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RADIUS OF 3-CONNECTED GRAPHS

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Abstract. We show that if G is a 3-connected graph with radius r, then $r \leq \frac{|V(G)|+15}{4}$.

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§1. Introduction

By a graph, we mean a finite, undirected, simple graph without loops or multiple edges. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v, w \in V(G)$, let $d(v, w)$ denote the usual distance between v and w. Set

$$r(G) := \min_{v \in V(G)} \max_{w \in V(G)} d(v, w).$$

The number $r(G)$ is called the radius of G. A vertex $z \in V(G)$ is called a central vertex of G if $\max_{w \in V(G)} d(z, w) = r(G)$.

In [1], Harant and Walther proved that the inequality $r < \frac{n}{4} + O(\log n)$ holds for a 3-connected graph with radius r containing precisely n vertices, where O denotes the order as n tends to infinity. The purpose of this paper is to prove the following theorem.

Theorem. *Let G be a 3-connected graph with radius r containing precisely n vertices. Then the following inequality holds:*

$$r \leq \frac{n+15}{4}.$$

§2. Preliminary Results

Throughout the rest of the paper, we let G, n, r be as in the Theorem. For a vertex $v \in V(G)$ and a nonnegative integer i, let

$$N_i(v) := \{w | w \in V(G), d(v, w) = i\}.$$

We write $N(v)$ for $N_1(v)$. Fix a central vertex z, and let

$$X_i := N_i(z) \qquad for \quad 0 \le i \le r.$$

Note that for each i with $1 \le i \le r-1$ and each $x \in X_i$, $N(x) \subset X_{i-1} \cup X_i \cup X_{i+1}$.

Lemma 2.1. *$|X_i| \ge 3$ for all i with $1 \le i \le r-1$.*

Proof. Since $G - X_i$ is disconnected, the desired conclusion immediately follows from the 3-connectedness of G. □

Lemma 2.2. *$n \ge 3r - 1$.*

Proof. By Lemma 2.1, $n = \sum_{i=0}^{r} |X_i| \ge 1 + 3(r-1) + 1 = 3r - 1$. □

Let i, j be integers with $0 \le i, j \le r$. For $v, w \in X_i$, we let

$$M_j(v) := N_{|j-i|}(v) \cap X_j \qquad and \qquad M_j(v, w) := M_j(v) \cup M_j(w).$$

Lemma 2.3. *Let $1 \le a < i \le r$. Suppose that $|X_a| = 3$, and write $X_a = \{u_1, u_2, u_3\}$. Let $j \in \{1, 2, 3\}$. Write $\{1, 2, 3\} = \{j, k, l\}$ and suppose that for each h ($a \le h < i$), $d(w_1, w_2) \ge 3$ for any $w_1 \in M_h(u_j)$ and any $w_2 \in M_h(u_k, u_l)$. Then the following hold.*

(1) (a) $|M_i(u_j)| \ge 1$. (b) If $i \ge a + 2$, $|M_i(u_j)| \ge 2$.

(2) $|M_i(u_k, u_l)| \ge 2$.

Proof. From the assumptions of the lemma, it follows that $G - (\{u_k, u_l\} \cup M_i(u_j))$ is disconnected, and hence (1)(a) follows from the assumption that G is 3-connected. Similarly, $G - (\{u_j\} \cup M_i(u_k, u_l))$ is disconnected, and, in the case where $i \ge a + 2$, $G - (\{u_j\} \cup M_i(u_j))$ is also disconnected, and hence (1)(b) and (2) also follow from the 3-connectedness of G. □

§3. Proof of the Theorem

We continue with the notation of the preceding section. The bulk of the proof of the Theorem is devoted to the verification of the following lemma, which roughly says that the average of the $|X_i|$ is only slightly less than four, if it is less than four:

Lemma 3.1. *Let a, b be integers with $a \geq 7$, $a+2 \leq b \leq r-6$, and suppose that $|X_a| = |X_b| = 3$ and $|X_i| > 3$ for all i with $a+1 < i < b$. Then*

$$\sum_{i=a}^{b-1} |X_i| \geq 4(b-a).$$

To prove the lemma, suppose, by way of contradiction, that $\sum_{i=a}^{b-1} |X_i| < 4(b-a)$. Then one of the following two situations must occur:
(A) $|X_i| = 4$ for all $a < i < b$; or
(B) $|X_{a+1}| = 3$, and $|X_i| = 4$ or 5 for each $a+1 < i < b$, and the number of those indices i with $a+1 < i < b$ for which $|X_i| = 5$ is at most one.

We define an integer C as follows. Fix $j \in \{1,2,3\}$ for the moment and write $\{j,k,l\} = \{1,2,3\}$. Set

$$Q_j := \left\{ i \,\middle|\, a \leq i < b, \begin{array}{c} \textit{there exists } w_1 \in M_i(u_j) \textit{ and there exists} \\ w_2 \in M_i(u_k,u_l) \textit{ such that } d(w_1,w_2) \leq 2 \end{array} \right\}.$$

If $Q_j = \emptyset$, then $|X_b| = |M_b(u_j)| + |M_b(u_k,u_l)| \geq 2+2 = 4$ by Lemma 2.3, which contradicts the assumption that $|X_b| = 3$. Thus Q_j is not an empty set.

Having this in mind, we define $C_j = \min Q_j$ for each $j \in \{1,2,3\}$, and let

$$C = \max\{C_1, C_2, C_3\}.$$

We now relabel u_1, u_2, u_3 so that $C = \max\{C_1, C_2, C_3\} = C_1$.

The following remarks immediately follow from the definition of C.

Remark 3.2. For each i with $a \leq i \leq C$, we have $X_i - M_i(u_1) = M_i(u_2,u_3)$.

Remark 3.3. For each i with $a+1 \leq i \leq C-1$, we have $N(x) \subset M_{i-1}(u_1) \cup M_i(u_1) \cup M_{i+1}(u_1)$ for any $x \in M_i(u_1)$, and $N(y) \subset M_{i-1}(u_2,u_3) \cup M_i(u_2,u_3) \cup M_{i+1}(u_2,u_3)$ for any $y \in M_i(u_2,u_3)$.

The following two claims also immediately follow from Lemma 2.3.

Claim 1. Suppose that $C \geq a+1$.
(1) If (A) holds, then $|M_{a+1}(u_1)| = 1$ or 2, and $|M_i(u_1)| = 2$ for each $a+2 \leq i \leq C$.
(2) If (B) holds, then $|M_{a+1}(u_1)| = 1$, $|M_i(u_1)| = 2$ or 3 for each $a+2 \leq i \leq C$, and the number of those indices i with $a+2 \leq i \leq C$ for which $|M_i(u_1)| = 3$ is at most one.

Claim 2. Suppose that $C \geq a+1$.
(1) If (A) holds, then $|M_{a+1}(u_2, u_3)| = 2$ or 3, and $|M_i(u_2, u_3)| = 2$ for each $a+2 \leq i \leq C$.
(2) If (B) holds, then $|M_i(u_2, u_3)| = 2$ or 3 for each $a+1 \leq i \leq C$, and the number of those indices i with $a+1 \leq i \leq C$ for which $|M_i(u_2, u_3)| = 3$ is at most one.

Claim 3. If $C \geq a+3$, then
$|M_{i-1}(u_1) \cup M_i(u_1) \cup M_{i+1}(u_1)| \leq 7$ for each $a+2 \leq i \leq C-1$.

Proof. Since Claim 1 implies that $|M_i(u_1)| \leq 3$ for each $a+1 \leq i \leq C$, and that the number of indices i with $a+1 \leq i \leq C$ such that $|M_i(u_1)| = 3$ is at most one, the desired inequality follows immediately. □

Claim 4. If $C \geq a+3$, then
$|M_{i-1}(u_2, u_3) \cup M_i(u_2, u_3) \cup M_{i+1}(u_2, u_3)| \leq 7$ for each $a+2 \leq i \leq C-1$.

Proof. Since Claim 2 implies that $|M_i(u_2, u_3)| \leq 3$ for each $a+1 \leq i \leq C$, and that the number of indices i with $a+1 \leq i \leq C$ such that $|M_i(u_2, u_3)| = 3$ is at most one, the desired inequality follows immediately. □

Claim 5. Suppose that $C \geq a+3$, and let $a+2 \leq i \leq C-1$.
(1) For any $x, x' \in M_i(u_1)$, $d(x, x') \leq 2$.
(2) For any $y, y' \in M_i(u_2, u_3)$, $d(y, y') \leq 2$.

Proof. Take $x, x' \in M_i(u_1)$. If $x = x'$ or $xx' \in E(G)$, then we clearly have $d(x, x') \leq 2$. Thus assume $x \neq x'$ and $xx' \notin E(G)$. Then

$$N(x) \cup N(x') \subset M_{i-1}(u_1) \cup M_i(u_1) \cup M_{i+1}(u_1) - \{x, x'\}.$$

Since $|M_{i-1}(u_1) \cup M_i(u_1) \cup M_{i+1}(u_1)| \leq 7$ by Claim 3, this implies

$$|N(x) \cup N(x')| \leq |M_{i-1}(u_1) \cup M_i(u_1) \cup M_{i+1}(u_1) - \{x, x'\}| \leq 7 - 2 = 5.$$

On the other hand, since G is 3-connected, $|N(x)| \geq 3$ and $|N(x')| \geq 3$. Consequently, $N(x) \cap N(x') \neq \emptyset$, and hence $d(x, x') \leq 2$. We can prove (2) in exactly the same way by using Claim 4 in place of Claim 3. □

Claim 6. Suppose that $C \geq a+3$.
(1) For any $x, x' \in M_C(u_1)$, $\quad d(x, x') \leq 4$.
(2) For any $y, y' \in M_C(u_2, u_3)$, $\quad d(y, y') \leq 4$.

Proof. Take $x, x' \in M_C(u_1)$. Then there exist $x_1, x_1' \in M_{C-1}(u_1)$ with $xx_1, x'x_1' \in E(G)$. By Claim 5, $d(x_1, x_1') \leq 2$. We get $d(x, x') \leq d(x, x_1) + d(x_1, x_1') + d(x_1', x') \leq 1+2+1 = 4$. Thus (1) is proved, and (2) can be proved in exactly the same way. □

Claim 7.
(1) If $C \leq a+2$, then there exists $u \in X_a$ such that $d(u, u') \leq 6$ for every $u' \in X_a$.
(2) If $C \geq a+3$, then there exists $w \in X_C$ such that $d(w, w') \leq 6$ for every $w' \in X_C$.

Proof. By the definition of C, there exist $w_1 \in M_C(u_1)$ and $w_2 \in M_C(u_2, u_3)$ such that $d(w_1, w_2) \leq 2$.

(1) Since $w_2 \in M_C(u_2, u_3)$, there is $u_j \in X_a$, $u_j \neq u_1$, such that $d(w_2, u_j) = C - a$. Then $d(u_1, w_1) = d(w_2, u_j) = C - a \leq 2$, and hence

$$d(u_1, u_j) \leq d(u_1, w_1) + d(w_1, w_2) + d(w_2, u_j) \leq 2+2+2 = 6.$$

Now take $u_k \in X_a$ so that $u_k \neq u_1$ and $u_k \neq u_j$. Since $C = \max C_i$, $C_k \leq C = a+2$. Thus arguing as above, we see that there exists $u_l \in X_a$ with $d(u_k, u_l) \leq 6$. Since $|X_a| = 3$, u_l is either u_1 or u_j. Set $u = u_l$. Then this u satisfies the desired condition.

(2) Set $w = w_1 \in M_C(u_1) \subset X_C$. Let $w' \in X_C$. We show that $d(w, w') \leq 6$. If $w' \in M_C(u_1)$, then Claim 6 implies that $d(w, w') \leq 4 \leq 6$. Thus we may assume $w' \in M_C(u_2, u_3)$. Let w_2 be as in the definition of C. Then $d(w, w_2) = d(w_1, w_2) \leq 2$ by the definition of w_1 and w_2. Since $w_2, w' \in M_C(u_2, u_3)$, we also get $d(w_2, w') \leq 4$ from Claim 6. Consequently, $d(w, w') \leq d(w, w_2) + d(w_2, w') \leq 2+4 = 6$, as desired. □

For convenience, we restate Claim 7 in the following form:

Claim 8. For some m ($a \leq m < b$) and some $v \in X_m$, $d(v, v') \leq 6$ for every $v' \in X_m$.

Proof of Lemma 3.1. Let m and v be as in Claim 8. Observe that $7 \leq a \leq m < b \leq r-6$.

- Case 1. $r-m \le m$.
Let z' be a vertex in X_{r-m} which is on a shortest $z-v$ path. Then $d(z',z) = r-m$ and $d(z',v) = m-(r-m) = 2m-r$. Take $x \in V(G)$, and let $x \in X_k$. First assume that $0 \le k < m$. Then

$$d(z',x) \le d(z',z) + d(z,x) = r-m+k < r-m+m = r.$$

Next assume that $m \le k \le r$. Let v' be a vertex in X_m which is on a shortest $z-x$ path. Then $d(v',x) = k-m \le r-m$. Since $d(v,v') \le 6$ by Claim 8, we get

$$d(z',x) \le d(z',v) + d(v,v') + d(v',x) \le m+6 < r.$$

Thus in either case, $d(z',x) < r$. Since x was arbitrary, this contradicts the fact that r is the radius of G.

- Case 2. $r-m > m$.
In this case, $2m < r$. Let $z' = v \in X_m$. Then $d(z',z) = m$. Take $x \in V(G)$, and let $x \in X_k$. First assume that $0 \le k < m$. Then

$$d(z',x) \le d(z',z) + d(z,x) = m+k < 2m < r.$$

Next assume that $m \le k \le r$. Let v' be a vertex in X_m which is on a shortest $z-x$ path. Then $d(v',x) = k-m$. Since $d(z',v') = d(v,v') \le 6$ by Claim 8, we get

$$d(z',x) \le d(z',v') + d(v',x) \le 6 + (k-m) \le r + (6-m) < r.$$

Thus in either case, $d(z',x) < r$. Since x was arbitrary, this contradicts the fact that r is the radius of G.

This completes the proof of Lemma 3.1. □

Lemma 3.4. *Suppose that $r \ge 14$. Then $\displaystyle\sum_{i=7}^{r-6} |X_i| \ge 4(r-12) - 2$.*

Proof. Let $I := \{\ i\ | 7 \le i \le r-6,\ \ |X_i| = 3\}$. We may assume $|I| \ge 3$. Let $I = \{i_1, i_2, \cdots, i_{|I|}\}$ with $i_1 < i_2 < \cdots < i_{|I|}$. From I we define a new sequence $j_1 < j_2 < \cdots < j_s$ inductively as follows. Set $j_1 = i_1$. For $l \ge 2$, set $j_l = \min\{i | i \in I,\ \ i \ge j_{l-1} + 2\}$ (if $\{i | i \in I,\ \ i \ge j_{l-1} + 2\} = \emptyset$, then we set $s = l-1$ and terminate this procedure). We have $j_s = j_{|I|}$ or $j_{|I|-1}$ by the definition.

By Lemma 3.1, $\sum_{i=j_{h-1}}^{j_h-1} |X_i| \geq 4(j_h - j_{h-1})$ for all $2 \leq h \leq s$. Taking the sum of these inequalities, we get

$$\sum_{i=j_1}^{j_s-1} |X_i| = \sum_{h=2}^{s} \sum_{i=j_{h-1}}^{j_h-1} |X_i| \geq 4(j_s - j_1).$$

Consequently,

$$\begin{aligned} \sum_{i=7}^{r-6} |X_i| &= \sum_{i=7}^{j_1-1} |X_i| + \sum_{i=j_1}^{j_s-1} |X_i| + \sum_{i=j_s}^{r-6} |X_i| \\ &\geq 4(j_1 - 7) + 4(j_s - j_1) + 4(r - 5 - j_s) - 2 = 4(r - 12) - 2. \end{aligned}$$

This completes the proof of Lemma 3.4. □

We are now in a position to complete the proof of the Theorem. If $r \leq 13$, the conclusion follows from Lemma 2.2. Thus we may assume $r \geq 14$. We clearly have $|X_0| = 1$ and $|X_r| \geq 1$ and, by Lemma 2.1, $|X_i| \geq 3$ for all $1 \leq i \leq 6$ and all $r-5 \leq i \leq r-1$. From Lemma 3.4, $\sum_{i=7}^{r-6} |X_i| \geq 4(r-12)-2$. Adding all $|X_i|$, we obtain

$$n = \sum_{i=0}^{r} |X_i| \geq 1 + 3 \times 6 + \{4(r - 12) - 2\} + 3 \times 5 + 1 = 4r - 15.$$

This completes the proof of the Theorem. □

§4. Remarks

Considering more carefully, we see that
(1) this proof can be extended to $a = 6$,
(2) it never happens that $|X_i| = 3$ for all $1 \leq i \leq 5$,
(3) it never happens that $|X_i| = 3$ for all $r - 5 \leq i \leq r - 1$.
Thus the inequality can be improved to $r \leq \frac{n+12}{4}$. On the other hand, as is constructed in [1], for each $n \geq 8$ with $n \equiv 0 \pmod 4$, there exists a 3-connected graph of order n having radius $\frac{n+4}{4}$.

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[1] J. Harant and H. Walther, *On the rdius of graph*, J. Comb. Theory Ser. B 30 (1981), 113-117.

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