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Statistical inference for parallelism hypothesis in growth curve model

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Abstract. Let $\boldsymbol{y} = (y_1, ..., y_p)'$ be a p-dimensional random vector measurable on the individuals drawn from each of k p-dimensional normal populations Π_i : $N_p(\boldsymbol{\mu}_i, \Sigma)$, $i = 1, \ldots, k$. In this paper we consider the growth curve model which has a mean structure as follows: $\boldsymbol{\mu}_i = X\boldsymbol{\theta}_i$, $i = 1, \ldots, k$, where X is a $p \times q$ given matrix with rank q and $\boldsymbol{\theta}_i$'s are unknown parameter vectors. First we derive an LR test for a parallelism hypothesis $H_1 : X\boldsymbol{\theta}_i - X\boldsymbol{\theta}_k = \gamma_i \mathbf{1}_p$, $i = 1, \ldots, k-1$, where γ_i's are unknown parameters, and $\mathbf{1}_p$ is the p-dimensional vector with all the elements 1. Next we obtain the MLE of $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{k-1})'$ and its distribution, and propose a simultaneous confidence interval for linear combinations of $\boldsymbol{\gamma}$.

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§1. Introduction

Suppose that a variable y is measured at p time points $t_1, t_2, \ldots, t_p$, and let the variable y measured at the t_i time point be denoted by y_i. Further, suppose that there are random samples of $\boldsymbol{y} = (y_1, \ldots, y_p)'$ for each of k different groups Π_i, $i = 1, \ldots, k$, and let the random samples be denoted by

$$\Pi_i : \ \boldsymbol{y}_{i1}, \ldots, \boldsymbol{y}_{in_i}, \tag{1.1}$$

which are independently distributed as $N_p(\boldsymbol{\mu}_i, \Sigma)$. For the observations, we assume the growth curve model which is described (see e.g. Potthoff and Roy (1964)) by

$$\boldsymbol{\mu}_i = X\boldsymbol{\theta}_i, \ i = 1, \ldots, k, \tag{1.2}$$

where X is a $p \times q$ given matrix with rank q and $\boldsymbol{\theta}_i$'s are unknown parameter vectors.

The purpose of this paper is to extend profile analysis, especially statistical inference on a parallelism hypothesis which is expressed as

$$(1.3) \qquad H_1 : \; X\boldsymbol{\theta}_i - X\boldsymbol{\theta}_k = \gamma_i \mathbf{1}_p, \quad i = 1, \ldots, k-1,$$

where γ_i's are unknown parameters, and $\mathbf{1}_p$ is the p-dimensional vector with all the elements 1. The profile analysis in the usual MANOVA model with $X = I_p$ has been studied by Greenhouse and Geisser (1959), Srivastava (1987), etc. Srivastava (1987) obtained the likelihood ratio (LR) criterion, and proposed a simultaneous confidence interval for linear combinations of $\boldsymbol{\gamma}$, based on an LR test for $\boldsymbol{\gamma} = \mathbf{0}$.

It may be noted that the parallelism hypothesis is assured if and only if $\mathbf{1}_p \in \mathcal{R}[X]$. Further, considering a practical point of view it is assumed that

C1: The first column of X is $\mathbf{1}_p$, i.e., $X = (\mathbf{1}_p, X_2)$.

Then, it is shown that the parallelism hypothesis is equivalent to

$$(1.4) \qquad H_1 \quad \Leftrightarrow \quad \boldsymbol{\theta}_i = \boldsymbol{\theta}_k + \gamma_i(1, 0, \ldots, 0)', \; i = 1, \ldots, k-1,$$

$$(1.5) \qquad \Leftrightarrow \quad \boldsymbol{\theta}_{12} = \cdots = \boldsymbol{\theta}_{k2},$$

where

$$\boldsymbol{\theta}_i = \begin{pmatrix} \theta_{i1} \\ \boldsymbol{\theta}_{i2} \end{pmatrix}, \quad \boldsymbol{\theta}_{i2} : (q-1) \times 1, \quad i = 1, \ldots, k.$$

In this paper we note that an LR test for the parallelism hypothesis is obtained by using a general theory of testing a general linear hypothesis in growth curve model. Further, we give a direct derivation based on a canonical form. The canonical form is also used for deriving the MLE of $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{k-1})'$ and its distribution. We propose a simultaneous confidence interval for linear combinations of $\boldsymbol{\gamma}$.

§2. LR Test for Parallelism Hypothesis

Let all the observations in (1.1) be denoted by

$$Y = (\boldsymbol{y}_{11}, \ldots, \boldsymbol{y}_{1n_1}, \boldsymbol{y}_{21}, \ldots, \boldsymbol{y}_{2n_2}, \ldots, \boldsymbol{y}_{k1}, \ldots, \boldsymbol{y}_{kn_k})'.$$

Then the growth curve model in (1.2) is

$$(2.1) \qquad M : \; \mathrm{E}(Y) = A\Theta X',$$

and the rows of Y are independently distributed as p-variate normal distributions with the same covariance matrix Σ, where $\Theta = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_k)'$, and A is the design matrix between individuals given by

$$A = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{n_k} \end{pmatrix}. \tag{2.2}$$

We have noted that the parallelism hypothesis H can be expressed as (1.4) or (1.5). This is shown as follows. Multiplying both sides of (1.3) by $(X'X)^{-1}X'$ from the left-hand side, we have

$$\boldsymbol{\theta}_i = \boldsymbol{\theta}_k + \gamma_i \tilde{\mathbf{1}}_q, \; i = 1, \ldots, k-1,$$

where $\tilde{\mathbf{1}}_q = (X'X)^{-1}X'\mathbf{1}_p$. Moreover, from the assumption C1 it holds that

$$\tilde{\mathbf{1}}_q = (X'X)^{-1}X'\mathbf{1}_p = (1, 0, \ldots, 0)',$$

since $(X'X)^{-1}X'X = I_q$ and the first column of X is $\mathbf{1}_p$. The converse is obtained, by multiplying the above equality by X from the left-hand side and using $P_X\mathbf{1}_p = \mathbf{1}_p$, where $P_X = X(X'X)^{-1}X'$. From (1.5) we can express the parallelism hypothesis as

$$C\Theta D = O, \tag{2.3}$$

where

$$C = (I_{k-1}, -\mathbf{1}_{k-1}), \quad D = \begin{pmatrix} \mathbf{0}' \\ I_{q-1} \end{pmatrix}. \tag{2.4}$$

Therefore, by using a result (see e.g. Kshirsagar and Smith (1995)) on the test of a general linear hypothesis we have the following results.

Theorem 2.1. *An LR test for H_1 in* (1.3) *under the growth curve model* (1.2) *satisfying condition* C1 *is based on*

$$\Lambda = \frac{|S_e|}{|S_e + S_h|}, \tag{2.5}$$

where

$$S_e = D'(X'S^{-1}X)^{-1}D, \quad S_h = (C\hat{\Theta}D)'(CRC')^{-1}C\hat{\Theta}D, \tag{2.6}$$

and $n = n_1 + \cdots + n_k$, $\bar{\boldsymbol{y}}_i = (1/n_i)\sum_{j=1}^{n_i} \boldsymbol{y}_{ij}$, *i=1, ..., k,*

$$
\begin{aligned}
S &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\boldsymbol{y}_{ij} - \bar{\boldsymbol{y}}_i)(\boldsymbol{y}_{ij} - \bar{\boldsymbol{y}}_i)', \\
\hat{\Theta} &= (A'A)^{-1}A'YS^{-1}X(X'SX)^{-1}, \qquad (2.7)\\
R &= (A'A)^{-1} + (A'A)^{-1}A'YS^{-1}\{S - X(X'S^{-1}X)^{-1}X'\} \\
&\quad \times S^{-1}Y'A(A'A)^{-1}.
\end{aligned}
$$

The null distribution of Λ *is a lambda distribution* $\Lambda_{q-1}(k-1, n-k-(p-q))$.

§3. A Canonical Form

The growth model (2.1) satisfying the parallelism hypothesis H_1 is expressed as

$$
M_1: \ \mathrm{E}(Y) = \mathbf{1}_n\boldsymbol{\theta}_k'X' + A_1\boldsymbol{\gamma}\mathbf{1}_p'. \qquad (3.1)
$$

where A_1 is a submatrix composed from the first $k-1$ columns of A. In order to obtain a canonical form, consider a transformation $Z = H'YB$ with an orthogonal matrix $H = (\boldsymbol{h}_1, H_2, H_3)$ and an orthogonal matrix $B = (\boldsymbol{b}_1, B_2, B_3)$, i.e.,

$$
\begin{aligned}
Z &= (\boldsymbol{h}_1, H_2, H_3)'Y(\boldsymbol{b}_1, B_2, B_3) \\
&= \begin{pmatrix} z_{11} & \boldsymbol{z}_{12}' & \boldsymbol{z}_{13}' \\ \boldsymbol{z}_{21} & Z_{22} & Z_{23} \\ \boldsymbol{z}_{31} & Z_{32} & Z_{33} \end{pmatrix}. \qquad (3.2)
\end{aligned}
$$

The orthogonal matrix H is defined as follows. We define $\boldsymbol{h}_1$ as $(1/\sqrt{n})\mathbf{1}_n$. The column vectors of H_2 consist of orthogonal bases for the space $\mathcal{R}[\mathbf{1}_n]^{\perp} \cap \mathcal{R}[A_1]$, and let H_2 be defined by $H_2 = (I_n - P_n)A_1\{A_1'(I_n - P_n)A_1\}^{-1/2}$, where $P_n = (1/n)\mathbf{1}_n\mathbf{1}_n'$. The column vectors of H_3 consist of orthogonal bases for $\mathcal{R}[A]^{\perp}$, and we may use a matrix satisfying $H_3H_3' = I_n - A(A'A)^{-1}A'$. Similarly the column vectors of B are defined by

$$
\boldsymbol{b}_1 = (1/\sqrt{p})\mathbf{1}_p, \quad B_2 = (I_p - P_p)X_2\{X_2'(I_p - P_p)X_2\}^{-1/2}
$$

and B_3 satisfying $B_3B_3' = I_p - X(X'X)^{-1}X'$. Then, the mean of Z under (2.1) is

$$
\mathrm{E}(Z) = \begin{pmatrix} \xi_{11} & \boldsymbol{\xi}_{12}' & \mathbf{0}' \\ \boldsymbol{\xi}_{21} & \Xi_{22} & O \\ \mathbf{0} & O & O \end{pmatrix}, \qquad (3.3)
$$

where

$$\Xi \equiv \begin{pmatrix} \xi_{11} & \boldsymbol{\xi}_{12}' \\ \boldsymbol{\xi}_{21} & \Xi_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}_1' \\ H_2' \end{pmatrix} A\Theta X'(\boldsymbol{b}_1, B_2).$$

The Ξ under the parallelism model (3.1) is

$$\begin{pmatrix} \xi_{11} & \boldsymbol{\xi}_{12}' \\ \boldsymbol{\xi}_{21} & \Xi_{22} \end{pmatrix} = \begin{pmatrix} \nu_1 & \boldsymbol{\nu}_2' \\ \boldsymbol{\delta} & O \end{pmatrix}, \tag{3.4}$$

where

$$\begin{aligned} \boldsymbol{\nu} &= (\nu_1, \boldsymbol{\nu}_2')' \\ &= (\boldsymbol{b}_1, B_2)'\{\sqrt{n}X\boldsymbol{\theta}_k + n^{-1/2}(n_1\gamma_1 + \cdots + n_{k-1}\gamma_{k-1})\mathbf{1}_p\}, \\ \boldsymbol{\delta} &= \sqrt{p}\{A_1'(I_n - P_0)A_1\}^{1/2}\boldsymbol{\gamma}. \end{aligned} \tag{3.5}$$

The rows of Z are independently normal, and have the same covariance matrix

$$\begin{aligned} \Psi &= (\boldsymbol{b}_1, B_2, B_3)'\Sigma(\boldsymbol{b}_1, B_1, B_3) \\ &= \begin{pmatrix} \psi_{11} & \boldsymbol{\psi}_{21}' & \boldsymbol{\psi}_{31}' \\ \boldsymbol{\psi}_{21} & \Psi_{22} & \Psi_{23} \\ \boldsymbol{\psi}_{31} & \Psi_{32} & \Psi_{33} \end{pmatrix}. \end{aligned}$$

As a matter of course, the resultant canonical form (3.3) for testing the parallelism hypothesis under the model (3.4) is essentially the same as that of the canonical form (Gleser and Olkin (1970)) for testing a general linear hypothesis under the growth curve model. However, it may be noted that in our canonical form the parameter vector $\boldsymbol{\gamma}$ under the parallelism model (3.1) is simply expressed as

$$\boldsymbol{\gamma} = (1/\sqrt{p})Q^{1/2}\boldsymbol{\delta}, \tag{3.6}$$

where

$$\begin{aligned} Q &\equiv \{A_1'(I_n - P_0)A_1\}^{-1} \\ &= \{\mathrm{diag}(n_1, \ldots, n_{k-1}) - \frac{1}{n}(n_1, \ldots, n_{k-1})'(n_1, \ldots, n_{k-1})\}^{-1} \\ &= \mathrm{diag}(\frac{1}{n_1}, \ldots, \frac{1}{n_{k-1}}) + \frac{1}{n_k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}'. \end{aligned} \tag{3.7}$$

§4. LR test and MLE in Canonical Form

The LR test for testing $\Xi_{22} = O$ under (3.3) can be obtained by using a general result (see e.g. Gleser and Olkin (1970), Fujikoshi et al. (1999), etc.) on the test of a general linear hypothesis under the growth curve model.

However, as we wish to derive an explicit expression for the MLE of $\boldsymbol{\gamma}$, we give a derivation for the LR test as well as the MLE.

Let the likelihood $L_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi)$ of Z under the parallelism model (3.1). Then

$$\begin{aligned} g_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) \equiv & -2\log L_1(\boldsymbol{\nu}, \boldsymbol{\delta}, \Psi) = n\log|\Psi| + np\log 2\pi \\ & +\mathrm{tr}\Psi^{-1}\left[(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13})'(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13})\right. \\ & \left.+(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)})'(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)}) + W\right], \end{aligned}$$

where $\boldsymbol{z}'_{1(12)} = (\boldsymbol{z}_{11}, \boldsymbol{z}'_{12})$, $Z_{2(23)} = (\boldsymbol{z}_{22}, Z_{23})$,

$$W = (\boldsymbol{z}_{31}, Z_{32}, Z_{33})'(\boldsymbol{z}_{31}, Z_{32}, Z_{33}) = \begin{pmatrix} w_{11} & \boldsymbol{w}'_{21} & W'_{31} \\ \boldsymbol{w}_{21} & W_{22} & W_{23} \\ \boldsymbol{w}_{31} & W_{32} & W_{33} \end{pmatrix}. \tag{4.1}$$

Similar notations are used for partition matrices of Ψ. We also use the following notations.

$$\Psi_{(12)(12)\cdot 3} = \Psi_{(12)(12)} - \Psi_{(12)3}\Psi_{33}^{-1}\Psi_{3(12)}, \text{ etc.}$$

The following formulas are used in our derivation.

$$\begin{aligned} & |\Psi| = \psi_{11\cdot 23}\cdot|\Psi_{(23)(23)}| = \psi_{11\cdot 23}\cdot|\Psi_{22\cdot 33}|\cdot|\Psi_{33}|, \\ & \mathrm{tr}\Psi^{-1}(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13})'(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}', \boldsymbol{z}'_{13}) = \mathrm{tr}\Psi_{33}^{-1}\boldsymbol{z}_{13}\boldsymbol{z}'_{13} \\ & \qquad +\mathrm{tr}\Psi_{(12)(12)\cdot 3}^{-1}(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}' - \boldsymbol{z}'_{13}\mathcal{C})'(\boldsymbol{z}'_{1(12)} - \boldsymbol{\nu}' - \boldsymbol{z}'_{13}\mathcal{C}), \\ & \mathrm{tr}\Psi^{-1}(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)})'(\boldsymbol{z}_{21} - \boldsymbol{\delta}, Z_{2(23)}) = \mathrm{tr}\Psi_{(23)(23)}^{-1}Z'_{2(23)}Z_{2(23)} \\ & \qquad +\psi_{11\cdot 23}^{-1}(\boldsymbol{z}_{21} - \boldsymbol{\delta} - Z_{2(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{21} - \boldsymbol{\delta} - Z_{2(23)}\boldsymbol{\eta}), \\ & \mathrm{tr}\Psi^{-1}W = \mathrm{tr}\Psi_{(23)(23)}^{-1}W_{(23)(23)} + \psi_{11\cdot 23}^{-1}(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta}), \end{aligned}$$

where $\mathcal{C} = \Psi_{33}^{-1}\Psi_{3(12)}$ and $\boldsymbol{\eta} = \Psi_{(23)(23)}^{-1}\boldsymbol{\psi}_{(23)1}$.

Note that there is one-to-one correspondence between Ψ and $\{\Psi_{(23)(23)}, \psi_{11\cdot 23}, \boldsymbol{\eta}\}$. Similarly there is one-to-one correspondence between $\Psi_{(23)(23)}$ and $\{\Psi_{33}, \Psi_{22\cdot 3}, \mathcal{B}\}$, where $\mathcal{B} = \Psi_{33}^{-1}\Psi_{32}$. It is easy to see that the MLE's of $\boldsymbol{\delta}$ and $\boldsymbol{\nu}$ are given by

$$\hat{\boldsymbol{\delta}} = \boldsymbol{z}_{21} - Z_{2(23)}\hat{\boldsymbol{\eta}}, \quad \hat{\boldsymbol{\nu}} = \boldsymbol{z}_{1(12)} - \hat{\mathcal{C}}'\boldsymbol{z}_{13} \tag{4.2}$$

and

$$\hat{\boldsymbol{\eta}} = (Z'_{3(23)}Z_{3(23)})^{-1}Z'_{3(23)}\boldsymbol{z}_{31} = W_{(23)(23)}^{-1}\boldsymbol{w}_{(23)1}. \tag{4.3}$$

These imply that

$$\begin{aligned}\min_{\boldsymbol{\nu},\boldsymbol{\delta},\Psi} g_1(\boldsymbol{\nu},\boldsymbol{\delta},\Psi) = &\min_{\psi_{11\cdot 23},\Psi_{(23)(23)}} \big[n\log\{\psi_{11\cdot 23}\cdot|\Psi_{(23)(23)}|\} \\ &+\psi_{11\cdot 23}^{-1}w_{11\cdot 23} + np\log 2\pi + \mathrm{tr}\Psi_{33}^{-1}\boldsymbol{z}_{13}\boldsymbol{z}_{13}' \\ &+\mathrm{tr}\Psi_{(23)(23)}^{-1}\left\{W_{(23)(23)} + Z_{2(23)}'Z_{2(23)}\right\}\big]. \end{aligned} \tag{4.4}$$

Here we use

$$\begin{aligned}\min_{\eta}(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta})'(\boldsymbol{z}_{31} - Z_{3(23)}\boldsymbol{\eta}) &= \boldsymbol{z}_{31}'(I_{n-k} - P_{Z_{3(23)}})\boldsymbol{z}_{31} \\ &= w_{11\cdot 3} - \boldsymbol{w}_{1(23)}'W_{(23)(23)}^{-1}\boldsymbol{w}_{1(23)} = w_{11\cdot 23}.\end{aligned}$$

Let

$$\begin{aligned} T &= W + (\boldsymbol{z}_{21}, Z_{22}, Z_{23})'(\boldsymbol{z}_{21}, Z_{22}, Z_{23}) \\ &= \begin{pmatrix} t_{11} & \boldsymbol{t}_{21}' & \boldsymbol{t}_{31}' \\ \boldsymbol{t}_{21} & T_{22} & T_{23} \\ \boldsymbol{t}_{31} & T_{32} & T_{33} \end{pmatrix}. \end{aligned} \tag{4.5}$$

Then, we have

$$\begin{aligned}\mathrm{tr}\Psi_{(23)(23)}^{-1}T_{(23)(23)} &= \Psi_{33}^{-1}T_{33} \\ &+\mathrm{tr}\Psi_{22\cdot 3}^{-1}\left\{(T_{33}^{-1}T_{32} - \mathcal{B})'T_{33}(T_{33}^{-1}T_{32} - \mathcal{B}) + T_{22\cdot 3}\right\},\end{aligned} \tag{4.6}$$

where $\mathcal{B} = \Psi_{33}^{-1}\Psi_{32}$. Substituting (4.6) to (4.4),

$$\begin{aligned}\min_{\boldsymbol{\nu},\boldsymbol{\delta},\Psi} g_1(\boldsymbol{\nu},\boldsymbol{\delta},\Psi) = &\min_{\psi_{11\cdot 23},\Psi_{22\cdot 3},\Psi_{33}} [n\log\{\psi_{11\cdot 23}\cdot|\Psi_{22\cdot 3}|\cdot|\Psi_{33}|\} \\ &+np\log 2\pi + \psi_{11\cdot 23}^{-1}w_{11\cdot 23} + \mathrm{tr}\Psi_{33}^{-1}(T_{33} + \boldsymbol{z}_{13}\boldsymbol{z}_{13}') + \mathrm{tr}\Psi_{22\cdot 3}^{-1}T_{22\cdot 3}] \\ &= n\log\{\hat{\psi}_{11\cdot 23}^{(\omega)}\cdot|\hat{\Psi}_{22\cdot 3}^{(\omega)}|\cdot|\hat{\Psi}_{33}^{(\omega)}|\} + np(\log 2\pi + 1),\end{aligned} \tag{4.7}$$

where

$$n\hat{\psi}_{11\cdot 23}^{(\omega)} = w_{11\cdot 23},\ n\hat{\Psi}_{22\cdot 3}^{(\omega)} = T_{22\cdot 3},\ n\hat{\Psi}_{33}^{(\omega)} = T_{33} + \boldsymbol{z}_{13}\boldsymbol{z}_{13}'. \tag{4.8}$$

Let $L(\Xi, \Psi)$ be the likelihood function of Z under (3.3). Then

$$\begin{aligned} g(\Xi,\Psi) \equiv\ & -2\log L(\Xi,\Psi) = n\log|\Psi| + np\log 2\pi \\ &+\mathrm{tr}\Psi^{-1}\left[(Z_{(12)(12)} - \Xi, Z_{(12)3})'(Z_{(12)(12)} - \Xi, Z_{(12)3}) + W\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\min_{\Xi,\Psi} g(\Xi,\Psi) = &\min_{\Psi_{(12)(12)\cdot 3},\Psi_{33}} \big[n\log\left\{|\Psi_{(12)(12)\cdot 3}|\cdot|\Psi_{33}|\right\} + np\log 2\pi \\ &+\mathrm{tr}\Psi_{(12)(12)\cdot 3}^{-1}W_{(12)(12)\cdot 3} + \mathrm{tr}\Psi_{33}^{-1}(W_{33} + Z_{(12)3}'Z_{(12)3})\big] \\ &= n\log\left\{|\Psi_{(12)(12)\cdot 3}^{(\Omega)}|\cdot|\Psi_{33}^{(\Omega)}|\right\} + np(\log 2\pi + 1),\end{aligned} \tag{4.9}$$

where

$$n\hat{\Psi}^{(\Omega)}_{(12)(12)\cdot 3} = W_{(12)(12)\cdot 3},\ n\hat{\Psi}^{(\Omega)}_{33} = W_{33} + Z'_{(12)3}Z_{(12)3} = n\hat{\Psi}^{(\omega)}_{33}. \tag{4.10}$$

From (4.7) and (4.9) we have the following results.

Theorem 4.1. *The LR criterion λ for H_1 in* (1.3) *under the growth curve model* (1.2) *satisfying condition* C1 *is given by*

$$\begin{aligned} \lambda^{2/n} &= \frac{|W_{(12)(12)\cdot 3}| \cdot |\hat{\Psi}^{(\Omega)}_{33}|}{w_{11\cdot 23} \cdot |T_{23\cdot}| \cdot |\hat{\Psi}^{(\omega)}_{33}|} \\ &= \frac{|W_{22\cdot 3}|}{|T_{22\cdot 3}|}. \end{aligned} \tag{4.11}$$

The null distribution of $\lambda^{2/n}$ is a lambda distribution $\Lambda_{q-1}(k-1, n-k-(p-q))$.

Proof The distribution result follows from Theorem 2.1, but here we give a direct proof. In order to obtain the null distribution of $\lambda^{2/n}$, we note that

(1) $T_{(23)(23)} = W_{(23)(23)} + Z'_{2(23)}Z'_{2(23)}$.

(2) $W_{(23)(23)}$ and $Z'_{2(23)}Z_{2(23)}$ are independently distributed as Whishart distributions $W_{p-1}(n-k, \Psi_{(23)(23)})$ and $W_{p-1}(k-1, \Psi_{(23)(23)})$, respectively.

Then, using a distributional result (see e.g. Rao (1973), Fujikoshi (1981), etc.) that

$$\frac{|W_{22\cdot 3}|}{|T_{22\cdot 3}|} \sim \Lambda_{q-1}(k-1, n-k-(p-q)).$$

In the process of deriving the distributional result Fujikoshi (1981) has shown that

$$T_{22\cdot 3} = W_{22\cdot 3} + V, \tag{4.12}$$

where

$$V = (Z_{22} - Z_{23}W_{33}^{-1}W_{32})'(I_{k-1} + Z_{23}W_{33}^{-1}Z'_{23})^{-1}(Z_{22} - Z_{23}W_{33}^{-1}W_{32}).$$

The result (4.12) is useful in showing that the two expressions (2.5) and (4.12) are the same. In fact, we can show the following relationships which implies the conclusion.

Lemma 4.1. *It holds that*

$$\begin{aligned} S_e &= \{X'_2(I_p - P_p)X_2\}^{-1/2}W_{22\cdot 3}\{X'_2(I_p - P_p)X_2\}^{-1/2}, \\ S_h &= \{X'_2(I_p - P_p)X_2\}^{-1/2}V\{X'_2(I_p - P_p)X_2\}^{-1/2}. \end{aligned} \tag{4.13}$$

Proof The first equality of (4.13) follows that

$$\begin{aligned} W_{22\cdot 3} &= B_2'\{S - SB_3(B_3'SB_3)^{-1}B_3S\}B_2 \\ &= B_2'X(X'S^{-1}X)^{-1}X'B_2 \end{aligned}$$

and

$$\begin{aligned} B_2'X &= \{X_2'(I_p - P_p)X_2\}^{-1/2}X_2(I_p - P_p)(\mathbf{1}_p, X_2) \\ &= (\mathbf{0}, \{X_2'(I_p - P_p)X_2\}^{1/2}). \end{aligned}$$

Here, we use a well known formula: Let $G = (G_1\ G_2)$ be a $p \times p$ nonsingular matrix such that $G_1'G_2 = O$. Then, for a $p \times p$ positive definite matrix Q,

$$G_2(G_2'QG_2)^{-1}G_2' = Q^{-1} - Q^{-1}G_1(G_1'Q^{-1}G_1)^{-1}G_1'Q^{-1}.$$

To see the second equality of (4.13), first note that

$$\begin{aligned} &Z_{22} - Z_{23}W_{33}^{-1}W_{32} = H_2'YB_2 - H_2'YB_3(B_3'SB_3)^{-1}B_3SB_2 \\ &\quad = H_2'YS^{-1}X(X'S\ -1X)^{-1}X'B_2. \end{aligned}$$

Further, using

$$\{A_1'(I_n - P_n)A_1\}^{-1} = \mathrm{diag}(1/n_1, \ldots, 1/n_{k-1}) + (1/n_k)\mathbf{1}_{k-1}\mathbf{1}_{k-1}',$$

we have

$$\begin{aligned} &\{A_1'(I_n - P_n)A_1\}^{-1/2}H_2'Y = \{A_1'(I_n - P_n)A_1\}^{-1}A_1'(I_n - P_n)Y \\ &\quad = (\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_k, \ldots, \bar{\boldsymbol{y}}_{k-1} - \bar{\boldsymbol{y}}_k)', \end{aligned}$$

and

$$\begin{aligned} &\{A_1'(I_n - P_n)A_1\}^{-1/2}(I_{k-1} + Z_{23}W_{33}^{-1}Z_{23}')\{A_1'(I_n - P_n)A_1\}^{-1/2} \\ &\quad = \{A_1'(I_n - P_n)A_1\}^{-1} + \{A_1'(I_n - P_n)A_1\}^{-1/2}H_2'Y \\ &\qquad \times B_3(B_3'SB_3)^{-1}B_3'Y'H_2\{A_1'(I_n - P_n)A_1\}^{-1/2} \\ &\quad = \mathrm{diag}(1/n_1, \ldots, 1/n_{k-1}) + (1/n_k)\mathbf{1}_{k-1}\mathbf{1}_{k-1}' \\ &\quad +(\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_k, \ldots, \bar{\boldsymbol{y}}_{k-1} - \bar{\boldsymbol{y}}_k)'S^{-1}\{S - X(X'S^{-1}X)^{-1}X'\}S^{-1} \\ &\qquad \times(\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_k, \ldots, \bar{\boldsymbol{y}}_{k-1} - \bar{\boldsymbol{y}}_k). \end{aligned}$$

From these we can obtain the final results by the help of

$$\begin{aligned} &C(A'A)^{-1}C' = \mathrm{diag}(1/n_1, \ldots, 1/n_{k-1}) + (1/n_k)\mathbf{1}_{k-1}\mathbf{1}_{k-1}', \\ &C(A'A)^{-1}A'Y = (\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_k, \ldots, \bar{\boldsymbol{y}}_{k-1} - \bar{\boldsymbol{y}}_k). \end{aligned}$$

§5. Estimation of γ

We have seen that the MLE of $\boldsymbol{\delta}$ is given by (4.2), and $\hat{\boldsymbol{\eta}}$ is given by (4.3). Therefore, we can write the MLE of $\boldsymbol{\gamma}$ as

$$\hat{\gamma} = (1/\sqrt{p})\{A_1'(I_n - P_0)A_1\}^{-1/2}(\boldsymbol{z}_{21} - Z_{2(23)}W_{(23)(23)}^{-1}\boldsymbol{w}_{(23)1}). \tag{5.1}$$

First we consider to express the MLE $\hat{\boldsymbol{\gamma}}$ in terms of the original observation matix Y. Note that

$$\begin{aligned}
\hat{\gamma} &= \frac{1}{p}\{A_1'(I_n - P_n)A_1\}^{-1}A_1'(I_n - P_n)Y \\
&\quad \times[I_p - (B_2, B_3)\{(B_2, B_3)'S(B_2, B_3)\}^{-1}(B_2.B_3)'S]\boldsymbol{b}_1 \\
&= \frac{1}{p}(\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_k, \ldots, \bar{\boldsymbol{y}}_{k-1} - \bar{\boldsymbol{y}}_k)'S^{-1}\boldsymbol{b}_1(\boldsymbol{b}_1'S^{-1}\boldsymbol{b}_1)^{-1}\boldsymbol{b}_1'\boldsymbol{b}_1.
\end{aligned}$$

This implies that

$$\hat{\gamma} = (\mathbf{1}_p'S^{-1}\mathbf{1}_p)^{-1}(\bar{\boldsymbol{y}}_1 - \bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_{k-1} - \bar{\boldsymbol{y}}_{k-1})'S^{-1}\mathbf{1}_p, \tag{5.2}$$

which is the same expression with the one (see Srivastava (1987)) in MANOVA, though their canonical forms are slightly different.

It is easy to see that $\hat{\boldsymbol{\gamma}}$ is an unbiased estimator, since S and $\{\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_k\}$ are independent. The expressions (5.1) or (5.2) shows that the distribution of $\hat{\gamma}$ can be obtained from the results in MANOVA case. Therefore, we can construct confidence intervals for $\boldsymbol{\gamma}$. In the following we explain the methods given in Fujikoshi (2009) which is based on the following result.

Theorem 5.1. *For a fixed vector* $\boldsymbol{a} = (a_1, \ldots, a_{k-1})'$,

$$X_{\boldsymbol{a}} = \frac{(\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)^{1/2}}{(\boldsymbol{a}'Q\boldsymbol{a})^{1/2}}\boldsymbol{a}'(\hat{\gamma} - \gamma) = VU, \tag{5.3}$$

where U *is distributed as* $N(0, 1)$,

$$V = \frac{(\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)^{1/2}(\mathbf{1}_p'S^{-1}\Sigma S^{-1}\mathbf{1}_p)^{1/2}}{(\mathbf{1}_p'S^{-1}\mathbf{1}_p)}, \tag{5.4}$$

and U *and* V *are independent. Further,* V^2 *is distributed as*

$$V^2 = 1 + \frac{\chi^2_{p-1}}{\chi^2_{m-p+2}},$$

where $m = n - k - (p - q)$, *and* χ^2_{p-1} *and* χ^2_{m-p+2} *are independent* χ^2 *variables with* $p - 1$ *and* $m - p + 2$ *degrees of freedom, respectively.*

For constructing a confidence interval of $\boldsymbol{a}'\boldsymbol{\gamma}$ for given $\boldsymbol{a}$, it is important to consider the distribution of $\hat{X}_{\boldsymbol{a}}$, which is defined from $X_{\boldsymbol{a}}$ by substituting S to Σ, i.e.,

$$\begin{aligned} \hat{X}_{\boldsymbol{a}} &= \frac{(\mathbf{1}_p' S^{-1}\mathbf{1})^{1/2}}{(\boldsymbol{a}'Q\boldsymbol{a})^{1/2}}\boldsymbol{a}'(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}) \\ &= \frac{(\mathbf{1}_p' S^{-1}\mathbf{1}_p)^{1/2}}{(\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)^{1/2}}\cdot VU \\ &= RU, \end{aligned} \tag{5.5}$$

where

$$R = \frac{(\mathbf{1}_p' S^{-1}\Sigma S^{-1}\mathbf{1}_p)^{1/2}}{(\mathbf{1}_p' S^{-1}\mathbf{1}_p)^{1/2}}. \tag{5.6}$$

For constructing a simultaneous confidence interval for $\boldsymbol{a}'\boldsymbol{\gamma}$ for all $\boldsymbol{a}$, it is natural to use

$$\begin{aligned} T &= \max_{\boldsymbol{a}} \hat{X}_{\boldsymbol{a}}^2 = (\mathbf{1}_p' S^{-1}\mathbf{1}_p)\max_{\boldsymbol{a}} \frac{(\boldsymbol{a}'(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}))^2}{\boldsymbol{a}'Q\boldsymbol{a}} \\ &= (\mathbf{1}_p' S^{-1}\mathbf{1})^2(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'Q^{-1}(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}) \\ &= R^2\chi^2_{k-1}. \end{aligned} \tag{5.7}$$

Here it is known (see e.g. Fujikoshi (2009)) that R^2 is distributed as

$$R^2 = \frac{m}{\chi^2_{m-p+1}}\left[1+\frac{\chi^2_{p-1}}{\chi^2_{m-p+2}}\right],$$

where $\chi^2_{p-1}, \chi^2_{m-p+1}$ and χ^2_{m-p+2} are independent χ^2 variables with $p-1, m-p+1$ and $m-p+2$ degees of freedom, respectively.

The statistic $\hat{X}_{\boldsymbol{a}}$ is a scale mixtures of the standard normal distribution with scale factor R, while T is a scale mixture of a chisquare variate χ^2_{k-1} with scale factor R^2. Using asymptotic expansions (see Fujikoshi (2009)) of their distributions, we can get confidence intervals.

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References

[1] Fujikoshi, Y. (1981). The power function of the likelihood ratio test for additional information in a multivariate linear model. *Ann. Inst. Statist. Math.*, **33**, 279-285.

[2] Fujikoshi, Y., Kanda, T. and Ohtaki, M. (1999). Growth curve models with hierarchical within-individuals design matrices. *Ann. Inst. Statist. Math.*, **51**, 707-721.

[3] Fujikoshi, Y. (2009). Confidence intervals and model selection criteria in profile analysis. submitted.

[4] Gleser, L. J and Olkin (1970). Linear models in multivariate analysis. *Essays in Prob. Statist.*, (R.C. Bose and Others, eds.), Univ. North Carolina Press, Chapel Hill, N.C., 267-292.

[5] Greenhouse and Geisser (1959). On the methods in the analysis of profile data. *Psychometrika*, **24**, 95-112.

[6] Kshirsagar, A. M. and Smith, W. B. (1995). *Grouth Curves.* Marcel Dekker.

[7] Rao, C.R. (1973). *Linear Statistical Inference and Its Applications*, 2nd ed. Wiley, New York. (299)

[8] Srivastava, M. S. (1987). Profile analysis of several groups. *Commun. Statist.-Theory Meth.*, **16**, 909-926.

[9] Srivastava, M. S. (2002). *Methods of Multivariate Statistics. Wiley, New York.*

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