SUT Journal of Mathematics
Vol. 32, No. 1 (1996), 91–99

WEIGHTED INEQUALITIES FOR MAXIMAL OPERATORS VIA ATOMIC DECOMPOSITIONS OF TENT SPACES

Yves Rakotondratsimba

(Received March 11, 1996)

Abstract. It is shown how the atomic decomposition of tent spaces can be used to get a characterization of the weight functions u, v for which the fractional maximal operators M_s sends the weighted Lebesgue spaces L^p_v into L^q_u with $1 < p < \infty$, $0 < q < \infty$.

AMS 1991 *Mathematics Subject Classification.* 42B25

Key words and phrases. Weighted inequalities, Maximal operators, Tent spaces

§1 Introduction and Results

The fractional maximal operator M_s of order s, $0 \leq s < n$, acts on locally integrable functions of $\mathbb{R}^n$ as

$$(M_s f)(x) = \sup\Big\{|Q|^{\frac{s}{n}-1} \int_Q |f(y)|dy; \quad Q \quad \text{cube with} \quad Q \ni x\Big\}.$$

The cubes considered have always their sides parallel to the coordinate axes. Here $M = M_0$ is the well known Hardy-Littlewood maximal operator.

Let $u(.)$, $v(.)$ be weight functions on $\mathbb{R}^n$, i.e. nonnegative locally integrable functions. For $1 < p \leq q < \infty$, the two weight norm inequality

$$(1.1) \quad \Big\|(M_s f)(.)\Big\|_{L^q_u} = \left(\int_{\mathbb{R}^n} (M_s f)^q(y)u(y)dy\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} f^p(y)v(y)dy\right)^{\frac{1}{p}} = C\Big\|f(.)\Big\|_{L^p_v} \quad \text{for all} \quad f(.) \geq 0$$

was first characterized by Sawyer [Sa] by the condition

$$(1.2) \quad \Big\|(M_s \sigma 1\!\!1_Q)(.)1\!\!1_Q(.)\Big\|_{L^q_u} \leq S\Big\|\sigma(.)1\!\!1_Q(.)\Big\|_{L^p_v} < \infty \quad \text{for all cubes} \quad Q.$$

Here C and S are nonnegative fixed constants, $\sigma(.) = v^{-\frac{1}{p-1}}(.)$, and $1\!\!1_E(.)$ is the characteristic function of the measurable set E.

Our purpose is to provide a proof of this famous Sawyer's theorem, by using the atomic decompositions of tent spaces introduced by Coifman, Meyer, Stein [Co-Me-St]. The idea of getting maximal inequalities from notions on tents spaces is known. However the systematic use for the two weight inequality (1.1) seems not clear in the literature. The motivation in writing this paper follows after investigations on vector valued inequalities and weighted inequalities in Lorentz and Orlicz spaces for maximal operators. So it appears (see a forthcoming paper) that atomic decomposition of tent spaces are powerful and convenient tools to tackle weighted inequalities. We present here the basic elements of this unified approach.

A condition required for the inequality (1.1) is

$$\frac{1}{p} - \frac{1}{q} \leq \frac{s}{n}.$$

Indeed (1.1) implies $|Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\frac{1}{|Q|}\int_Q u(y)dy\Big)^{\frac{1}{q}} \leq C\Big(\frac{1}{|Q|}\int_Q v(y)dy\Big)^{\frac{1}{p}}$ for all cubes Q, and taking $Q \ni x$ with $|Q| \to 0$ then, by the Lebesgue differentiation theorem, the above restriction on s, n, p, q holds. Consequently for the Hardy-Littlewood maximal operator $M = M_0$ (i.e. $s = 0$) the two weight norm inequality (1.1) has only a sense when $q \leq p$. The problem for $p = q$ is treated by the Sawyer's theorem quoted above, and the case $q < p$ is solved by Verbitsky [Ve].

Now we first see how the necessary condition (1.2) is also sufficient to obtain inequality (1.1). As proved in [Sa], the real problem remains to get a dyadic version of (1.1) which can be stated as

Theorem 1. *Let $1 < p \leq q < \infty$. Assume the condition* (1.2) *is satisfied for all dyadic cubes. Then there is $C > 0$ such that*

$$\Big\|(M_s^{dya}f)(.)\Big\|_{L^q_u} \leq CS\,\|f(.)\|_{L^p_v} \quad \textit{for all} \quad f(.) \geq 0 \tag{1.3}$$

where $(M_s^{dya}f)(x) = \sup\Big\{|Q|^{\frac{s}{n}-1}\int_Q |f(y)|dy;\quad Q \textit{ dyadic cube with } Q \ni x\Big\}.$

The constant C only depends on s, n, p and q. This result was first due to Sawyer [Sa], but here we provide a proof (see the next paragraph) based on atomic decomposition of tent spaces.

For weight functions $v(.)$ with $\sigma(.) = v^{-\frac{1}{p-1}}(.)$ satisfying the usual Muckenhoupt condition A_∞ [Ga-FR], Pérez [Pe] proved that (1.1) is equivalent to the simpler condition

$$|Q|^{\frac{s}{n}-1}\Big(\int_Q u(y)dy\Big)^{\frac{1}{q}}\Big(\int_Q \sigma(y)dy\Big)^{1-\frac{1}{p}} \leq A > 0 \quad \text{for all cubes } Q. \tag{1.4}$$

Clearly (1.4) is also a characterizing condition for (1.1) whenever there is $C > 0$ such that

$$(\mathcal{C}) \qquad (M_s\sigma 1\!\!1_Q)(.)1\!\!1_Q(.) \le C|Q|^{\frac{s}{n}-1}\Big(\int_Q \sigma(y)dy\Big)1\!\!1_Q(.).$$

Weight functions $\sigma(.)$ satisfying $(\mathcal{C})$ can be given by

Proposition 2. *Inequality* $(\mathcal{C})$ *is satisfied whenever* $\sigma(.) \in RD_\rho$ *with* $\rho > 0$ *and* $1 - \frac{s}{n} \le \rho$.

The condition $\sigma(.) \in RD_\rho$ means $\int_{Q'} \sigma(y)dy \le R\Big(\frac{|Q'|}{|Q|}\Big)^\rho \int_Q \sigma(y)dy$ for all cubes Q', Q with $Q' \subset Q$. Any doubling weight functions $\sigma(.)$ (and in particular any A_∞ weight function) satisfies the RD_ρ condition for some $\rho > 0$.

Clearly a necessary condition for (1.1) is

$$(1.5) \qquad \Big\|\Big(M_s\big[\sigma\sum_j \alpha_j 1\!\!1_{Q_j}\big]\Big)(.)1\!\!1_{\bigcup_j Q_j}(.)\Big\|_{L^q_u} \le S\Big\|\sigma(.)\sum_j \alpha_j 1\!\!1_{Q_j}(.)\Big\|_{L^p_v}$$

for all dyadic cubes Q_j and all $\alpha_j > 0$. This condition can be seen as a generalization of the Sawyer's one (1.2). By density argument, in considering $\varepsilon\sigma(.)1\!\!1_{[-N,N]^n}(.) + \sigma(.)\sum_j \alpha_j 1\!\!1_{Q_j}(.)$ with $\varepsilon \to 0$ and $N \to \infty$, then (1.5) becomes also a sufficient condition for $M_s\colon L^p_v \to L^q_u$.

Although (1.5) is a characterizing condition for (1.1) for $q < p$, it is unfortunately too complicated for any practical use. Studies of more convenient conditions for computations and their connections with results in [Ve] have been made by the author in [Ra].

Theorem 3. *Let* $1 < p < \infty$, $0 < q < \infty$. *Suppose for a constant* $S > 0$

$$(1.6) \qquad \Big\|\sum_j \alpha_j(M_s\sigma 1\!\!1_{Q_j})(.)1\!\!1_{Q_j}(.)\Big\|_{L^q_u} \le S\Big\|\sigma(.)\sum_j \alpha_j 1\!\!1_{Q_j}(.)\Big\|_{L^p_v}$$

for all cubes Q_j *and all* $\alpha_j > 0$. *Then the embedding* (1.1) *holds. Also* (1.6) *becomes a necessary condition for* (1.1) *whenever* $u(.)$ *satisfies the* A_∞ *condition. In particular for* $0 < s < n$ *then*

$$(1.7) \qquad \big\|(I_sf)(.)\big\|_{L^q_u} \le C\big\|f(.)\big\|_{L^p_v} \quad \text{for all } f(.) \ge 0,$$

if and only if (1.6) *is true, and where* I_s *is the fractional integral operator* $(I_sf)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n}f(y)dy$.

If moreover $\sigma(.)$ satisfies $(\mathcal{C})$ then, by Proposition 2, the condition (1.6) in this result can be replaced by

$$(1.8) \qquad \Big\|\sum_j \alpha_j\Big(|Q_j|^{\frac{s}{n}-1}\int_{Q_j}\sigma(y)dy\Big)1\!\!1_{Q_j}(.)\Big\|_{L^q_u} \le S\Big\|\sigma(.)\sum_j \alpha_j 1\!\!1_{Q_j}(.)\Big\|_{L^p_v}.$$

Condition like (1.8) have been introduced by Chanillo, Strömberg, Wheeden in [Ch-St-Wh] in order to derive similar results, by using atomic decomposition of weighted Hardy spaces.

§2 Proofs of Results

Proof of Theorem 1

To prove Theorem 1, as done in [Sa] by applying translations and reflections of the cone $[0,\infty[^n$, it is sufficient to find $C>0$ such as

$$\Big\|(M_s^Rf)(.)1\!\!1_{[0,\infty[^n}(.)\Big\|_{L_u^q} \le CS\Big\|f(.)\Big\|_{L_v^p} \quad \text{for all} \quad R>0, \tag{2.1}$$

where $(M_s^Rf)(x) = \sup\Big\{|Q|^{\frac{s}{n}-1}\int_Q |f(y)|dy\ ;\ Q$ dyadic cubes with $Q\ni x$ and $Q\subset (]0,R[)^n\Big\}$. The estimates which will be obtained do not depend on $R>0$. We first state a basic lemma on which lies the above results.

Lemma 1. *Let $1<p<\infty$, and $0\le s<n$. There is $C>0$ such that, for all $f(.)\in L_v^p$ and $R>0$, one can find scalars $\lambda_j>0$, and dyadic cubes Q_j for which*

$$\Big(\sum_j \lambda_j^p\Big)^{\frac{1}{p}} \le C\Big\|f(.)\Big\|_{L_v^p} \tag{2.2}$$

$$\Big\|\sum_j \lambda_j |Q_j|_\sigma^{-\frac{1}{p}} 1\!\!1_{Q_j}(.)\Big\|_{L_\sigma^p} \le C\Big\|f(.)\Big\|_{L_v^p} \tag{2.3}$$

$$(M_s^Rf)^r(.) \le \sum_j \lambda_j^r |Q_j|_\sigma^{-\frac{r}{p}}(M_s\sigma 1\!\!1_{Q_j})^r(.)1\!\!1_{Q_j}(.) \quad \textit{for all } r>0, \tag{2.4}$$

here $|Q|_\sigma$ is defined as $\int_Q \sigma(y)dy$.

Inequality (2.4) holds for almost $x\in]0,\infty[^n$, and precisely for non dyadic points (see below). Technical difficulties due to $0\times\infty$ can be avoided by truncating the weight function $v(.)$, and by observing that all estimates do not call on the bound of this weight. For the convenience, formal operations will be done and as usual $0\times\infty$ is taken as 0.

Lemma 1 contains the whole philosophy of weighted inequalities (1.1). Indeed inequality (2.4) yields a cut off of $(M_s^Rf)(.)$. Summation of the resulting pieces is ensured by (2.2) [resp. (2.3)]. Deferring after the proof of this Lemma, the remaining of the proof of Theorem 1 is very easy.

Remind that $p\le q$ or $\frac{q}{p}\ge 1$. Using the fact that $u(.)$ does not charge (dyadic) points, then

$$\begin{aligned}\Big\|(M_s^Rf)(.)1\!\!1_{[0,\infty[^n}(.)\Big\|_{L_u^q}^p &= \Big\|(M_s^Rf)^p(.)1\!\!1_{[0,\infty[^n}(.)\Big\|_{L_u^{\frac{q}{p}}}\\ &\le \sum_j \lambda_j^p\Big(|Q_j|_\sigma^{-\frac{1}{p}}\Big\|(M_s\sigma 1\!\!1_{Q_j})(.)1\!\!1_{Q_j}(.)\Big\|_{L_u^q}\Big)^p \quad \text{by (2.4)}\end{aligned}$$

$$\leq S^p \sum_j \lambda_j^p \leq CS^p \Big\| f(.) \Big\|^p_{L^p_v} \text{ by using condition (1.2) and (2.2).}$$

So Theorem 1 can be derived from these estimates.

Proof of Theorem 3

As above, instead of (1.1), it is sufficient to get the dyadic inequality (1.3). And this last is obtained as follows

$$\begin{aligned}
\|(M_s^R f)(.)\mathbb{1}_{[0,\infty[^n}(.)\|_{L^q_u} &\leq \Big\| \sum_j \lambda_j |Q_j|_\sigma^{-\frac{1}{p}} (M_s \sigma \mathbb{1}_{Q_j})(.)\mathbb{1}_{Q_j}(.) \Big\|_{L^q_u} \quad \text{by (2.4)} \\
&\leq S \Big\| \sigma(.) \sum_j \lambda_j |Q_j|_\sigma^{-\frac{1}{p}} \mathbb{1}_{Q_j}(.) \Big\|_{L^p_v} \quad \text{by condition (1.6)} \\
&= S \Big\| \sum_j \lambda_j |Q_j|_\sigma^{-\frac{1}{p}} \mathbb{1}_{Q_j}(.) \Big\|_{L^p_\sigma} \quad \text{recall that } \sigma(.) = v^{-\frac{1}{p-1}}(.) \\
&\leq CS \Big\| f(.) \Big\|_{L^p_v} \quad \text{by (2.3).}
\end{aligned}$$

By a Muckenhoupt-Wheeden's inequality [Mu-Wh], for $0 < s < n$ and $u(.)$ satisfying the A_∞ condition then $\|(M_s f)(.)\|_{L^q_u} \approx \|(I_s f)(.)\|_{L^q_u}$, so the embedding (1.1) becomes equivalent to (1.7). Consequently the necessity of condition (1.6) appears by taking $f(.) = \sigma(.) \sum_j \alpha_j \mathbb{1}_{Q_j}(.)$ in (1.7) and by using the linearity of I_s and also the fact that $(M_s g)(.) \leq c(s,n)(I_s g)(.)$.

Proof of Proposition 2

To get inequality ($\mathcal{C}$) let Q_0 be a cube and $x \in Q_0$. It is sufficient to estimate $\mathcal{Q} = |Q|^{\frac{s}{n}-1} \int_{Q \cap Q_0} \sigma(y)dy$ by $C|Q_0|^{\frac{s}{n}-1} \int_{Q_0} \sigma(y)dy$. Here $Q \cap Q_0 \ni x$ and $C > 0$ is a constant which depends only on s, n and the constant on the RD_ρ condition.

If $\frac{1}{3}|Q_0|^{\frac{1}{n}} \leq |Q|^{\frac{1}{n}}$ then clearly $\mathcal{Q} \leq c(s,n)|Q_0|^{\frac{s}{n}-1} \int_{Q_0} \sigma(y)dy$. Now suppose $|Q|^{\frac{1}{n}} \leq \frac{1}{3}|Q_0|^{\frac{1}{n}}$. If $Q \subset Q_0$ then $\mathcal{Q} \leq C|Q_0|^{\frac{s}{n}-1} \int_{Q_0} \sigma(y)dy$, by the condition $\sigma(.) \in RD_\rho$ with $n - s \leq n\rho$. Otherwise there is another cube $Q_1 \subset Q_0$ such that $|Q_1| = |Q|$ and $Q \cap Q_0 \subset Q_1$. By using the same condition, then $\mathcal{Q} \leq |Q_1|^{\frac{s}{n}-1} \int_{Q_1} \sigma(y)dy \leq C|Q_0|^{\frac{s}{n}-1} \int_{Q_0} \sigma(y)dy$.

Preliminaries for the Proof of Lemma 1

For the convenience the essential notions on tent spaces [Co-Me-St], used for the proof of Lemma 1, are reminded.

Let X be the cone $[0,\infty[^n$ minus the set of dyadics points, i.e. $X = [0,\infty[^n - \{(2^{-l}k_j)_j; l \in \mathbb{Z} \text{ and } (k_j)_j \in \mathbb{N}^n\}$. The upper half-space is defined by $\widetilde{X} = X \times \{2^{-l};\ \ l \in \mathbb{Z}\}$. For each couple $(y,w) \in \widetilde{X}$ there is an unique (open) dyadic cube $Q = Q_{yw}$ which contains y and with the side length $w = 2^{-l}$. We write

$$(y,w) \in \widetilde{\Gamma}(x) \quad \text{if and only if} \quad x \in Q_{yw}. \tag{2.5}$$

And

$$\widehat{\Omega} = \left(\bigcup\{\widetilde{\Gamma}(x); \quad x \in \Omega^c\}\right)^c \quad \text{for each set } \Omega \subset [0, \infty[^n. \tag{2.6}$$

Thus

$$(y, w) \in \widehat{\Omega} \quad \text{if and only if} \quad Q_{yw} \subset \Omega. \tag{2.7}$$

Finally the functional $\mathcal{A}_\infty$ acting on each measurable function $\widetilde{f}$ on $\widetilde{X}$ is defined by

$$(\mathcal{A}_\infty \widetilde{f})(x) = \sup\{|\widetilde{f}(y, w)|; \ (y, w) \in \widetilde{\Gamma}(x)\}. \tag{2.8}$$

Then we have the following (atomic decomposition)

Lemma 2. *Let $0 < p < \infty$. There is $C > 0$ such that, for all functions $\widetilde{f}(y, w)$ with support contained in $\widehat{(]0, R[^n)}$ $(R > 0)$ and $\|(\mathcal{A}_\infty \widetilde{f})(.)\|_{L^p_\sigma} < \infty$, one can find scalars $\lambda_j > 0$, dyadic cubes Q_j, and functions $\widetilde{a}_j(y, w)$ which satisfy:*

(2.9)

$$\text{the supports of the } \widetilde{a}_j \text{ are disjoint and } \quad |\widetilde{a}_j(y, w)| \leq |Q_j|_\sigma^{-\frac{1}{p}} \widetilde{\mathrm{1\!I}}_{\widehat{Q}_j}(y, w);$$

$$\widetilde{f}(y, w) = \sum_j \lambda_j \widetilde{a}_j(y, w) \quad \text{a.e.}; \tag{2.10}$$

$$\left(\sum_j \lambda_j^p\right)^{\frac{1}{p}} \leq C \left\|(\mathcal{A}_\infty \widetilde{f})(.)\right\|_{L^p_\sigma}; \tag{2.11}$$

$$\sum_j \left(\lambda_j |Q_j|_\sigma^{-\frac{1}{p}}\right)^r \mathrm{1\!I}_{Q_j}(.) \leq C(\mathcal{A}_\infty \widetilde{f})^r(.) \quad \text{for all} \quad r > 0. \tag{2.12}$$

Here $\widetilde{\mathrm{1\!I}}_{\widehat{Q}}(., .)$ is the characteristic function of the tent $\widehat{Q}$.

The notions introduced in [Co-Me-St] are not exactly quoted above, since here the dyadic version is presented. Lemma 2 can be obtained by doing a slight modification of the proof given in [Co-Me-St], and outlines of proof will be given below.

Proof of Lemma 1

Let $f(.) \in L^p_v$ and $R > 0$. First observe that

$$(M_s^R f)(x) = \sup\left\{|Q_{yw}|^{\frac{s}{n}-1} \int_{Q_{yw}} |f(y)| dy; \ Q_{yw} \ni x \text{ and } Q_{yw} \subset (]0, R[)^n\right\}$$

$$= \sup\Big\{\Big(|Q_{yw}|^{\frac{s}{n}-1}|Q_{yw}|_\sigma\Big)\widetilde{f}(y,w);\ Q_{yw}\ni x \text{ and } Q_{yw}\subset (]0,R[)^n\Big\}$$

for each $x\in X$. Here $\widetilde{f}(y,w)$ can be considered as a function supported by $\widehat{]0,R[^n}$, and on this support

$$\widetilde{f}(y,w)=\widetilde{f}_\sigma(y,w)=|Q_{yw}|_\sigma^{-1}\int_{Q_{yw}}|(\sigma^{-1}f)(y)|\sigma(y)dy. \tag{2.13}$$

By (2.8) and (2.5) then

$$(\mathcal{A}_\infty\widetilde{f})(.)\le (N_\sigma[\sigma^{-1}f])(.) \tag{2.14}$$

with $(N_\sigma g)(x)=\sup\Big\{|Q|_\sigma^{-1}\int_Q|g(y)|\sigma(y)dy;\ Q \text{ is a dyadic cube with } Q\ni x\Big\}$. Since (by interpolation) N_σ is bounded on L^p_σ (see [Ga-RF]) and $\sigma^{1-p}(.)=v(.)$, then for a constant $c>0$ which depends only on n and p:

(2.15)
$$\Big\|(\mathcal{A}_\infty\widetilde{f})(.)\Big\|_{L^p_\sigma}\le\Big\|(N_\sigma[\sigma^{-1}f])(.)\Big\|_{L^p_\sigma}\le c\Big\|\sigma^{-1}(.)f(.)\Big\|_{L^p_\sigma}=c\Big\|f(.)\Big\|_{L^p_v}<\infty.$$

Consequently by Lemma 2, $\widetilde{f}(.,.)$ can be decomposed as in (2.9)— (2.12).

Inequalities (2.11) [resp. (2.12) with $r=1$] and (2.15) yield (2.2) [resp. (2.3)]. To get (2.4) let $r>0$ and $Q_{yw}\ni x$. Then

$$\begin{aligned}
&\Big(|Q_{yw}|^{\frac{s}{n}-1}|Q_{yw}|_\sigma\Big)^r\widetilde{f}^r(y,w)\\
&\quad=\Big(|Q_{yw}|^{\frac{s}{n}-1}|Q_{yw}|_\sigma\Big)^r\sum_j\lambda_j^r\widetilde{a}_j^r(y,w)\quad\text{(the support of the } \widetilde{a}_j\text{'s are disjoint)}\\
&\quad\le\sum_j\lambda_j^r|Q_j|_\sigma^{-\frac{r}{p}}\Big[|Q_{yw}|^{\frac{s}{n}-1}|Q_{yw}|_\sigma\widetilde{1\!\mathrm{I}}_{\widehat{Q}_j}(y,w)\Big]^r\quad\text{by (2.9)}\\
&\quad\le\sum_j\lambda_j^r|Q_j|_\sigma^{-\frac{r}{p}}\Big[|Q_{yw}|^{\frac{s}{n}-1}|Q_{yw}\cap Q_j|_\sigma\widetilde{1\!\mathrm{I}}_{\widehat{Q}_j}(y,w)\Big]^r\\
&\qquad\qquad\text{by (2.7): } Q_{yw}\subset Q_j\\
&\quad=\sum_j\lambda_j^r|Q_j|_\sigma^{-\frac{r}{p}}\Big[\Big(|Q_{yw}|^{\frac{s}{n}-1}\int_{Q_{yw}}\sigma(y)1\!\mathrm{I}_{Q_j}(y)dy\Big)\widetilde{1\!\mathrm{I}}_{\widehat{Q}_j}(y,w)\Big]^r\\
&\quad\le\sum_j\lambda_j^r|Q_j|_\sigma^{-\frac{r}{p}}(M_s\sigma1\!\mathrm{I}_{Q_j})^r(x)1\!\mathrm{I}_{Q_j}(x)\quad\text{recall that } x\in Q_{yw}\subset Q_j.
\end{aligned}$$

Taking the supremum on cubes $Q_{yw}\ni x$ with $Q_{yw}\subset]0,R[^n$, then (2.4) appears.

Proof of Lemma 2

Let $\Omega_k = \{(A_\infty \widetilde{f})(.) > 2^k\}$, where k is an integer. Since $(A_\infty \widetilde{f})(.)$ is supported by $[0, R[^n$ then one can find dyadic cubes such that $\Omega_k = \bigcup_j Q_{jk}$, with disjoint interiors. Then $\widehat{\Omega}_k = \bigcup_j \widehat{Q}_{jk}$, and $\widehat{\Omega}_{k+1} \subset \widehat{\Omega}_k$. On the other hand $|\widetilde{f}(y,w)| \le 2^{k+1}$ on $\widehat{\Omega}^c_{k+1}$. So define scalars and functions

$$\lambda_{jk} = 2^{(k+1)}|Q_{jk}|_\sigma^{\frac{1}{p}},$$
$$\widetilde{a}_{jk}(y,w) = 2^{-(k+1)}|Q_{jk}|_\sigma^{-\frac{1}{p}} \times \widetilde{f}(y,w) \times \widetilde{1\!\!\mathrm{I}}_{\widehat{Q}_{jk} - \widehat{\Omega}_{k+1}}(y,w).$$

Clearly the supports $\widetilde{E}_{jk} = \widehat{Q}_{jk} - \widehat{\Omega}_{k+1}$ of the $\widetilde{a}_{jk}(.,.)$ are disjoint and moreover the estimate in (2.9) holds. The a.e. equality (2.10) also appears since

$$\widetilde{f}(y,w) = \sum_k \widetilde{f}(y,w)\widetilde{1\!\!\mathrm{I}}_{\widehat{\Omega}_k - \widehat{\Omega}_{k+1}}(y,w) = \sum_k \sum_j \widetilde{f}(y,w)\widetilde{1\!\!\mathrm{I}}_{\widehat{Q}_{jk} - \widehat{\Omega}_{k+1}}(y,w)$$
$$= \sum_k \sum_j \lambda_{jk}\widetilde{a}_{jk}(y,w).$$

Inequality (2.11) can be obtained as follows

$$\sum_k \sum_j \lambda^p_{jk} = \sum_k 2^{(k+1)p} \sum_j |Q_{jk}|_\sigma$$
$$\le c \sum_k 2^{kp} |(A_\infty \widetilde{f})(.) > 2^k|_\sigma \le c \Big\| (A_\infty \widetilde{f})(.) \Big\|^p_{L^p_\sigma}.$$

In order to get (2.12), let $r > 0$. Then

$$\sum_k \sum_j \Big[\lambda_{jk}|Q_{jk}|_\sigma^{-\frac{1}{p}}\Big]^r 1\!\!\mathrm{I}_{Q_{jk}}(.) = \sum_k 2^{(k+1)r} \sum_j 1\!\!\mathrm{I}_{Q_{jk}}(.)$$
$$= \sum_k 2^{(k+1)r} 1\!\!\mathrm{I}_{\{(A_\infty \widetilde{f})(.) > 2^k\}}(.)$$
$$= c \sum_{l=0}^{\infty} 2^{-l} \sum_k 2^{(k+l)r} 1\!\!\mathrm{I}_{\{2^{(k+l)} < (A_\infty \widetilde{f})(.) \le 2^{(k+l+1)}\}}(.)$$
$$< c \sum_{l=0}^{\infty} 2^{-l} (A_\infty \widetilde{f})^r(.) = c' (A_\infty \widetilde{f})^r(.).$$

Acknowldegment

I would like to thank Professors J.L. Torrea and R. Macias for encouraging me to write this paper, during my visit at the University Autonoma of Madrid. Also I would like to thank the referee for his helpful comments and suggestions.

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Yves RAKOTONDRATSIMBA
Institut Polytechnique St-Louis, EPMI
13 Bd de l'Hautil 95 092 Cergy-Pontoise cedex, France