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NONLINEAR BOUNDARY STABILIZATION OF ISOTROPIC ELASTICITY SYSTEMS

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Abstract. We study the energy decay rate for the isotropic elasticity systems in a bounded domain under weak growth assumptions on the feedback function. This work improves some earlier results of Lagnese [9] and Komornik [5].

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1. Introduction

The problem of proving the energy decay rates for solutions of systems of evolution equations with dissipation at the boundary has been treated by several authors. Indeed, in the case of wave or plate equations we can mention Komornik [3], Komornik-Zuazua [7], Lagnese [8], Lasiecka [10], Lasiecka-Tataru [11], Lions [12], and Zuazua [13], among others.

Very little is known for the isotropic elasticity systems. To our knowledge, uniform decay estimates for two-dimensional homogeneous isotropic systems by applying either linear or nonlinear boundary feedbacks was studied by Lagnese [9], and quite recently Komornik [5] has obtained exponential decay for three dimensional case when the boundary dissipation is linear.

In this paper we consider the problem of nonlinear boundary stabilization for isotropic elasticity systems. More precisely, we consider the following problem

$$(P) \qquad \begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\operatorname{grad}\operatorname{div} u = 0 \quad \text{in} \quad \Omega\times(0,+\infty), \\ u = 0 \qquad \text{on} \qquad \Gamma_0\times(0,+\infty), \\ \mu\frac{\partial u}{\partial\nu} + (\lambda+\mu)(\operatorname{div} u)\nu + (m\cdot\nu)g(u') = 0 \quad \text{on} \quad \Gamma_1\times(0,+\infty), \\ u(0) = u_0, \quad \text{and} \quad u'(0) = u_1 \quad \text{in} \quad \Omega. \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ having a boundary Γ of class C^2, $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ denotes the outward unit normal vector to Γ, λ and μ (the

Lamé constants in the physical interpretation of the model) are two positive constants, $\{\Gamma_0, \Gamma_1\}$ is a partition of the boundary Γ such that

$$\Gamma_0 \neq \emptyset \quad \text{and} \quad \bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset, \tag{1.1}$$

$m(x) = x - x_0,\ x \in \mathbb{R}^n$, with x_0 is a fixed point in $\mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is such that

$$g \text{ is globally lipschitz continuous;} \tag{1.2}$$

$$g(0) = 0 \quad \text{and} \quad x \cdot g(x) \geq 0 \text{ for all } x \in \mathbb{R}^n; \tag{1.3}$$

$$\text{there exists } a \geq 1 \text{ such that } |g(x)||x| \leq a g(x) \cdot x \text{ for all } x \in \mathbb{R}^n \tag{1.4}$$

(the dot denotes the usual inner product in $\mathbb{R}^n$). The boundary velocity feedback denotes the surface traction $\mu\{(\nabla u) + (\nabla u)^T\} + \lambda(\operatorname{div} u)\nu$ (we refer to [2] for more explanations on the physical meaning of (P)).

Let us denote by $H^1_{\Gamma_0}(\Omega)$ the set of functions $v \in H^1(\Omega)$ satisfying $v = 0$ on Γ_0. Using the standard nonlinear semi-group theory, the problem (P) is well posed in the following sense: for every $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega)^n \times L^2(\Omega)^n$ arbitrarily, there exists a unique mild solution

$$u \in C(\mathbb{R}_+, H^1(\Omega)^n) \cap C^1(\mathbb{R}_+, L^2(\Omega)^n). \tag{1.5}$$

Moreover, if $(u_0, u_1) \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right)^n \times H^1_{\Gamma_0}(\Omega)^n$ and if the compatibility condition

$$\mu \frac{\partial u_0}{\partial \nu} + (\lambda + \mu)(\operatorname{div} u_0)\nu + (m \cdot \nu) g(u_1) = 0 \quad \text{on} \quad \Gamma_1 \tag{1.6}$$

holds, then we have the following regularity property

$$u \in C(\mathbb{R}_+, H^2(\Omega)^n) \cap C^1(\mathbb{R}_+, H^1(\Omega)^n) \cap C^2(\mathbb{R}_+, L^2(\Omega)^n) \tag{1.7}$$

we say in this case that u is a strong solution. In particular we have $\sup_{0<t<\infty} \|\nabla u'(t)\|_{L^2(\Omega)^n} < +\infty$.

Let us define the energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ of the solutions by the formula

$$E(t) := \frac{1}{2} \int_\Omega |u'|^2 + \mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 \, dx. \tag{1.8}$$

Assume that

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0 \quad \text{and} \quad m \cdot \nu \geq 0 \text{ on } \Gamma_1 \tag{1.9}$$

then the energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing function. Indeed, if u is a strong solution, then we have

$$E'(t) = \int_\Omega u' \cdot u'' + \mu \nabla u \cdot \nabla u' + (\lambda+\mu)(\mathrm{div} u)(\mathrm{div} u')\, dx$$

$$= \int_\Omega \mu(\nabla u' \cdot \nabla u + u' \cdot \Delta u) + (\lambda+\mu)\big((\mathrm{div} u)(\mathrm{div} u') + u' \cdot \nabla(\mathrm{div} u)\big)\, dx$$

$$= \int_{\Gamma_1} u' \cdot \Big(\mu \frac{\partial u}{\partial \nu} + (\lambda+\mu)(\mathrm{div} u)\nu\Big)\, d\Gamma$$

and then

$$E'(t) = -\int_{\Gamma_1} (m\cdot\nu) u' \cdot g(u')\, d\Gamma \ \le 0, \tag{1.10}$$

hence

$$E(S) - E(T) = \int_S^T \int_{\Gamma_1} (m\cdot\nu) u' \cdot g(u')\, d\Gamma dt \tag{1.11}$$

for all $0 \le S < T < +\infty$.

The inequality (1.10) remains valid for all mild solutions by an easy density argument.

In [9] and [5], Lagnese and Komornik, respectively, have studied the energy decay rate when g is such that

$$c_1|x|^p \le |g(x)| \le c_2|x|^{\frac{1}{p}} \quad \text{if} \quad |x| \le 1 \tag{1.12}$$

$$c_3|x| \le |g(x)| \le c_4|x| \quad \text{if} \quad |x| > 1 \tag{1.13}$$

c_i $(1 \le i \le 4)$ are four positive constants and $p \ge 1$.

These works have a serious drawback: they never apply for bounded functions g (because of $c_3 > 0$ in (1.13)). The purpose of this paper is to obtain a variant of Lagnese and Komornik's results for bounded feedback functions. The case of scalar wave equation was treated by Komornik in [3].

Our main result is the following

Main theorem. *In addition to (1.1)-(1.4) and (1.9), assume that g satisfies*

$$c_1|x|^p \le |g(x)| \le c_2|x|^{\frac{1}{p}} \quad \textit{if} \quad |x| \le 1 \tag{1.14}$$

$$c_3 \le |g(x)| \le c_4|x| \quad \textit{if} \quad |x| > 1 \tag{1.15}$$

where c_i $(1 \leq i \leq 4)$ *are four positive constants and* p *is such that*

$$p = 1 \quad \text{if} \quad n = 1, \tag{1.16}$$

$$p > 1 \quad \text{if} \quad n = 2, \tag{1.17}$$

$$p \geq n - 1 \quad \text{if} \quad n \geq 3. \tag{1.18}$$

Then for every

$$(u_0, u_1) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega))^n \times H^1_{\Gamma_0}(\Omega)^n \tag{1.19}$$

satisfying

$$\mu\frac{\partial u_0}{\partial \nu} + (\lambda + \mu)(div\, u_0)\nu + (m \cdot \nu)g(u_1) = 0 \quad \text{on} \quad \Gamma_1, \tag{1.20}$$

the solution of (P) *satisfies the estimates*

$$E(t) \leq ce^{-\omega t} \ \ \forall t > 0 \ \ (\omega > 0), \quad \text{if} \quad n = 1 \tag{1.21}$$

$$E(t) \leq ct^{\frac{2}{1-p}} \ \ \forall t > 0, \quad \text{if} \quad n \geq 2 \tag{1.22}$$

with a constant c *depending on the initial data* (u_0, u_1).

The proof of the theorem will be based on an integral inequality proved in Komornik [4]

Lemma 1.1. *(Th. 9.1 [3]) Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a decreasing function, and assume that there exists a nonnegative number* α *and a positive number* A *such that*

$$\int_t^{+\infty} E^{\alpha+1}(s)\, ds \leq AE(t) \quad \text{for all} \quad t \geq 0. \tag{1.23}$$

Then putting

$$T := AE(0)^{-\alpha}, \tag{1.24}$$

we have

$$E(t) \leq E(0)\left(\frac{T + \alpha T}{T + \alpha t}\right)^{\frac{1}{\alpha}} \quad \text{for all} \quad t \geq T \tag{1.25}$$

if $\alpha > 0$ *and*

$$E(t) \leq E(0)e^{1-\frac{t}{T}} \quad \text{for all} \quad t \geq T \tag{1.26}$$

if $\alpha = 0$.

2. Proof of the main theorem

From now on we denote by c various positive constants which may be different at different occurencies. Putting for brevity $Mu := (\nabla u)2m + (n-1)u$, we have for any fixed $0 \leq S < T < +\infty$

Lemma 2.1. *We have*

$$2\int_S^T E^{\frac{p+1}{2}}(t)\,dt \leq cE(S) + c\int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_1} (m\cdot\nu)\{|u'|^2 + |g(u')|^2\}\,d\Gamma dt.$$

Proof. We have

$$0 = \int_S^T E^{\frac{p-1}{2}} \int_\Omega (Mu)\cdot(u'' - \mu\Delta u - (\lambda+\mu)\text{grad div}u)\,dxdt$$

$$= \Big[E^{\frac{p-1}{2}} \int_\Omega u'\cdot Mu\,dx\Big]_S^T - \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u'\cdot Mu\,dxdt$$

$$-\int_S^T E^{\frac{p-1}{2}} \int_\Omega u'\cdot Mu' + \mu(Mu)\cdot(\Delta u) + (\lambda+\mu)(Mu)\cdot(\text{grad div}u)\,dxdt.$$

By the definition of the energy and its non-increasigness, it follows easily that

$$\left|\int_\Omega (Mu)\cdot u'\,dx\right| \leq cE(S), \tag{2.1}$$

$$\left|E^{\frac{p-1}{2}}\int_\Omega u'\cdot Mu\,dx\right| \leq cE^{\frac{p+1}{2}} \leq cE(S), \tag{2.2}$$

and

$$\left|E^{\frac{p-3}{2}}E'\int_\Omega u'\cdot Mu\,dx\right| \leq -cE^{\frac{p-1}{2}}E' \leq -c(E^{\frac{p+1}{2}})'. \tag{2.3}$$

Multiplying the first equation in (P) with $E^{\frac{p-1}{2}}2m\cdot\nabla u$, we have

$$0 = \int_S^T E^{\frac{p-1}{2}} \int_\Omega (2m_k\partial_k u_i)(u_i'' - \mu\partial_j^2 u_i - (\lambda+\mu)\partial_i\partial_j u_j)\,dxdt$$

$$= \Big[E^{\frac{p-1}{2}} \int_\Omega (2m_k\partial_k u_i)u_i'\,dx\Big]_S^T - \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega 2m_k u_i'\partial_k u_i\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}} \int_\Omega -m_k\partial_k(u_i')^2 + \mu m_k\partial_k(\partial_j u_i)^2 + 2\mu(\partial_j m_k)(\partial_k u_i)(\partial_j u_j)$$

$$+(\lambda+\mu)m_k\partial_k(\partial_j u_j)^2+2(\lambda+\mu)(\partial_i m_k)(\partial_k u_i)(\partial_j u_j)\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_\Gamma -2\mu\nu_j m_k(\partial_k u_i)(\partial_j u_i)-2(\lambda+\mu)\nu_i m_k(\partial_k u_i)(\partial_j u_i)\,d\Gamma dt$$

$$=\Big[E^{\frac{p-1}{2}}\int_\Omega(2m_k\partial_k u_i)u_i'\,dx\Big]_S^T-\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega 2m_k u_i'\partial_k u_i\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_\Omega(\partial_k m_k)\Big((u_i')^2-\mu(\partial_j u_i)^2-(\lambda+\mu)(\partial_j u_j)^2\Big)$$

$$+2\mu(\partial_j m_k)(\partial_k u_i)(\partial_j u_i)+2(\lambda+\mu)(\partial_i m_k)(\partial_k u_i)(\partial_j u_j)\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_\Gamma -2\mu\nu_j m_k(\partial_k u_i)(\partial_j u_i)-2(\lambda+\mu)\nu_i m_k(\partial_k u_i)(\partial_j u_j)$$

$$+(m_k\nu_k)\Big(-(u_i')^2+\mu(\partial_j u_i)^2+(\lambda+\mu)(\partial_j u_j)^2\Big)\,d\Gamma dt.$$

Since $\partial_k m_k=n,\ \partial_i m_k=\delta_{ik},\ (u_i')^2=|u'|^2,\ (\partial_j u_i)^2=|\nabla u|^2,\ \partial_j u_j=\operatorname{div}u$ and $m_k\nu_k=m\cdot\nu$, we can rewrite this identity in the following form

$$\int_S^T E^{\frac{p-1}{2}}\int_\Omega n|u'|^2+(2-n)\mu|\nabla u|^2+(2-n)(\lambda+\mu)(\mathrm{div}u)^2\,dxdt$$

$$=\Big[E^{\frac{p-1}{2}}\int_\Omega(2m\cdot\nabla u)\cdot u'\,dx\Big]_T^S-\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega 2m_k u_i'\partial_k u_i\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_\Gamma(m\cdot\nu)\Big(|u'|^2-\mu|\nabla u|^2-(\lambda+\mu)(\mathrm{div}u)^2\Big)$$

$$+(2m_k\partial_k u_i)(\mu\partial_\nu u_i+(\lambda+\mu)\nu_i\operatorname{div}u)\,d\Gamma dt.$$

Next we multiply the first equation in (P) with $E^{\frac{p-1}{2}}u$, we obtain

$$0=\int_S^T E^{\frac{p-1}{2}}\int_\Omega u_i(u_i''-\mu\partial_j^2 u_i-(\lambda+\mu)\partial_i\partial_j u_j)\,dxdt$$

$$=\Big[E^{\frac{p-1}{2}}\int_\Omega u_i u_i'\,dx\Big]_S^T-\frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'\cdot u\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_\Omega -(u_i')^2+\mu(\partial_j u_i)^2+(\lambda+\mu)(\partial_i u_i)^2\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_\Gamma -\mu u_i\partial_\nu u_i-(\lambda+\mu)(\nu_i u_i)\partial_j u_j\,d\Gamma dt,$$

and hence

$$\int_S^T E^{\frac{p-1}{2}} \int_\Omega -|u'|^2 + \mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2 \, dxdt$$

$$= \Big[E^{\frac{p-1}{2}} \int_\Omega u \cdot u' \, dx\Big]_T^S - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u' \cdot u \, dxdt$$

$$+ \int_S^T E^{\frac{p-1}{2}} \int_\Gamma u_i(\mu\partial_\nu u_i + (\lambda+\mu)\nu_i \mathrm{div}\, u) \, d\Gamma dt.$$

Multiplying this by $(n-1)$ and adding to the preceding identity, we obtain that

$$\int_S^T E^{\frac{p-1}{2}} \int_\Omega |u'|^2 + \mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2 \, dxdt$$

$$= \Big[E^{\frac{p-1}{2}} \int_\Omega u' \cdot Mu \, dx\Big]_T^S - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u' \cdot Mu \, dxdt$$

$$+ \int_S^T E^{\frac{p-1}{2}} \int_\Gamma (m\cdot\nu)(|u'|^2 - \mu|\nabla u|^2 - (\lambda+\mu)(\mathrm{div} u)^2)$$

$$+ (Mu_i)(\mu\partial_\nu u_i + (\lambda+\mu)\nu_i \mathrm{div} u) \, d\Gamma dt.$$

On Γ_0, we have $u = 0$, whence $u' = 0$, $Mu_i = 2m \cdot \nabla u_i = 2(m\cdot\nu)\partial_\nu u_i$ and $\mathrm{div} u = (\partial_\nu u)\cdot\nu$. Hence the integral on Γ_0 is equal to

$$(m\cdot\nu)\Big(\mu|\frac{\partial u}{\partial\nu}|^2 + (\lambda+\mu)(\mathrm{div} u)^2\Big). \tag{2.4}$$

Furthermore on Γ_1, we have

$$\mu\frac{\partial u}{\partial\nu} + (\lambda+\mu)\nu \mathrm{div} u = -(m\cdot\nu)g(u'). \tag{2.5}$$

Hence

$$\int_S^T E^{\frac{p-1}{2}} \int_\Omega |u'|^2 + \mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2 \, dxdt$$

$$= \Big[E^{\frac{p-1}{2}} \int_\Omega u' \cdot Mu \, dx\Big]_T^S - \frac{p-1}{2} \int_S^T E^{\frac{p-3}{2}} E' \int_\Omega u' \cdot Mu \, dxdt$$

$$+ \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_0} (m\cdot\nu)\Big(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u\Big)^2 \, d\Gamma dt$$

$$+ \int_S^T E^{\frac{p-1}{2}} \int_{\Gamma_1} (m\cdot\nu)\Big(|u'|^2 - \mu|\nabla u|^2 - (\lambda+\mu)(\mathrm{div} u)^2\Big) - (Mu)(m\cdot\nu)g(u') \, d\Gamma dt.$$

Using the fact that $(m\cdot\nu)\le 0$ on Γ_0, we conclude that

$$2\int_S^T E^{\frac{p+1}{2}}(t)\,dt \le \Big[E^{\frac{p-1}{2}}\int_\Omega u'\cdot Mu\,dx\Big]_T^S - \frac{p-1}{2}\int_S^T E^{\frac{p-3}{2}}E'\int_\Omega u'\cdot Mu\,dxdt$$

$$+\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)\{|u'|^2-\mu|\nabla u|^2-(\lambda+\mu)(\mathrm{div}u)^2\}-(m\cdot\nu)(Mu)\cdot g(u')\,d\Gamma dt.$$

As we have

$$-(Mu)\cdot g(u') \le 2|m_k\partial_k u||g(u')| - (n-1)u\cdot g(u')$$

$$\le 2R|\nabla u||g(u')| + |1-n|u\cdot g(u') \quad \text{where} \quad R := \sup_{x\in\Omega}|m(x)|$$

$$\le \mu|\nabla u|^2 + c_\varepsilon|g(u')|^2 + \varepsilon|u|^2$$

for every $\varepsilon > 0$.

Since

$$\int_{\Gamma_1}|u|^2\,d\Gamma \le c\int_\Omega |\nabla u|^2\,dx \le 2cE(t),$$

we get

$$\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)|u|^2\,d\Gamma dt \le c\int_S^T E^{\frac{p+1}{2}}(t)\,dt.$$

Hence, we conclude from (2.1)-(2.3), by taking ε small enough, that

$$(2.6)\quad 2\int_S^T E^{\frac{p+1}{2}}(t)\,dt \le cE(S) + c\int_S^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(m\cdot\nu)(|u'|^2+|g(u')|^2)\,d\Gamma dt.$$

Proof of the main theorem completed. First, we assume that $n=1$, then by (1.10), (1.14), (1.15) and (2.6) we have

$$\int_S^T E(t)\,dt \le cE(S) + c\int_S^T\int_{\Gamma_1}(m\cdot\nu)|u|^2\,d\Gamma dt$$

$$\le cE(S) + c\int_S^T\int_{\Gamma_1,|u'|\ge 1}(m\cdot\nu)|u'|(u'\cdot g(u'))\,d\Gamma dt + c\int_S^T\int_{\Gamma_1,|u'|\le 1}(m\cdot\nu)u'\cdot g(u')\,d\Gamma dt$$

$$\le cE(S) + c(\|u'\|_\infty + 1)\int_S^T(-E'(t))\,dt.$$

Applying the trace theorem $(H^1_{\Gamma_0}(\Omega)\subset)H^1(\Omega)\hookrightarrow L^\infty(\Gamma)$ we have

$$\int_S^T E(t)\,dt \le cE(S) + c(\|u'\|_{H^1(\Omega)}+1)(E(S)-E(T))$$

$$\leq cE(S)$$

and then, we conclude by applying the lemma 1.1.

Assume now that $n \geq 2$, we have by (1.14)

$$\int_{|u'|\leq 1} (m\cdot\nu)(|u'|^2 + |g(u')|^2)\, d\Gamma \leq c \int_{|u'|\leq 1} (m\cdot\nu)(u'\cdot g(u'))^{\frac{2}{p+1}}\, d\Gamma \tag{2.7}$$

$$\leq c\left(\int_{|u'|\leq 1} (m\cdot\nu) u'\cdot g(u')\, d\Gamma\right)^{\frac{2}{p+1}} \leq c(-E')^{\frac{2}{p+1}},$$

hence, the Young inequality gives for every $\varepsilon > 0$ the estimate

$$\int_S^T E^{\frac{p-1}{2}} \int_{|u'|\leq 1} (m\cdot\nu)(|u'|^2 + |g(u')|^2)\, d\Gamma dt \leq c \int_S^T E^{\frac{p-1}{2}} (-E')^{\frac{2}{p+1}}\, dt$$

$$\leq c \int_S^T \varepsilon E^{\frac{p+1}{2}} - c(\varepsilon)E' \leq \varepsilon \int_S^T E^{\frac{p+1}{2}}(t)\, dt + c(\varepsilon)E(S).$$

Whence, (2.6) becomes

$$(2-\varepsilon)\int_S^T E^{\frac{p+1}{2}}(t)\, dt \leq c(\varepsilon)E(S)$$

$$+c\int_S^T E^{\frac{p-1}{2}} \int_{|u'|>1} (m\cdot\nu)(|u'|^2 + |g(u')|^2)\, d\Gamma dt. \tag{2.8}$$

On the other hand we have

$$E^{\frac{p-1}{2}} \int_{|u'|>1} (m\cdot\nu)(|u'|^2 + |g(u')|^2)\, d\Gamma \leq cE^{\frac{p-1}{2}} \int_{|u'|>1} (m\cdot\nu)|u'|^2\, d\Gamma$$

$$\leq cE^{\frac{p-1}{2}} \int_{|u'|>1} (m\cdot\nu)|u'|^{2-s}(u'\cdot g(u'))^s\, d\Gamma, \quad \text{where} \quad s := \frac{2}{p+1}\ (0 < s < 1)$$

$$\leq cE^{\frac{p-1}{2}} \| |u'|^{2-s} \|_{\frac{1}{1-s}} \| (u'\cdot g(u'))^s \|_{\frac{1}{s}}$$

$$= cE^{\frac{p-1}{2}} \|u'\|_\alpha^{(1-s)\alpha} \|u'\cdot g(u')\|_1^s \quad \text{with} \quad \alpha := \frac{2-s}{1-s}$$

$$= cE^{\frac{p-1}{2}} \|u'\|_\alpha^{(1-s)\alpha} (-E')^s \leq \varepsilon E^{\frac{p-1}{2(1-s)}} \|u'\|_\alpha^\alpha - c(\varepsilon)E'$$

$$= \varepsilon E^{\frac{p+1}{2}} \|u'\|_\alpha^\alpha - c(\varepsilon)E'.$$

Using the trace theorem

$$H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-1}}(\Gamma) = L^\alpha(\Gamma)$$

we conclude that

$$E^{\frac{p-1}{2}} \int_{|u'|>1} (m \cdot \nu)|u'|^2 \, d\Gamma \leq c\varepsilon E^{\frac{p+1}{2}} - c(\varepsilon)E',$$

and hence (2.8) becomes

$$(2 - c\varepsilon) \int_S^T E^{\frac{p+1}{2}}(t)\, dt \leq c(\varepsilon)E(S). \tag{2.9}$$

Choosing $\varepsilon = \frac{1}{c}$ (for example), (2.9) yields

$$\int_S^T E^{\frac{p+1}{2}}(t)\, dt \leq cE(S) \tag{2.10}$$

and lemma 1.1 gives the desired decay estimate.

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