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POSITIVE SOLUTIONS OF IMPULSIVE INTEGRODIFFERENTIAL BOUNDARY VALUE PROBLEMS*

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§1. INTRODUTION

Impulsive differential equations arise naturally and often in engineering and physics, see [1]–[4] for example. Recently, various existence principles of such problems are obtained. Among these, Guo and Liu in [1] proved that at least two solutions exist for superlinear impulsive boundary value problems. In this paper, we consider the existence of positive solutions for impulsive integrodifferential boundary value problems, where the nonlinear term is sublinear at infinity and may have singular nature at the origin. Our results are new even in the non-impulsive case. Specifically, consider the following problem:

$$(1.1)\qquad \begin{cases} (p(t)x'(t))' + p(t)f(t, x(t), (Hx)(t), (Sx)(t)) = 0, \\ \quad t \in (0,1), t \neq t_k, k = 1, 2, ..., m, \\ \lim\limits_{\varepsilon \to +0} [x(t_k + \varepsilon) - x(t_k - \varepsilon)] = I_k(x(t_k)), \\ x(t) \text{ is left continuous at } t = t_k, k = 1, 2, ..., m, \\ \alpha x(0) - \beta \lim\limits_{t\to 0} p(t)x'(t) = \gamma x(1) + \delta \lim\limits_{t \to 1} p(t)x'(t) = 0. \end{cases}$$

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where $f \in C[(0,1) \times R^+ \times R^1 \times R^1, R^+]$, $R^+ = (0,\infty)$, $p \in C[0,1] \bigcap C^1(0,1)$, $p(t) > 0$ for $t \in (0,1)$. We shall assume the following conditions throughout this paper:

$$\int_0^1 \frac{1}{p(t)} dt < \infty.$$

The operators H and S are given by

$$(Hx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Sx)(t) = \int_0^1 k_1(t,s)x(s)ds \tag{1.2}$$

with $k, k_1 \in C[[0,1] \times [0,1], [0,\infty)]$, and $\alpha, \beta, \gamma, \delta \geq 0, \beta\gamma + \alpha\delta + \alpha\gamma > 0, \beta\delta = 0, I_k \in C[[0,\infty),[0,\infty)], k = 1,2,...,m, 0 < t_1 < t_2 < ... < t_m < 1$. Note that the nonlinear term $f(t,x,y,z)$ may be singular at $t = 0,1$ and $x = 0$, i.e., it may be unbounded when t tends to $0,1$ or when x tends to 0. Let $J = [0,1], PC(J) = \{x : x$ is a function from J to R^1, continuous at $t \neq t_k$, left continuous at $t = t_k$, and right hand limit at $t = t_k$ exist for $k = 1,2,...,m\}$. Recall that $PC(J)$ is a Banach space with norm $\|x\| = \sup_{t\in J} |x(t)|$. Denote the normal cone of $PC(J)$ by $P = \{x : x \in PC(J), x(t) \geq 0, t \in [0,1]\}$. A function x is called a positive solution of (1.1) if $x(t) > 0, t \in (0,1), x \in PC(J)$ and satisfies (1.1). Throughout this paper, we use C to denote a generic constant, and $C(\varepsilon)$ a constant dependent of ε.

§2. THE x-NONSINGULAR CASE

In this section, we assume that $f(t,x,y,z)$ is nonsingular with resect to x at $x = 0$, and we shall prove the existence of positive solutions. Denote

$$\tau_1(t) = \int_t^1 \frac{1}{p(t)} dt, \quad \tau_0(t) = \int_0^t \frac{1}{p(t)} dt \tag{2.1}$$

then we have $\tau_1, \tau_0 \in C[0,1]$. Let $\rho^2 = \beta\gamma + \alpha\delta + \alpha\gamma \int_0^1 \frac{1}{p(t)} dt$, and write

$$u(t) = \frac{1}{\rho}[\delta + \gamma\tau_1(t)], \quad v(t) = \frac{1}{\rho}[\beta + \alpha\tau_0(t)]. \tag{2.2}$$

Note that $\gamma v + \alpha u \equiv \rho$. Define

$$G(t,s) = \begin{cases} u(t)v(s)p(s), & 0 \leq s \leq t \leq 1, \\ v(t)u(s)p(s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Denote $\theta(s) = \tau_1(s)$ for $s \in (0,1)$ when $\beta > 0, \delta = 0$; $\theta(s) = \tau_0(s)$ for $s \in (0,1)$ when $\beta = 0, \delta > 0$; $\theta(s) = \tau_0(s)$ for $s \in [0, \frac{1}{2}]$, and $\theta(s) = \tau_1(s)$ for $s \in (\frac{1}{2}, 1]$ when $\beta = 0, \delta = 0$. Write

$$\triangle(px')\Big|_{t_k} = \lim_{\varepsilon\to+0}[p(t_k+\varepsilon)x'(t_k+\varepsilon) - p(t_k-\varepsilon)x'(t_k-\varepsilon)],$$

and introduce the following condition (see [1]):

$$\triangle(px')\Big|_{t_k} = -\frac{\gamma I_k(x(t_k))}{\delta+\gamma\tau_1(t_k)}, \quad k=1,2,...,m. \tag{2.4}$$

Similar to Lemma 1 of [1], we have the following Lemma.

Lemma 2.1. *If $x \in P$ is a solution of the following integral equation*

$$\begin{aligned} x(t) = (Ax)(t) &= \int_0^1 G(t,s)f(s,x(s),(Hx)(s),(Sx)(s))ds \\ &\quad + (\delta+\gamma\tau_1(t))\sum_{0<t_k<t}\frac{I_k(x(t_k))}{\delta+\gamma\tau_1(t_k)} \end{aligned} \tag{2.5}$$

then x is a positive solution of (1.1) satisfying condition (2.4).

Lemma 2.2. *The following estimate holds*

$$G(t,s) \le \theta(s)p(s), \quad t,s\in[0,1], s\neq 0,1.$$

Proof. It is straight forward, see [5], or see Appendix. □

In order to show the existence of positive solutions, we now make the following assumptions:

(H_1) $f(t,x,y,z) \le \psi(t)\phi(x,y,z)$, where $\psi \in C[(0,1),R^+], \phi \in C[R^+ \times R^1 \times R^1, R^+]$ and $\int_0^1 \theta(s)p(s)\psi(s)ds < \infty$.

(H_2) $\theta(s)p(s)$ is bounded for $s\in(0,1)$.

(H_3) $\lim_{x\to\infty}\frac{I_k(x)}{x} = 0, k=1,2,...,m.$

(H_4) $\lim_{|x|+|y|+|z|\to\infty}\frac{\phi(x,y,z)}{|x|+|y|+|z|} \le \lambda, \lambda>0.$

(H_5) $\lambda[1+M+M_1]\int_0^1\theta(s)\psi(s)p(s)ds < \frac{1}{m+1}$, where $M=\max\{k(t,s): t,s\in[0,1]\}, M_1=\max\{k_1(t,s): t,s\in[0,1]\}$.

(H_6) For any $h>0$, there exist $y\in C[0,1]$ with $y(t)\ge 0$ for $t\in[0,1]$ and $y(t)\not\equiv 0$ such that $f(t,x,y,z)\ge y(t)$ for $t\in(0,1), x,y,z\in(0,h]$.

Lemma 2.3. *Assume (H$_1$) holds, and $\phi(x,y,z)$ is bounded on $(0,1)\times[0,M]$ $\times[0,M]$, where $M>0$ is arbitrary. Then the operator A maps P^* into P and is completely continuous, where $P^*=\{x\in P : x(t)>0, t\in(0,1)\}$.*

Proof. For any $x\in Q$, where Q is a bounded subset of P, we have that

$$f(s,x(s),Hx,Sx)\le C\psi(s), \tag{2.6}$$

where C is a constant. Define

$$y_1(t)=\int_0^1 G(t,s)f(s,x(s),(Hx)(s),(Sx)(s))ds=(A_1x)(t),$$

$$y_2(t)=(\delta+\gamma\tau_1(t))\sum_{0<t_k<t}\frac{I_k(x(t_k))}{\delta+\gamma\tau_1(t_k)}=(A_2x)(t).$$

From [5] we know that $y_1\in C[0,1]$ and A is continuous and maps bounded sets into bounded sets, where P^* and P have induced topology from $PC(J)$, see Appendix for complete proof. When $t\in(0,1), t\ne t_k$, we can directly get

$$-p(t)y_1'(t)=\frac{\gamma}{\rho}\int_0^t vpf ds-\frac{\alpha}{\rho}\int_t^1 upf ds. \tag{2.7}$$

Because the proofs of other cases are similar, we now only consider the case of $\beta=\delta=0, \alpha=\gamma=1$. Then from (2.6) we have

$$|p(t)y_1'(t)|\le C\int_0^t \tau_0 p\psi ds+C\int_t^1 \tau_1 p\psi ds.$$

Notice the fact that

$$\int_0^1\frac{1}{p(t)}(\int_0^t \tau_0 p\psi ds)dt=\int_0^1\tau_0\tau_1\psi p dt<\infty,$$

$$\int_0^1\frac{1}{p(t)}(\int_t^1 \tau_1 p\psi ds)dt=\int_0^1\tau_0\tau_1\psi p dt<\infty.$$

Hence $\{y_1(t)\}$ is pre-compact for $x\in Q$. Similarly we can prove that $\{y_2(t)\}$ is pre-compact for $x\in Q$. As a result, A is completely continuous. The proof is complete. □

Theorem 2.4. *Suppose (H$_1$)–(H$_5$) hold and $\phi(x,y,z)$ is bounded on $(0,1)\times[0,M]\times[0,M]$ for arbitrary $M>0$. Then problem (1.1) has a positive solution $x\in PC(J)$.*

Proof. From the definition of A_2 and condition (H_3), it is easy to show that

$$\|A_2x\| \leq \varepsilon C\|x\| + C(\varepsilon) \tag{2.8}$$

Let M and M_1 be as in condition (H_5). From our assumptions we know that

$$\phi(x,y,z) \leq (\lambda+\varepsilon)(|x|+|y|+|z|) + C(\varepsilon), \qquad x,y,z>0.$$

Hence

$$\begin{aligned} \|A_1x\| &\leq \int_0^1 \theta\psi p\phi(x,Hx,Sx)ds \\ &\leq \int_0^1 \theta\psi pds(\lambda+\varepsilon)(\|x\|+M\|x\|+M_1\|x\|)+C(\varepsilon). \end{aligned}$$

Choose ε such that $\int_0^1 \theta\psi pds(\lambda+\varepsilon)(1+M+M_1)<1$ and we obtain

$$\lim_{\|x\|\to\infty} \frac{\|A_1x\|}{\|x\|} < 1.$$

Then the fixed point theorem of cone compression (see [6]) yields the required solution. The proof is complete. □

§3. THE x-SINGULAR CASE

In this section we will give an existence principle when the function $f(t,x,y,z)$ is unbounded. First, consider the following approximate problem:

$$\begin{cases} (p(t)x'(t))' + p(t)f_n(t,x(t),(Hx)(t),(Sx)(t)) = 0, \\ \quad t\in(0,1), \quad t\neq t_k, k=1,2,...,m, \\ \lim\limits_{\varepsilon\to+0}[x(t_k+\varepsilon)-x(t_k-\varepsilon)] = I_k(x(t_k)), \\ x(t) \text{ is left continuous at } t=t_k, \quad k=1,2,...,m, \\ \alpha x(0) - \beta\lim\limits_{t\to 0}p(t)x'(t) = \gamma x(1)+\delta\lim\limits_{t\to 1}p(t)x'(t) = 0 \end{cases} \tag{3.1}$$

where $f_n(t,x,y,z) = f(t,\max\{\frac{1}{n},x\},y,z)$. Suppose ($H_1$)–($H_5$) hold. Then from Theorem 2.4 we know that problem (3.1) is solvable for any integer $n\leq 1$. Moreover, the solutions of (3.1) satisfy (2.4).

Lemma 3.1. *Suppose (H_1)–(H_5) hold. Then there exists a constant $R>0$ independent of n such that $0\leq x(t)\leq R, t\in[0,1]$ for any positive solution x of (3.1).*

Proof. Let x be a positive solution of (3.1).

(1) Suppose $\|x\| = x(0)$. From the boundary condition we know $\beta \lim_{t\to 0} p(t)x'(t) = \alpha x(0) \geq 0$. Obviously $\lim_{t\to 0} p(t)x'(t) \leq 0$, thus $\beta \lim_{t\to 0} p(t)x'(t) = 0$. As a result we can deduce $\alpha = 0$, and furthermore, $\beta > 0$ with $\lim_{t\to 0} p(t)x'(t) = 0$. As stated above, in this case $\delta = 0$, henceforth $x(1) = 0$. Then from (3.1) and (2.4) we get $x'(t) < 0$ for $t \in (0,1), t \neq t_k$. Choose $T > 1$ such that

$$\phi(x,y,z) \leq (\lambda+\varepsilon)(|x|+|y|+|z|) \text{ for } |x|+|y|+|z| \geq T$$

Without loss of generality we can assume that there exists $t_k^* \in (t_{k-1}, t_k]$ with $x(t_k^*) = T$, and $\|x\| > T$. In the following we assume that $t_0 = 0, t_{m+1} = 1$ for convenience. Now we begin with the first interval $[0, t_1]$. In the case of $x(t_1) \leq T$, we then choose $t_1^* \in (0, t_1]$ such that $x(t_1^*) = T$. By integration we get for $t < t_1^*$ that

$$\begin{aligned} -p(t)x'(t) &\leq \int_0^t p(s)f_n(s,x(s),Hx,Sx)ds \\ &\leq (\lambda+\varepsilon)\int_0^t p(s)\psi(s)(x+Hx+Sx)ds \\ &\leq (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^t p(s)\psi(s)ds. \end{aligned} \tag{3.3}$$

Thus we have

$$\begin{aligned} x(0)-T &\leq (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1^*} p(s)\psi(s)\tau_1(s)ds \\ &\leq (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1} p(s)\psi(s)\theta(s)ds. \end{aligned} \tag{3.4}$$

From condition (H_5), by letting ε be small enough we can get the required constant $R > 0$ such that $\|x\| \leq R$. If on the other hand $x(t_1) > T$, then we integrate (3.3) on $[0, t_1]$. Thus we get

$$x(0) - x(t_1) \leq (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1} \psi(s)\theta(s)p(s)ds. \tag{3.5}$$

Because x satisfies (2.4) we know $px'\Big|_{t_1+0} \leq 0$. Then integration on $[t_1, t]$ with $t < t_2^*$ yields

$$-p(t)x'(t) \leq -px'\Big|_{t_1+0} + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_{t_1}^t p\psi ds \tag{3.6}$$

where t_2^* belongs to (t_1, t_2) or $t_2^* = t_2$. From (2.4) we get

$$-px'\Big|_{t_1+0} = -px'\Big|_{t_1-0} + \frac{\gamma I_1(x(t_1))}{\delta+\gamma\tau_1(t_1)}. \tag{3.7}$$

From (H_3) we have $C(\varepsilon) > 0$ dependent of ε such that

$$I_k(x) < \varepsilon x + C(\varepsilon), \qquad x > 0, k = 1, 2, ..., m. \tag{3.8}$$

Moreover (3.3) yields

$$-px'\Big|_{t_1-0} \le (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1}\psi p ds. \tag{3.9}$$

Together with (3.7) we can deduce that

$$-px'\Big|_{t_1+0} \le (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1}\psi p ds + \varepsilon C\|x\| + C(\varepsilon). \tag{3.10}$$

Combining (3.6) and (3.10) yields (Note that $\theta = \tau_1$ in this case):

$$\begin{aligned} & x(t_1+0) - x(t_2^*) \\ & \le (\lambda+\varepsilon)\|x\|(1+M+M_1)(\int_{t_1}^{t^2}\frac{1}{p}ds)(\int_0^{t_1}\psi p ds) + \varepsilon C\|x\| + C(\varepsilon) \\ & \qquad + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_{t_1}^{t_2}\Big[\frac{1}{p(t)}\int_{t_1}^{t}\psi p(s) ds\Big]dt \\ & \le (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1}\psi p\theta ds + \varepsilon C\|x\| + C(\varepsilon) \\ & \qquad + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_{t_1}^{t_2}\psi p\theta ds. \end{aligned}$$

Using $\triangle x\Big|_{t_1} > 0$, (2.4) and (3.5) we then obtain

$$\begin{aligned} x(0) \quad \le \quad & x(t_2^*) + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_2}\psi p\theta ds + \varepsilon C\|x\| \\ & + C(\varepsilon) + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_1}\psi p\theta ds. \end{aligned} \tag{3.11}$$

If $x(t_2^*) = T$, then the proof is complete by letting ε be small enough. If $x(t_2) > T$ and $t_2^* = t_2$ we can get (3.11) for $t_2^* = t_2$. Then inequalities similar to (3.5)–(3.11) hold. Thus by induction we finally have

$$\begin{aligned} x(0) \quad \le \quad & x(t_k^*) + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_k}\psi p\theta ds + \varepsilon C\|x\| \\ & + C(\varepsilon) + (k-1)(\lambda+\varepsilon)\|x\|(1+M+M_1)\int_0^{t_k}\psi p\theta ds. \end{aligned} \tag{3.12}$$

By (H_5) we can choose ε small, and the required R exists.

(2) Suppose $\|x\| = x(1)$, then $\gamma = 1, \delta = 0$, and $\lim_{t\to 1} p(t)x'(t) = 0$. The rest of the proof is similar to step (1).

(3) Suppose $\|x\| = x(t_0)$, where $t_0 \in (t_{k-1}, t_k)$. First assume $\alpha = 0, \beta > 0$, then $x'(t_0) = 0$, and $\theta(s) = \tau_1(s)$. So the proof is similar to case (1). If $\beta = \delta = 0, \alpha = \gamma = 1$, then we can distinguish between two cases: $t_0 \le \frac{1}{2}$ and $t_0 \ge \frac{1}{2}$. Integration on $[t_0, 1]$ and $[0, t_0]$ respectively will yield the required estimate.

(4) Suppose $\|x\| = x(t_k+0), 1 \le k \le m$. Then $x'(t_k+0) \le 0$, and $x'(t+0) < 0$ for $t \in (t_k, 1)$. If $\lim_{t\to 0} p(t)x'(t) = 0$ we can prove as in case (1) that $x(0) \le R$. Thus condition (H_3) implies that $\|x\|$ is bounded.

Now assume that x' has one zero to in $[0, t_k]$ (including right limit zeros). From equation (3.1) and conditon (2.4), we know that $x'(t) \le 0$ for $t \in (t_0, 1) \setminus \{t_1, t_2, ..., t_m\}$. Then from the impulse conditions $\lim_{\varepsilon\to+0}[x(t_j + \varepsilon) - x(t_j - \varepsilon)] = I_j(x(t_j))$ for $j = 1, 2, ..., m$ and (3.8) we get

$$\|x\| = x(t_k + 0) \le m\varepsilon\|x\| + C(\varepsilon) + x(t_{**} + 0), \quad t_{**} \in [t_0, t_k). \tag{3.13}$$

By essentially the same way as in the proof of (3.12) of step (1), we get

$$\begin{aligned} x(t_0 + 0) \ \le\ & x(t_k^*) + (\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_0^{t_k} \psi p\theta ds + \varepsilon C\|x\| + C(\varepsilon) \\ & + (k-1)(\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_0^{t_k} \psi p\theta ds \end{aligned} \tag{3.14}$$

where t_k^* satisfies $t_k^* = t_k$, or $x(t_k^*) = T$ and $t_k^* \in (t_0, t_k)$. If $x(t_k^*) = T$ then (3.14) becomes

$$\begin{aligned} x(t_0 + 0) \ \le\ & T + (\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_0^{t_k} \psi p\theta ds + \varepsilon C\|x\| + C(\varepsilon) \\ & + (k-1)(\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_0^{t_k} \psi p\theta ds. \end{aligned} \tag{3.15}$$

Thus the inequalities (3.13) and (3.15) yield

$$\begin{aligned} \|x\| \ \le\ & T + (\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_0^{t_k} \psi p\theta ds \\ & + \varepsilon C\|x\| + C(\varepsilon) + m\varepsilon\|x\| \\ & + (k-1)(\lambda + \varepsilon)\|x\|(1 + M + M_1) \int_0^{t_k} \psi p\theta ds. \end{aligned} \tag{3.16}$$

If on the other hand $t_k^* = t_k$, then let $t_{**} = t_k$ in (3.13), and together with (3.14) we get that (3.16) holds. By letting ε be small enough we know that the required estimate holds.

Finally let x' have no zeros in $[0, t_k]$. If $x'(t_k - 0) \ge 0$, then from (2.4) and (3.8) we have

$$0 \le -px'\Big|_{t_k+0} \le -\triangle(px')\Big|_{t_k} \le C\varepsilon\|x\| + C(\varepsilon), \tag{3.17}$$

$$0 \le -px'\Big|_{t_k-0} \le -\triangle(px')\Big|_{t_k} \le C\varepsilon\|x\| + C(\varepsilon). \tag{3.18}$$

Since the proofs are similar we only consider the case of $\beta > 0, \delta = 0$. Thus $\gamma = 0$ and $x(1) = 0$. By integration, instead of $\lim_{t\to 0} p(t)x'(t) = 0$ (or $x'(t_0) = 0$), we can use inequality (3.17). Then similar to (3.3)–(3.12) we can easily obtain

$$\begin{aligned}\|x\| \quad &\le \quad T + (\lambda+\varepsilon)\|x\|(1+M+M_1)\int_{t_k}^{1}\psi p\theta ds + \varepsilon C\|x\| + C(\varepsilon) \\ &\qquad + mC\varepsilon\|x\| + (m-k)(\lambda+\varepsilon)\|x\|(1+M+M_1)\int_{t_k}^{1}\psi p\theta ds.\end{aligned}$$

Consequently our lemma is ture. Next from the boundary condition, we have $\alpha x(0) = \beta \lim_{t\to 0} p(t)x'(t)$. If $\alpha = 0$, then $\beta > 0$ and $\lim_{t\to 0} p(t)x'(t) = 0$. Thus from equation (3.1) and condition (2.4) we know that $\|x\| = x(0)$, and this is exactly case (1). If $\alpha > 0, x(0) = 0$, then $\lim_{t\to 0} p(t)x'(t) \ge 0$. When $\lim_{t\to 0} p(t)x'(t) = 0$, this is again case (1). Now we need to consider the case of $\lim_{t\to 0} p(t)x'(t) > 0$. If on the other hand $\alpha > 0, x(0) > 0$, then $\beta > 0$, and $\lim_{t\to 0} p(t)x'(t) > 0$. In both cases we finally know that $x'(t) > 0$ when t belongs to some neighbourhood of zero. Therefore we can choose t_j with $1 \le j \le k-1$ such that $x'(t_j+0) \le 0$ and $x'(t_j - 0) \ge 0$ (Notice that we have already assumed that $x'(t)$ has no zeros). Moreover $x'(t) \le 0$ for $t \in (t_j, 1) \setminus \{t_1, t_2, ..., t_m\}$. Note that we can use inequalities (3.17) and (3.18) instead of the condition $x'(t_0) = 0$. Let $t_0 = t_j$ in inequality (3.13). Then by essentially the same way as in the proof of inequalities (3.13)–(3.16), we can prove that (3.13)–(3.16) still hold. The only difference is using t_0 instead of t_j. The proof is complete. □

Lemma 3.2. *Let* (H_1)–(H_6) *hold. Then there exists* x^* *such that* $x^*(t) > 0$ *for* $t \in (0,1)$ *and* $x(t) \ge x^*(t), t \in (0,1)$, *where* x *is any solution of the problem (3.1).*

Proof. From (2.5), Lemma 3.1 and (H_6), let $h = 1 + R + M + M_1$ and

$$x^*(t) = \int_0^1 G(t,s)y(s)ds.$$

Then it is easy to show that x^* is the required function. □

Now we assume that (H_1)–(H_6) hold, and $\phi(x,y,z) \ge 1$ without loss of generality. Let $a(x,y,z) = \dfrac{\phi(x,y,z)}{x}$, and

$$b(u) = \sup\{a(x,y,z) : x \in (u, R+1], y, z \in [0, (R+1)(1+M+M_1)]\}.$$

$$T(u) = \int_0^u \frac{1}{b(v)}dv.$$

Lemma 3.3. *Let x be a solution of (3.1). Denote by t_x^0 the zeros of $x'(t)$. Then there exists η independent of n such that*

(i) $t_x^0 \le 1-\eta$, *when* $\beta > 0, \delta = 0$.

(ii) $t_x^0 \ge \eta$, *when* $\delta > 0, \beta = 0$.

(iii) $\eta \le t_x^0 \le 1-\eta$, *when* $\beta = \delta = 0$.

where n is any integer.

Proof. The proof of this lemma is essentially the same as that of Lemma 4.4 of [5]. Thus it is omitted. □

Lemma 3.4. *Assume (H$_1$)–(H$_6$) hold. Then there exists $w(t) \in L^1(0,1)$ such that*

$$\Big|\frac{d}{dt}T(x(t))\Big| \le w(t), \quad t \neq t_k, k = 1, 2, ..., m$$

where x is any solution of (3.1).

Proof. From the proof of Lemma 3.1 we know that there are only three cases of $x(t)$ to be considered.

(1) $\lim_{t\to 0} p(t)x'(t) = 0$ or $\lim_{t\to 1} p(t)x'(t) = 0$. Since the proofs of two cases are similar, we only consider the case of $\lim_{t\to 0} p(t)x'(t) = 0$, hence $\alpha = 0, \beta > 0$, and x decreases on $(t_{k-1}, t_k), k = 1, 2, ..., m$. Integration on $(0, t_1)$ yields

$$-px' \le C\int_0^t p\psi b(x)ds \le Cb(x)\int_0^t \psi pds.$$

Let $z(t) = T(x(t))$. Then from (H$_1$) we know

$$|z'(t)| \le \frac{C}{p(t)}\int_0^t \psi pds \in L^1(0, t_1).$$

From (2.4) we have $-px'\Big|_{t_1+0} \le C$, hence for $t \in (t_1, t_2)$ we have

$$|px'| \le C + \int_{t_1}^{t_2} \psi\phi pds \le C.$$

Similarly $|x'| \le C$ for $t \in [t_1, t_m]$ and $|z'| \le C$. For $t \in (t_m, 1)$, we have (Notice that $b \ge \frac{1}{1+R}$)

$$|px'| \le C + C\int_{t_m}^t \psi\phi pds \le C + Cb(x)\int_{t_m}^t \psi pds,$$

$$|z'| \le \frac{C}{p(t)} + \frac{C}{p(t)}\int_{t_m}^t \psi pds \in L^1(0,1).$$

(2) x' has one zero t^0 in $(0,1)$, including limit zeros. According to the boundary conditions, we have the following three cases to be considered:

(i) $\beta > 0, \delta = 0$,

(ii) $\delta > 0, \beta = 0$,

(iii) $\beta = \delta = 0$.

However, proofs of the lemma in these three cases are similar. So for brevity we only consider the third case, i.e., the case of $\beta = \delta = 0$ and $\alpha = \gamma = 1$. Then Lemma 3.3 yields $\eta \le t^0 \le 1-\eta$. Now we can assume $\eta \le t_1, 1-\eta \ge t_m$ for convenience. By integration and condition (3.4) we get $C > 0$ independent of n such that

$$|x'(t)| \le C, \quad t \in [\eta, 1-\eta]. \tag{3.19}$$

Since b decreases and x increases, integration of (3.1) on $(t, t_1]$ yields:

$$\begin{aligned} &p(t)x'(t) - p(t_1)x'(t_1) \\ &\le \int_t^{t_1} p(s)\psi(s)\max\{\frac{1}{n}, x(s)\}a(\max\{\frac{1}{n}, x(s)\}, (Hx)(s), (Sx)(s))ds \\ &\le (R+1)\int_t^{t_1} p(s)\psi(s)b(x(s))ds. \end{aligned} \tag{3.20}$$

Thus from (3.19) we have

$$|px'| \le C + C\int_t^{t_1} \psi pb(x)ds \le C + Cb(x)\int_t^{t_1} \psi pds. \tag{3.21}$$

Let $z(t) = T(x(t))$, then

$$|z'| \le C + C\frac{1}{p}\int_t^{t_1} \psi pds \in L^1(0, t_1). \tag{3.22}$$

Similarly we can get

$$|z'| \le C + C\frac{1}{p}\int_{t_m}^{t} \psi pds \in L^1(t_m, 1). \tag{3.23}$$

(3) There exists $t_k, 1 \le k \le m$, such that $x'(t_k+0) < 0, x'(t_k-0) > 0$. As in the proof of Lemma 3.1, we know in this case that

$$|x'(t_k \pm 0)| \le C. \tag{3.24}$$

where C is independent of n. Thus we can also prove that (3.19) is true. Following the above steps (3.20)–(3.23), it is easy to get the required estimate. The proof is complete. □

Now we come to our main theorem of this section.

Theorem 3.5. *Suppose* (H_1)*–*(H_6) *hold. Then problem (1.1) has a positive solution satisfying (2.4).*

Proof. Let x_n be solutions of (3.1), and $z_n = T(x_n)$. From the above Lemma 3.1, Lemma 3.4 and the Arzela-Ascoli theorem we may assume z_n converges. Thus we know $x_n \to x$ in C[0,1]. Because $p(t)$ is positive on (0,1) we can assume without loss of generality that $x_n \to x$ in $PC^1[\varepsilon, 1-\varepsilon]$, where $\varepsilon \in (0,1)$ is arbitrary, and where $PC^1[\varepsilon, 1-\varepsilon] = \{x : x$ is continuously differentiable at $t \neq t_k$, and left continuous at $t = t_k$, and $x(t_k+0), x(t_k-0), x'(t_k+0), x'(t_k-0)$ exist for $k = 1, 2, ..., m\}$, equipped with the norm

$$\|x\|_1 = \max\{\|x\|, \|x'\|\}.$$

Hence x satisfies (2.4) and the impulsive conditions. If $\beta = \delta = 0, \alpha = \gamma = 1$, then x satisfies the boundary conditions and by integration we have

$$p(\eta)x_n'(\eta) - p(t)x_n'(t) = \int_\eta^t p(s)f_n(s, x_n, Hx_n, Sx_n)ds.$$

From the continuity of f we obtain

$$p(\eta)x'(\eta) - p(t)x'(t) = \int_\eta^t p(s)f(s, x, Hx, Sx)ds.$$

Thus x is a solution of (1.1). If $\beta > 0, \delta = 0$, then similar to [5] we can prove x is a solution of (1.1). □

Remark 3.6. Our conditions are weaker than those of [5], [7] even in the non-impulsive cases.

APPENDIX

In this appendix, we will give detailed proof of Lemma 2.2, and will prove that the oprator A in page 70 is continuous and maps bounded sets into bounded sets.

Proof of Lemma 2.2. First from the definiton of $u(t), v(t)$, and the condition $\int_0^1 \frac{1}{p(t)}dt < \infty$, we know that $u(t)$ is decreasing and $v(t)$ is increasing, and $u, v \in C[0,1]$. Thus from the definition of $G(t,s)$ we have that

$$(A_1) \qquad G(t,s) \leq u(s)v(s)p(s), \quad \text{for } t, s \in [0,1].$$

Case (1). Suppose that $\beta > 0, \delta = 0$. Then

$$\rho^2 = \beta\gamma + \alpha\gamma \int_0^1 \frac{1}{p(t)} dt,$$

$$u(s) = \frac{\gamma}{\rho}\tau_1(s), \quad v(s) = \frac{1}{\rho}[\beta + \alpha\tau_0(s)],$$

and $\theta(s) = \tau_1(s)$. Thus from (A_1) we have that

$$G(t,s) \le \frac{1}{\rho^2}[\beta + \alpha\tau_0(s)]\gamma\tau_1(s)p(s) \le p(s)\theta(s).$$

Case (2). Suppose that $\beta = 0, \delta > 0$. Proof of this case is similar to that of the case (1).

Case (3). Suppose that $\beta = \delta = 0$. Then

$$\rho^2 = \alpha\gamma \int_0^1 \frac{1}{p(t)} dt, \quad u(s) = \frac{\gamma}{\rho}\tau_1(s), v(s) = \frac{\alpha}{\rho}\tau_0(s),$$

and $\theta(s) = \tau_0(s)$, for $s \in (0, \frac{1}{2}); \theta(s) = \tau_1(s)$, for $s \in (\frac{1}{2}, 1)$. Thus from (A_1) we have that

$$G(t,s) \le \frac{1}{\rho^2}\alpha\gamma\tau_0(s)\tau_1(s)p(s) \le p(s)\theta(s).$$

The proof is complete.

Proof of the properties of the oprator A in page 70: Now we shall prove that the oprator A in page 70 is continuous and maps bounded sets into bounded sets, where the domain of A is P^*, and the range of A is P; both have induced topology from $PC(J)$. Let the oprators A_1, A_2 be the same as those in the proof of Lemma 2.3. We will give the whole proof in the following three steps:

(1) A_1 maps P^* into P.

In fact, let $x \in P^*, y_1(t) = (A_1x)(t)$. Because $\phi(x,y,z)$ is bounded on $(0,1) \times [0,M] \times [0,M]$, where M is arbitrary, we can choose constant C such that

$$(A_2) \qquad G(t,s)f(s,x(s),(Hx)(s),(Sx)(s)) \le C\theta(s)p(s)\psi(s) \in L^1(0,1).$$

It is clear that $G(t,s)$ is continuous on $[0,1] \times [0,1]$. Thus from Lebesgue's convergence theorem of dominance we know that $y_1(t)$ is continuous, and thus belongs to P.

(2) A_1 is continuous and maps bounded sets into bounded sets.

In fact, let E be a bounded set in P^*. Then we can choose a constant C as in step (1) such that the estimate (A_2) holds, which immediately yields that $A_1(E)$ is a bounded set in P. Next let $x_0 \in P^*$ be fixed and $x \in P^*$ such that

$\|x - x_0\| \leq 1$. Let $\varepsilon \in (0, \frac{1}{2})$. Then for $t \in (0,1)$ there exists constant C such that

$$\begin{aligned}
&|(A_1x)(t) - (A_1x_0)(t)| \\
&\quad \leq C\int_0^{\varepsilon} p(s)\theta(s)\psi(s)ds + C\int_{1-\varepsilon}^{1} p(s)\theta(s)\psi(s)ds \\
&\qquad + \int_{\varepsilon}^{1-\varepsilon} p(s)\theta(s)\psi(s)\,|f(s,x,Hx,Sx) - f(s,x_0,Hx_0,Sx_0)|ds.
\end{aligned}$$

Since $f(s,x,y,z)$ is continuous on $(0,1)\times(0,\infty)\times(0,\infty)\times[0,\infty)$, it is easy to show that A_1 is continuous.

(3) Clearly A_2 maps P^* into P, and from the continuity of I_k, $k = 1,2,...,m$ we know that there are constants such that for $x, x_0 \in P^*$ the following estimates hold:

$$|(A_2x)(t) - (A_2x_0)(t)| \leq C\sum_{k=1}^{m} |I_k(x(t_k)) - I_k(x_0(t_k))|$$

$$|(A_2x)(t)| \leq C\sum_{k=1}^{m} |I_k(x(t_k))|$$

Hence it is easy to show that A_2 is continuous and maps bounded sets into bounded sets. The proof is complete.

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