. C. De and Avijit Sarkar [9] introduced the notion of $\phi$-Ricci symmetric Sasakian manifolds. From the definitions given in Section 3, it follows that every locally $\phi$-symmetric Sasakian manifold is locally $\phi$-Ricci symmetric. But, the converse is not in general true . Also a $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is Sasakian manifold if $k = 1$. Considering the above facts we generalize the notion of $\phi$-symmetric Sasakian manifolds and study $\phi$-Ricci symmetric $N(k)$-contact metric manifolds. In the present paper we study $\phi$-Ricci symmetric $N(k)$-contact metric manifolds. The paper is organized as follows: In section 2 we recall $N(k)$-contact metric manifolds. $\phi$-Ricci symmetric $N(k)$-contact metric manifolds have been studied in section 3. We prove that a $\phi$-Ricci symmetric $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is Sasakian. Also we prove that if a $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is an $\eta$-Einstein manifold with constant coefficients,then the manifold is locally $\phi$-Ricci symmetric. In section 4, we prove that a three-dimensional $N(k)$-contact metric manifold is locally $\phi$-Ricci symmetric if and only if the scalar curvature is constant. Finally we give examples of $\phi$-Ricci symmetric $N(k)$-contact metric manifolds.

## §2. $N(k)$- contact metric manifolds

An $(2n + 1)$-dimensional manifold $M^{2n+1}$ is said to admit an almost contact structure if it admits a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$(2.1)\qquad (a)\ \phi^2 = -I + \eta \otimes \xi,\ \ (b)\ \eta(\xi) = 1,\ \ (c)\ \phi\xi = 0,\ \ (d)\ \eta \circ \phi = 0.$$

An almost contact structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M^{2n+1} \times \mathbf{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbf{R}$ and $f$ is a smooth function on $M \times \mathbf{R}$. Let $g$ be a compatible Riemannian metric with almost contact structure $(\phi, \xi, \eta)$, that is,

$$(2.2)\qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then $M$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$. From (2.2) it can be easily seen that

$$(2.3)\qquad (a)\ \ g(X, \xi) = \eta(X), \qquad (b)\ \ g(X, \phi Y) = -g(\phi X, Y),$$

for all vector fields $X$, $Y$. An almost contact metric structure becomes a contact metric structure if

$$(2.4)\qquad g(X, \phi Y) = d\eta(X, Y),$$

for all vector fields $X$, $Y$. The 1-form $\eta$ is then a contact form and $\xi$ is its characteristic vector field. We define a $(1,1)$ tensor field $h$ by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the Lie-differentiation. Then $h$ is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.5}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X, \quad X, Y \in T_pM, \tag{2.6}$$

where $\nabla$ is Levi-Civita connection of the Riemannian metric $g$. A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ for which $\xi$ is a killing vector is said to be a $K$-contact manifold. A Sasakian manifold is $K$-contact but not conversely. However a 3-dimensional $K$-contact manifold is Sasakian ([10]). It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ ([3]). On the other hand, on a Sasakian manifold the following holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

The $k$-nullity distribution $N(k)$ of a Riemannian manifold $M^{2n+1}$ ([13]) is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

$k$ being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold ([5]). If $k = 1$, then $N(k)$-contact metric manifold is Sasakian and if $k = 0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $k < 1$, the scalar curvature is $r = 2n(2n - 2 + k)$.

In [2], $N(k)$-contact metric manifold were studied in some detail. For more details we refer to [6], [4].

In $N(k)$-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.8}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.9}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad +[2(1-n)+2nk]\eta(X)\eta(Y), \qquad n \geq 1, \end{aligned} \tag{2.12}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \tag{2.13}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{2.14}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{2.15}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{2.16}$$

**Lemma 2.1.** ([12]) *Let $M^{2n+1}$ be an $\eta-$Einstein manifold of dimension $(2n+1)$ $(n \geq 1)$. If $\xi$ belongs to the $k-$nullity distribution, then $k = 1$ and the structure is Sasakian.*

The above results will be used in the following sections.

## §3. $\phi$-Ricci symmetric $N(k)$-contact metric manifolds

**Definition 3.1.** *A $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be $\phi$-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2(\nabla_X Q)(Y) = 0,$$

*for all vector fields $X, Y \in \chi(M)$ and $S(X,Y) = g(QX,Y)$. In particular, if $X, Y$ are orthogonal to $\xi$ then the manifold is said to be locally $\phi$-Ricci symmetric.*

**Definition 3.2.** *If the Ricci tensor $S$ of the manifold $M^{2n+1}$ is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions on $M^{2n+1}$ , and $X, Y \in \chi(M)$, then the manifold is called an $\eta$-Einstein manifold.*

Let us suppose that the manifold is $\phi$-Ricci symmetric. Then by definition

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Using (2.1) we have from above

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{3.1}$$

or,

$$-g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0.$$

Then we obtain

$$-g(\nabla_X Q(Y) - Q(\nabla_X Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0,$$

which induces

$$-g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{3.2}$$

Putting $Y = \xi$ we get from (3.2)

$$-g(\nabla_X Q(\xi), Z) + S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{3.3}$$

In view of (2.11) and (2.5) it follows from (3.3)

$$\begin{aligned} 2nkg(\phi X, Z) + 2nkg(\phi hX, Z) - S(\phi X, Z) \\ -S(\phi hX, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \end{aligned} \tag{3.4}$$

Replacement of $Z$ by $\phi Z$ in (3.4) yields

$$2nkg(\phi X, \phi Z) + 2nkg(\phi hX, \phi Z) - S(\phi X, \phi Z) - S(\phi hX, \phi Z) = 0. \tag{3.5}$$

In a $N(k)$-contact metric manifold we have,

$$g(\phi X, \phi Z) = g(X, Z) - \eta(X)\eta(Z). \tag{3.6}$$

Replacing $X$ by $hX$ and multiplying both sides of (3.6) with $2nk$ we get

$$2nkg(\phi hX, \phi Z) = 2nkg(hX, Z) - 2nk\eta(hX)\eta(Z). \tag{3.7}$$

Again in a contact metric manifold we have

$$\eta(hX) = g(hX, \xi) = g(X, h\xi) = 0. \tag{3.8}$$

Using (3.8) in (3.7) we obtain

$$2nkg(\phi hX, \phi Z) = 2nkg(hX, Z). \tag{3.9}$$

Replacing $X$ by $hX$ and using (3.8) we obtain from (2.13)

$$S(\phi hX, \phi Y) = S(hX, Y) - 4(n-1)g(h^2 X, Y). \tag{3.10}$$

Using (2.1) and (2.8) in (3.10) we obtain,

$$\begin{aligned} S(\phi hX, \phi Y) &= S(hX, Y) + 4(n-1)(k-1)g(X, Y) \\ &\quad -4(n-1)(k-1)\eta(X)\eta(Y). \end{aligned} \tag{3.11}$$

Using (3.6), (3.9), (2.13) and (3.11) in (3.5) we get

$$\begin{aligned} S(X,Y)+S(hX,Y) &= [2nk-4(n-1)(k-1)]g(X,Y) \\ &\quad +[2nk+4(n-1)]g(hX,Y) \\ &\quad +4(n-1)(k-1)\eta(X)\eta(Y). \end{aligned} \tag{3.12}$$

Again from (2.12) we get

$$\begin{aligned} S(hX,Y) &= 2(n-1)g(hX,Y)-2(n-1)(k-1)g(X,Y) \\ &\quad +2(n-1)(k-1)\eta(X)\eta(Y). \end{aligned} \tag{3.13}$$

Using (3.13) in (3.12) we obtain

$$\begin{aligned} S(X,Y) &= 2(n+k-1)g(X,Y)+2(nk+n-1)g(hX,Y) \\ &\quad +2(n-1)(k-1)\eta(X)\eta(Y). \end{aligned} \tag{3.14}$$

Now comparing the value of $S(X,hY)$ from (2.12) and (3.14) we get

$$\begin{aligned} 2(n-1)g(X,hY)+2(n-1)g(X,h^2Y) &= 2(n+k-1)g(X,hY) \\ &\quad +2(nk+n-1)g(X,h^2Y) \\ &\quad +2(n-1)(k-1)\eta(X)\eta(hY). \end{aligned} \tag{3.15}$$

Using (2.1), (2.8) and (3.8) in (3.16) we obtain

$$g(X,hY)=g(hX,Y)=n(k-1)g(X,Y)-n(k-1)\eta(X)\eta(Y). \tag{3.16}$$

Using (3.16) in (3.15) we get

$$S(X,Y)=Ag(X,Y)+B\eta(X)\eta(Y), \tag{3.17}$$

where $A=2[(n+k-1)+n(k-1)(nk+n-1)]$ and $B=2(k-1)[(n-1)-n(nk+n-1)]$.

Hence the manifold is an $\eta$-Einstein manifold. Thus we can state the following:

**Proposition 3.1.** *A $(2n+1)$-dimensional $\phi$-Ricci symmetric $N(k)$-contact metric manifold is an $\eta$-Einstein manifold.*

If $k=1$, then the manifold reduces to a Sasakian manifold. Therefore from Proposition 3.1 for $k=1$ we can state the following:

**Proposition 3.2.** *A $(2n+1)$-dimensional $\phi$-Ricci symmetric Sasakian manifold is an Einstein manifold.*

The above Proposition 3.2 have been proved by De and Sarkar [9].

Now from Lemma 1 and Proposition 3.1 we get the following:

**Theorem 3.1.** *A $\phi$-Ricci symmetric $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is Sasakian.*

**Definition 3.3.** ([12]) *A Sasakian manifold is said to be a locally $\phi$-symmetric manifold if*

$$\phi^2\left((\nabla_W R)(X,Y)Z\right) = 0,$$

*for all vector fields $X, Y, Z, W$ orthogonal to $\xi$. If $X, Y, Z, W$ are not orthogonal to $\xi$, then the manifold is called $\phi$-symmetric.*

**Remark**: *Since $\phi$-symmetry implies $\phi$-Ricci symmetry, therefore the above Propositions and Theorem* 3.1 *also hold for $\phi$-symmetric $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$.*

Next suppose that the $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is an $\eta$-Einstein manifold with constant coefficients. Then

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.18}$$

where $S(X,Y) = g(QX,Y)$ and $a$ and $b$ are constants $M^{2n+1}$. Hence

$$QX = aX + b\eta(X)\xi. \tag{3.19}$$

Using (3.19) we get

$$(\nabla_X Q)(Y) = b(\nabla_X \eta)(Y)\xi + b\eta(Y)(\nabla_X \xi). \tag{3.20}$$

Applying $\phi^2$ on both sides of (3.20) and using (2.1)($c$) we get

$$\phi^2((\nabla_X Q)(Y)) = b\eta(Y)\phi^2(\nabla_X \xi). \tag{3.21}$$

If Y is orthogonal to $\xi$, then (3.21) yields

$$\phi^2((\nabla_X Q)(Y)) = 0.$$

Hence the manifold is locally $\phi$-Ricci symmetric. This helps us to conclude the following:

**Theorem 3.2.** *If a $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ is an $\eta$-Einstein manifold with constant coefficients, then the manifold is locally $\phi$-Ricci symmetric.*

## §4. Three-dimensional $\phi$-Ricci symmetric $N(k)$-contact metric manifolds

In a 3-dimensional Riemannian manifold we have ([14])

$$(4.1)\quad \begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad + \tfrac{r}{2}[g(X,Z)Y - g(Y,Z)X], \end{aligned}$$

where $Q$ is the Ricci-operator, that is, $g(QX,Y) = S(X,Y)$ and $r$ is the scalar curvature of the manifold. Now putting $Z = \xi$ in (4.1) and using (2.11), we get

$$(4.2)\quad \begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY + 2k[\eta(Y)X - \eta(X)Y] \\ &\quad + \tfrac{r}{2}[\eta(X)Y - \eta(Y)X]. \end{aligned}$$

Using (2.15) in (4.2), we have

$$(4.3)\quad (k - \frac{r}{2})[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX.$$

Putting $Y = \xi$ in (4.3) and using (2.1)($b$), we obtain

$$(4.4)\quad QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi.$$

Differentiating (4.4) covariantly with respect to $W$ we obtain

$$(4.5)\quad \begin{aligned} (\nabla_W Q)(X) &= \tfrac{1}{2}\{dr(W)X + (6k - r)(\nabla_W \eta)(X)\xi \\ &\quad + (6k - r)\eta(X)(\nabla_W \xi) - dr(W)\eta(X)\xi\}. \end{aligned}$$

Applying $\phi^2$ on both sides of above and using (2.1) we have

$$(4.6)\quad \phi^2((\nabla_W Q)(X)) = \frac{1}{2}\{dr(W)(-X + \eta(X)\xi) + (6k - r)\eta(X)\phi^2(\nabla_W \xi)\}.$$

Hence (4.5) yields the following:

**Proposition 4.1.** *If the scalar curvature $r$ of a three-dimensional $N(k)$-contact metric manifold equal to $6k$ then the manifold is $\phi$-Ricci symmetric.*

If we take the vector field $X$ orthogonal to $\xi$, then (4.6) gives

$$\phi^2((\nabla_W Q)(X)) = -\frac{1}{2}dr(W)X.$$

Thus we are in a position to conclude the following:

**Theorem 4.1.** *A three-dimensional $N(k)$-contact metric manifold is locally $\phi$-Ricci symmetric if and only if the scalar curvature is constant.*

## §5. Examples

**Example 5.1.** ([1]): Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with $\xi$ belonging to the $k$-nullity distribution. If the curvature tensor is harmonic,then for $k \neq 0$, the manifold $M^{2n+1}$is an Einstein Sasakian manifold. Hence a $N(k)$-contact metric manifold with $k \neq 0$ whose curvature tensor is harmonic is $\phi$-Ricci symmetric which is not $\phi$-symmetric.

**Example 5.2.** ([11]): Ricci symmetric ($\nabla S = 0$) $N(k)$-contact metric manifold with $k \neq 0$ is an Einstein manifold. Hence a Ricci symmetric $N(k)$-contact metric manifold with $k \neq 0$ is $\phi$-Ricci symmetric which is not $\phi$-symmetric.

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