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On Kenmotsu manifolds satisfying certain curvature conditions

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Abstract. The object of the present paper is to study some curvature conditions on Kenmotsu manifolds. Also, we classify Kenmotsu manifolds which satisfy $P \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{C} = 0$, $\tilde{Z} \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{Z} = 0$ and $C \cdot \tilde{C} = 0$, where P is the projective curvature tensor, $\tilde{Z}$ is the concircular curvature tensor, $\tilde{C}$ is the quasi-conformal curvature tensor and C is the conformal curvature tensor.

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§1. Introduction

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes M to be an almost contact metric manifold and supposes that the product metric G on $M \times \mathbb{R}$ is Kaehlerian, then the structure on M is cosymplectic ([6]) and not Sasakian. On the other hand Oubina [9] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinate on $\mathbb{R}$, is Kaehlerian, then M is Sasakian and conversely.

In [11], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold M, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, M is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M is a warped product space

$\mathbb{R} \times_f \mathbb{C}^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions ([8]). We call it Kenmotsu manifold.

The notion of the *quasi-conformal curvature tensor* was given by Yano and Sawaki [12]. According to them a quasi-conformal curvature tensor $\tilde{C}$ is defined by

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{\tau}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.1}$$

where a and b are constants and R, S, Q and τ are the Riemannian curvature tensor type of $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-2}$ then (1.1) takes the form

$$\begin{aligned} \tilde{C}(X,Y)Z &= R(X,Y)Z \\ &\quad - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{\tau}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z, \end{aligned} \tag{1.2}$$

where C is the conformal curvature tensor ([5]). Thus the conformal curvature tensor C is a particular case of the tensor $\tilde{C}$. For this reason $\tilde{C}$ is called the quasi-conformal curvature tensor. A manifold (M^n, g), $n > 1$, shall be called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. It is known ([2]) that the quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or, Einstein if $a = 0$ and $b \neq 0$. Since, they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

We next define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$R(X,Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W,$$

$$(X \wedge_A Y)W = A(Y,W)X - A(X,W)Y,$$

respectively, where $X, Y, W \in \chi(M)$ and A is the symmetric $(0,2)-$tensor.

On the other hand, the *projective curvature tensor* P and the *concircular curvature tensor* $\tilde{Z}$ in a Riemannian manifold (M^n, g) are defined by

$$P(X,Y)W = R(X,Y)W - \frac{1}{n-1}(X \wedge_S Y)W, \tag{1.3}$$

(1.4) $$\tilde{Z}(X,Y)W = R(X,Y)W - \frac{\tau}{n(n-1)}(X \wedge_g Y)W,$$

respectively.

An almost contact metric manifold is said to be an $\eta-$Einstein manifold if the Ricci tensor S satisfies the condition

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y),$$

where λ_1, λ_2 are certain scalars. A Riemannian or a semi-Riemannian manifold is said to semisymmetric if $R(X,Y) \cdot R = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y.

Kenmotsu manifolds have been studied by many authors such as De and Pathak [3], Jun, De and Pathak [7], Ozgür and De [10] and many others.

In the present paper we have studied some curvature conditions on Kenmotsu manifolds. We have classified Kenmotsu manifolds which satisfy $P \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{C} = 0$, $\tilde{Z} \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{Z} = 0$ and $C \cdot \tilde{C} = 0$, where P is the projective curvature tensor, $\tilde{Z}$ is the concircular curvature tensor, $\tilde{C}$ is the quasi-conformal curvature tensor and C is the conformal curvature tensor.

§2. Preliminaries

Let $(M^n, \phi, \xi, \eta, g)$ be an n-dimensional (where $n = 2m + 1$) almost contact metric manifold, where ϕ is a $(1,1)-$tensor field, ξ is the structure vector field, η is a $1-$form and g is the Riemannian metric. It is well known that the (ϕ, ξ, η, g) structure satisfies the conditions ([1])

(2.1) $$\phi^2 X = -X + \eta(X)\xi, \;\; g(X,\xi) = \eta(X),$$

$$\phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X and Y on M^n.

If moreover

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,$$

$$\nabla_X \xi = X - \eta(X)\xi,$$

where ∇ denotes the Riemannian connection of g hold, then $(M^n, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold. In this case, it is well known ([8]) that

(2.2) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.3}$$

where S denotes the Ricci tensor. From (2.2), it easily follows that

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$$R(X,\xi)\xi = \eta(X)\xi - X. \tag{2.5}$$

In a Kenmotsu manifold, using (2.3) and (2.4), equations (1.3), (1.4), (1.2), and (1.1) reduce to

$$P(\xi,X)Y = -g(X,Y)\xi - \frac{1}{n-1}S(X,Y)\xi, \tag{2.6}$$

$$\tilde{Z}(\xi,X)Y = (1+\frac{\tau}{n(n-1)})(-g(X,Y)\xi + \eta(Y)X), \tag{2.7}$$

$$\begin{aligned} C(\xi,Y)W &= \frac{n-1+\tau}{(n-1)(n-2)}\{g(Y,W)\xi - \eta(W)Y\} \\ &\quad -\frac{1}{n-2}\{S(Y,W)\xi - \eta(W)QY\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \tilde{C}(\xi,Y)W &= K\{\eta(W)Y - g(Y,W)\xi\} \\ &\quad +b\{S(Y,W)\xi - \eta(W)QY\}, \end{aligned} \tag{2.9}$$

respectively, where $K = a + (n-1)b + \frac{\tau}{n}(\frac{a}{n-1} + 2b)$.

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point. Then the Ricci tensor and the scalar curvature of M are defined by

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i),$$

and

$$\tau = \sum_{i=1}^{n} S(e_i,e_i),$$

respectively.

Since $S(X,Y) = g(QX,Y)$, we have

$$S(\phi X,\phi Y) = g(Q\phi X,\phi Y),$$

where Q is the Ricci operator. Using the properties $g(X, \phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (2.1) and (2.3), we get

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y).$$

Also we have ([1])

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y).$$

A Kenmotsu manifold M^n is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),$$

for any vector fields X and Y, where $\lambda_1 = 1 + \frac{\tau}{n-1}$ and $\lambda_2 = -(n + \frac{\tau}{n-1})$.

Now, we define $P(X,Y) \cdot \tilde{C}$, $\tilde{Z}(X,Y) \cdot \tilde{C}$, $\tilde{C}(X,Y) \cdot \tilde{C}$, $\tilde{C}(X,Y) \cdot \tilde{Z}$ and $C(X,Y) \cdot \tilde{C}$ as

$$\begin{aligned}(P(X,Y) \cdot \tilde{C})(U,V)W &= P(X,Y)\tilde{C}(U,V)W - \tilde{C}(P(X,Y)U,V)W \\ &\quad -\tilde{C}(U,P(X,Y)V)W - \tilde{C}(U,V)P(X,Y)W,\end{aligned} \tag{2.10}$$

$$\begin{aligned}(\tilde{Z}(X,Y) \cdot \tilde{C})(U,V)W &= \tilde{Z}(X,Y)\tilde{C}(U,V)W - \tilde{C}(\tilde{Z}(X,Y)U,V)W \\ &\quad -\tilde{C}(U,\tilde{Z}(X,Y)V)W - \tilde{C}(U,V)\tilde{Z}(X,Y)W,\end{aligned} \tag{2.11}$$

$$\begin{aligned}(\tilde{C}(X,Y) \cdot \tilde{C})(U,V)W &= \tilde{C}(X,Y)\tilde{C}(U,V)W - \tilde{C}(\tilde{C}(X,Y)U,V)W \\ &\quad -\tilde{C}(U,\tilde{C}(X,Y)V)W - \tilde{C}(U,V)\tilde{C}(X,Y)W,\end{aligned} \tag{2.12}$$

$$\begin{aligned}(\tilde{C}(X,Y) \cdot \tilde{Z})(U,V)W &= \tilde{C}(X,Y)\tilde{Z}(U,V)W - \tilde{Z}(\tilde{C}(X,Y)U,V)W \\ &\quad -\tilde{Z}(U,\tilde{C}(X,Y)V)W - \tilde{Z}(U,V)\tilde{C}(X,Y)W,\end{aligned} \tag{2.13}$$

$$\begin{aligned}(C(X,Y) \cdot \tilde{C})(U,V)W &= C(X,Y)\tilde{C}(U,V)W - \tilde{C}(C(X,Y)U,V)W \\ &\quad -\tilde{C}(U,C(X,Y)V)W - \tilde{C}(U,V)C(X,Y)W,\end{aligned} \tag{2.14}$$

respectively, where $X, Y, U, V, W \in \chi(M)$.

§3. Kenmotsu manifolds satisfying $P(\xi, Y)\cdot\tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$P(\xi, Y)\cdot\tilde{C} = 0. \tag{3.1}$$

From (2.10), we have

$$\begin{aligned}(P(\xi,Y)\cdot\tilde{C})(Z,U)W &= P(\xi,Y)\tilde{C}(Z,U)W - \tilde{C}(P(\xi,Y)Z,U)W\\ &\quad -\tilde{C}(Z,P(\xi,Y)U)W - \tilde{C}(Z,U)P(\xi,Y)W = 0.\end{aligned} \tag{3.2}$$

Taking the inner product with X and using (2.6) in (3.2), we have

$$\begin{aligned}&g(Y,\tilde{C}(Z,U)W)\eta(X) - g(Y,Z)g(\tilde{C}(\xi,U)W,X)\\ &-g(Y,U)g(\tilde{C}(Z,\xi)W,X) - g(Y,W)g(\tilde{C}(Z,U)\xi,X)\\ &+\frac{1}{n-1}\{S(Y,\tilde{C}(Z,U)W)\eta(X) - S(Y,Z)g(\tilde{C}(\xi,U)W,X)\\ &-S(Y,U)g(\tilde{C}(Z,\xi)W,X) - S(Y,W)g(\tilde{C}(Z,U)\xi,X)\} = 0.\end{aligned} \tag{3.3}$$

Taking $U = \xi$ in (3.3), we have

$$\begin{aligned}&g(Y,\tilde{C}(Z,\xi)W)\eta(X) - g(Y,W)g(\tilde{C}(Z,\xi)\xi,X)\\ &+\frac{1}{n-1}\{S(Y,\tilde{C}(Z,\xi)W)\eta(X) - S(Y,W)g(\tilde{C}(Z,\xi)\xi,X)\} = 0.\end{aligned} \tag{3.4}$$

Using (2.9) in (3.4), we get

$$\begin{aligned}&K\{g(Y,Z)\eta(X)\eta(W) + \frac{1}{n-1}S(Y,Z)\eta(X)\eta(W)\\ &+g(Y,W)\eta(X)\eta(Z) - g(Y,W)g(X,Z)\\ &+\frac{1}{n-1}S(Y,W)\eta(X)\eta(Z) - \frac{1}{n-1}S(Y,W)g(X,Z)\}\\ &-b\{S(Y,Z)\eta(X)\eta(W) + \frac{1}{n-1}S(QY,Z)\eta(X)\eta(W)\\ &-S(X,Z)g(Y,W) - (n-1)g(Y,W)\eta(X)\eta(Z)\\ &-\frac{1}{n-1}S(Y,W)S(X,Z) - S(Y,W)\eta(X)\eta(Z)\} = 0,\end{aligned} \tag{3.5}$$

where $S(QY,Z) = S^2(Y,Z)$.

Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (3.5) for $Y = W = e_i$ gives

$$\{\tau + n(n-1)\}[bS(X,Z) - Kg(X,Z) + \{K + (n-1)b\}\eta(X)\eta(Z)] = 0. \tag{3.6}$$

Let U_1 and U_2 be a part of M satisfying $\tau + n(n-1) = 0$ and

$$bS(X,Z) - Kg(X,Z) + \{K + (n-1)b\}\eta(X)\eta(Z) = 0, \tag{3.7}$$

respectively. In the case of $\tau + n(n-1) \neq 0$, if $b = 0$, from (3.7) we get $a = 0$. This is the contradiction. Thus we find $b \neq 0$. By virtue of (3.7), we obtain $\frac{K}{b} = 1 + \frac{\tau}{n-1}$, which yields

$$S(X,Z) = (1 + \frac{\tau}{n-1})g(X,Z) - (n + \frac{\tau}{n-1})\eta(X)\eta(Z).$$

Hence we have the following:

Theorem 1. *Let M^n be an n-dimensional $(n > 1)$ Kenmotsu manifold satisfying the condition $P(\xi, Y) \cdot \tilde{C} = 0$. Then M is a part of*

1. *$\tau = -n(n-1)$, that is, the scalar curvature is the negative constant, or*
2. *an $\eta-$Einstein manifold.*

§4. Kenmotsu manifolds satisfying $\tilde{C}(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$\tilde{C}(\xi, Y) \cdot \tilde{C} = 0.$$

From (2.12), we have

$$\begin{aligned} (\tilde{C}(\xi,Y)\cdot\tilde{C})(U,V)W &= \tilde{C}(\xi,Y)\tilde{C}(U,V)W - \tilde{C}(\tilde{C}(\xi,Y)U,V)W \\ &\quad -\tilde{C}(U,\tilde{C}(\xi,Y)V)W - \tilde{C}(U,V)\tilde{C}(\xi,Y)W = 0. \end{aligned} \tag{4.1}$$

Taking the inner product with X and using $U = \xi$ in (4.1), we obtain

$$\begin{aligned} &g(\tilde{C}(\xi,Y)\tilde{C}(\xi,V)W,X) - g(\tilde{C}(\tilde{C}(\xi,Y)\xi,V)W,X) \\ &-g(\tilde{C}(\xi,\tilde{C}(\xi,Y)V)W,X) - g(\tilde{C}(\xi,V)\tilde{C}(\xi,Y)W,X) = 0. \end{aligned} \tag{4.2}$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ an orthonormal basis of the tangent space at any point. Now we put $X = W = e_i$ in (4.2). Straightforwardly we calculate the equation $\sum_{i=1}^{n} g((\tilde{C}(\xi,Y)\cdot\tilde{C})(\xi,e_i)W,e_i) = 0$. Then we obtain

$$\begin{aligned} &g(\tilde{C}(\xi,Y)\tilde{C}(\xi,e_i)W,e_i) - g(\tilde{C}(\tilde{C}(\xi,Y)\xi,e_i)W,e_i) \\ &-g(\tilde{C}(\xi,\tilde{C}(\xi,Y)e_i)W,e_i) - g(\tilde{C}(\xi,e_i)\tilde{C}(\xi,Y)W,e_i) = 0. \end{aligned} \tag{4.3}$$

Using (1.1) and (2.9) in (4.3), we get

$$\begin{aligned} \{a+(n-2)b\}[bS(QY,W) &- \frac{1}{n(n-1)}\{a(\tau+n(n-1)) + 2(n-1)b\tau\}S(Y,W) \\ &-(n-1)Kg(Y,W)] = 0. \end{aligned}$$

Thus we have $a + (n-2)b = 0$, or

$$(4.4) \qquad bS(QY,W) - \frac{1}{n(n-1)}\{a(\tau + n(n-1)) + 2(n-1)b\tau\}S(Y,W) \\ -(n-1)Kg(Y,W) = 0.$$

If $b = 0$, then we get

$$a\{\tau + n(n-1)\}\{S(Y,W) + (n-1)g(Y,W)\} = 0.$$

We can easily verify that

$$S(Y,W) = -(n-1)g(Y,W).$$

Therefore we have the following:

Theorem 2. *Let M^n be an n-dimensional $(n > 1)$ Kenmotsu manifold satisfying the condition $\tilde{C}(\xi, Y) \cdot \tilde{C} = 0$. Then we get*

1. *$a + (n-2)b = 0$, or*
2. *we find*

   i) *if $b = 0$, then M is an Einstein manifold,*
   ii) *if $b \neq 0$, then we get*

$$S(QY,W) = (\frac{K}{b} - n + 1)S(Y,W) + (n-1)\frac{K}{b}g(Y,W).$$

Now we need the following:

Lemma 1. *([4]) Let A be a symmetric $(0,2)$-tensor at a point x of a semi-Riemannian manifold (M^n, g), $n > 1$, and let $T = g \,\overline{\wedge}\, A$ be the Kulkarni-Nomizu product of g and A. Then, the relation*

$$T \cdot T = \alpha Q(g,T), \quad \alpha \in \mathbb{R}$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \quad \lambda \in \mathbb{R}$$

holds at x.

From Theorem 2 and Lemma 1 we get the following:

Corollary 1. *Let M^n be an $n-$dimensional $(n > 1)$ Kenmotsu manifold satisfying the condition $\tilde{C}(\xi, Y) \cdot \tilde{C} = 0$, then $T \cdot T = \alpha Q(g,T)$, where $T = g \,\overline{\wedge}\, A$ and $\alpha = \frac{K}{b} - n + 1$.*

§5. Kenmotsu manifolds satisfying $\tilde{Z}(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$\tilde{Z}(\xi, Y) \cdot \tilde{C} = 0.$$

From (2.11), we have

$$\begin{aligned} (\tilde{Z}(\xi, Y) \cdot \tilde{C})(U, V)W &= \tilde{Z}(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(\tilde{Z}(\xi, Y)U, V)W \\ &\quad -\tilde{C}(U, \tilde{Z}(\xi, Y)V)W - \tilde{C}(U, V)\tilde{Z}(\xi, Y)W = 0. \end{aligned} \tag{5.1}$$

Now using $U = \xi$ in (5.1), we have

$$\begin{aligned} &\tilde{Z}(\xi, Y)\tilde{C}(\xi, V)W - \tilde{C}(\tilde{Z}(\xi, Y)\xi, V)W \\ &-\tilde{C}(\xi, \tilde{Z}(\xi, Y)V)W - \tilde{C}(\xi, V)\tilde{Z}(\xi, Y)W = 0. \end{aligned} \tag{5.2}$$

Taking the inner product with X in (5.2) and using (2.7), we get

$$\begin{aligned} &\{1 + \frac{\tau}{n(n-1)}\}\{g(X, Y)\eta(\tilde{C}(\xi, V)W) - g(Y, \tilde{C}(\xi, V)W)\eta(X) \\ &-g(X, \tilde{C}(Y, V)W) + g(\tilde{C}(\xi, V)W, X)\eta(Y) \\ &-g(\tilde{C}(\xi, Y)W, X)\eta(V) - g(\tilde{C}(\xi, V)Y, X)\eta(W) \\ &+g(Y, W)g(\tilde{C}(\xi, V)\xi, X)\} = 0. \end{aligned} \tag{5.3}$$

Again from (2.7), we have $\tau \neq -n(n-1)$. Thus

$$\begin{aligned} &g(X, Y)\eta(\tilde{C}(\xi, V)W) - g(Y, \tilde{C}(\xi, V)W)\eta(X) - g(X, \tilde{C}(Y, V)W) \\ &+g(\tilde{C}(\xi, V)W, X)\eta(Y) - g(\tilde{C}(\xi, Y)W, X)\eta(V) - g(\tilde{C}(\xi, V)Y, X)\eta(W) \\ &+g(Y, W)g(\tilde{C}(\xi, V)\xi, X) = 0. \end{aligned} \tag{5.4}$$

Using (2.9) in (5.4), we get

$$\begin{aligned} &-a\{g(X, Y)g(V, W) + g(R(Y, V)W, X) - g(X, V)g(Y, W)\} \\ &-b(n-1)\{g(X, Y)g(V, W) - g(X, Y)\eta(V)\eta(W) - g(X, V)g(Y, W) \\ &+g(Y, W)\eta(X)\eta(V)\} + b\{S(Y, W)g(X, V) - S(X, Y)g(V, W) \\ &-S(Y, W)\eta(X)\eta(V) + S(X, Y)\eta(V)\eta(W)\} = 0. \end{aligned} \tag{5.5}$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (5.5) for $Y = W = e_i$ gives

$$\begin{aligned} (b-a)S(X, V) = \{(n-1)a + (n-1)^2 b + b\tau\}g(X, V) \\ -b\{\tau + n(n-1)\}\eta(X)\eta(V). \end{aligned} \tag{5.6}$$

If $a = b(\neq 0)$, then we have $a\{\tau + n(n-1)\}\{g(X,V) - \eta(X)\eta(V)\} = 0$. Because of (2.7), we find $\tau + n(n-1) \neq 0$. Thus $a \neq b$ holds. We obtain $\{a+(n-2)b\}\{\tau+n(n-1)\} = 0$ from (5.6), which means that $a+(n-2)b = 0$. Thus equation (5.6) can be rewritten as follows:

$$S(X,V) = (1 + \frac{\tau}{n-1})g(X,Y) - (n + \frac{\tau}{n-1})\eta(X)\eta(Y).$$

Hence we have the following:

Theorem 3. *An $n-$dimensional $(n > 1)$ Kenmotsu manifold M^n satisfying the condition $\tilde{Z}(\xi, Y) \cdot \widetilde{C} = 0$ is an $\eta-$Einstein manifold.*

§6. Kenmotsu manifolds satisfying $\tilde{C}(\xi, Y) \cdot \tilde{Z} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$\tilde{C}(\xi, Y) \cdot \tilde{Z} = 0.$$

From (2.13), we have

$$\begin{aligned} (\tilde{C}(\xi,Y)\cdot\tilde{Z})(U,V)W &= \tilde{C}(\xi,Y)\tilde{Z}(U,V)W - \tilde{Z}(\tilde{C}(\xi,Y)U,V)W \\ &\quad -\tilde{Z}(U,\tilde{C}(\xi,Y)V)W - \tilde{Z}(U,V)\tilde{C}(\xi,Y)W = 0. \end{aligned} \tag{6.1}$$

Putting $U = \xi$ in (6.1), we have

$$\begin{aligned} &\tilde{C}(\xi,Y)\tilde{Z}(\xi,V)W - \tilde{Z}(\tilde{C}(\xi,Y)\xi,V)W \\ &-\tilde{Z}(\xi,\tilde{C}(\xi,Y)V)W - \tilde{Z}(\xi,V)\tilde{C}(\xi,Y)W = 0. \end{aligned} \tag{6.2}$$

Taking the inner product with $X \in \chi(M)$ in (6.2) and using (2.9), we get

$$\begin{aligned} &K\{g(Y,\tilde{Z}(\xi,V)W)\eta(X) - \eta(\tilde{Z}(\xi,V)W)g(Y,X) - g(\tilde{Z}(\xi,V)W,X)\eta(Y) \\ &+ g(\tilde{Z}(Y,V)W,X) + g(\tilde{Z}(\xi,Y)W,X)\eta(V) - g(Y,W)g(\tilde{Z}(\xi,V)\xi,X) \\ &+ g(\tilde{Z}(\xi,V)Y,X)\eta(W)\} - b\{S(Y,\tilde{Z}(\xi,V)W)\eta(X) - \eta(\tilde{Z}(\xi,V)W)S(Y,X) \\ &+ (n-1)g(\tilde{Z}(\xi,V)W,X)\eta(Y) + g(\tilde{Z}(QY,V)W,X) + g(\tilde{Z}(\xi,QY)W,X)\eta(V) \\ &- S(Y,W)g(\tilde{Z}(\xi,V)\xi,X) + g(\tilde{Z}(\xi,V)QY,X)\eta(W)\} = 0. \end{aligned}$$

Using (1.4) and (2.7) in the above equation, we obtain

$$\begin{aligned} &K\{g(R(Y,V)W,X) + g(Y,X)g(V,W) - g(X,V)g(Y,W)\} \\ &-b\{g(R(QY,V)W,X) + S(Y,X)g(V,W) - S(Y,W)g(X,V)\} = 0. \end{aligned} \tag{6.3}$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (6.3) for $X = V = e_i$ gives

$$K\{S(Y,W) + (n-1)g(Y,W)\} - b\{S^2(Y,W) + (n-1)S(Y,W)\} = 0.$$

When $b = 0$, the above equation can be rewritten as follows:

$$K\{S(Y,W) + (n-1)g(Y,W)\} = 0,$$

which means that $K\{\tau + n(n-1)\} = 0$. From (2.7), we find $\tau + n(n-1) \neq 0$. Thus we get $K = 0$, namely, $a = 0$. Therefore we get $b \neq 0$ and

$$S(QY,W) = (\frac{K}{b} - n + 1)S(Y,W) + (n-1)\frac{K}{b}g(Y,W). \tag{6.4}$$

This leads to the following:

Theorem 4. *In an n-dimensional* $(n > 1)$ *Kenmotsu manifold M if the condition* $\tilde{C}(\xi,Y) \cdot \tilde{Z} = 0$ *holds on M, then the equation (6.4) is satisfied on M.*

From Theorem 4 and Lemma 1 we get the following:

Corollary 2. *Let M be an n-dimensional* $(n > 1)$ *Kenmotsu manifold satisfying the condition* $\tilde{C}(\xi,Y) \cdot \tilde{Z} = 0$*, then* $T \cdot T = \alpha Q(g,T)$*, where* $T = g \barwedge A$ *and* $\alpha = \frac{K}{b} - n + 1$*.*

§7. Kenmotsu manifolds satisfying $C(\xi,Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$C(\xi,Y) \cdot \tilde{C} = 0.$$

From (2.14), we have

$$\begin{aligned}(C(\xi,Y) \cdot \tilde{C})(U,V)W &= C(\xi,Y)\tilde{C}(U,V)W - \tilde{C}(C(\xi,Y)U,V)W \\ &\quad -\tilde{C}(U,C(\xi,Y)V)W - \tilde{C}(U,V)C(\xi,Y)W = 0.\end{aligned} \tag{7.1}$$

Taking the inner product with X and using $U = \xi$ in (7.1), we obtain

$$\begin{aligned}&g(C(\xi,Y)\tilde{C}(\xi,V)W,X) - g(\tilde{C}(C(\xi,Y)\xi,V)W,X) \\ &-g(\tilde{C}(\xi,C(\xi,Y)V)W,X) - g(\tilde{C}(\xi,V)C(\xi,Y)W,X) = 0.\end{aligned} \tag{7.2}$$

Let $\{e_i\}$ $(1 \leq i \leq n)$ be an orthonormal basis of the tangent space at any point. Now we put $X = V = e_i$ in (7.2). Straightforwardly we calculate the equation $\sum_{i=1}^{n} g((C(\xi, Y) \cdot \tilde{C})(\xi, e_i)W, e_i) = 0$. Then from (1.1), (2.8) and (2.9), we obtain

$$\{\frac{a}{n-2} + b\}\{S(QY, W) - [1 + \frac{\tau}{n-1} - (n-1)]S(Y, W) - [\tau + n - 1]g(Y, W)\} = 0.$$

Let U_1 and U_2 be a part of M satisfying $a + b(n-2) = 0$ and

$$S(QY, W) = [\frac{\tau}{n-1} + 2 - n]S(Y, W) + [\tau + n - 1]g(Y, W). \quad (7.3)$$

This leads to the following:

Theorem 5. *In $n-$dimensional $(n > 1)$ Kenmotsu manifold M^n satisfying the condition $C(\xi, Y) \cdot \tilde{C} = 0$. Then we get*

1. $a + (n-2)b = 0$, *or*
2. $a + b(n-2) \neq 0$, *then the equation (7.3) holds on M.*

From Theorem 5 and Lemma 1 we get the following:

Corollary 3. *Let M be an $n-$dimensional $(n > 1)$ Kenmotsu manifold satisfying the condition $C(\xi, Y) \cdot \tilde{C} = 0$, then $T \cdot T = \alpha Q(g, T)$, where $T = g \barwedge A$ and $\alpha = \frac{\tau}{n-1} + 2 - n$.*

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