SUT Journal of Mathematics
Vol. 32, No. 1 (1996), 59–66

RESIDUE FREE DIFFERENTIALS AND DIFFERENTIALS OF THE SECOND KIND

Takeo OHI and Hirokazu NAGAI

(Received December 21, 1995)

Abstract. Let k be a field of characteristic 0, V an n-dimensional non-singular algebraic variety over k, K the function field of V and $\Omega^1_{K/k}$ the module of differentials of K over k. A closed differential $\omega \in \Omega^1_{K/k}$ is called residue free if $res_W(\omega) = 0$ for any prime divisor W of V and a differential ω is called second kind if for any prime divisor W, there exists an element $\theta_W \in K$ such that $\nu_W(\omega - d\theta_W) \geq 0$, where ν_W is the canonical valuation with respect to W. In this paper, we prove the following theorem: Let ω be a closed element of $\Omega^1_{K/k}$. Then ω is residue free if and only if ω is of second kind.

AMS 1991 *Mathematics Subject Classification.* Primary 13N05, 12H05.

Key words and phrases. Differential module, residue map, residue free, differential of second kind.

§1. Introduction, notations and basic definitions.

Throughout this paper, k denotes a field of characteristic 0. Let V be an n-dimensional non-singular algebraic variety over k, where the word "algebraic variety over k" means an integral separated scheme (V, O_V) of finite type over k, and the word "non-singular" means the stalk $O_{V,P}$ at each point P of V is a regular local ring. We denote the function field of V by K. Let $\Omega^1_{K/k}$ be the module of differentials of K over k and let $d : K \longrightarrow \Omega^1_{K/k}$ be the universal derivation.

Let W be a prime divisor of V. Let $U = \mathrm{Spec}(A)$ be an open affine subset of V such that $U \cap W \neq \phi$. Then there exists a prime ideal $\wp$ of A such that $U \cap W = V(\wp) := \{\wp^* \in \mathrm{Spec}(A) | \wp^* \supseteq \wp\}$. For this prime ideal $\wp$, we have that $A_\wp$ is a discrete valuation ring and $\wp A_\wp = (t_1)$ for some element t_1 of $A_\wp$. We denote the valuation of $A_\wp$ by $\nu_\wp$.

Set $D_\wp = Q(A/\wp) = A_\wp/\wp A_\wp$. Then there exist $t_2, \cdots, t_n \in A_\wp$ such that $\bar{t}_2, \cdots, \bar{t}_n \in D_\wp$ form a transcendental basis of $D_\wp$ over k, where $\bar{t}_i$ denotes the canonical image of t_i in $D_\wp$. The differential module $\Omega^1_{A_\wp/k}$ of $A_\wp/k$ and the differential module $\Omega^1_{K/k}$ of K/k are given by the following equalities (see [3]):

$$(1.1)\qquad \begin{cases} \Omega^1_{A_\wp/k} = A_\wp dt_1 \oplus A_\wp dt_2 \oplus \cdots \oplus A_\wp dt_n, \\ \\ \Omega^1_{K/k} = K dt_1 \oplus K dt_2 \oplus \cdots \oplus K dt_n. \end{cases}$$

For a differential

$$\omega = f_1 dt_1 + f_2 dt_2 + \cdots + f_n dt_n,$$

where $f_i \in K$, we define $d\omega$ as an element of $\Omega^1_{K/k} \wedge \Omega^1_{K/k}$ (over K) in the following:

$$d\omega = df_1 \wedge dt_1 + df_2 \wedge dt_2 + \cdots + df_n \wedge dt_n \text{ (see [3])}.$$

Definition 1.1. A differential ω is called *closed* if $d\omega = 0$.

Let $\widehat{A}_\wp$ be the $\wp A_\wp$-adic completion of $A_\wp$. By the structure theorem of complete local rings, there exists a unique coefficient field C of $\widehat{A}_\wp$ such that
(1) $k(t_2, \cdots, t_n) \subset C \subset \widehat{A}_\wp$,
(2) $C \simeq D_\wp$, (obtained from the natural surjection $\widehat{A}_\wp \longrightarrow D_\wp$)
(3) $\widehat{A}_\wp = C[[t_1]]$.

Extending the natural injection $A_\wp \longrightarrow \widehat{A}_\wp$ to a ring homomorphism $K = Q(A_\wp) \longrightarrow Q(\widehat{A}_\wp)$, we get an injection $* : K \longrightarrow D_\wp((t_1))$ from the isomorphism $C \simeq D_\wp$. By f^*, we denote the image of $f \in K$ by the injection $*$.

For a differential $\omega = f_1 dt_1 + f_2 dt_2 + \cdots + f_n dt_n$, we set

$$f_1^* = \alpha_{-m} t_1^{-m} + \alpha_{-m+1} t_1^{-m+1} + \cdots + \alpha_{-1} t_1^{-1} + \alpha_0 + \alpha_1 t_1 + \alpha_2 t_1^2 + \cdots,$$

where $\alpha_j \in D_\wp$.

Then we may define the residue of ω, $res_{W;t_1,t_2,\cdots,t_n}(\omega)$, with respect to W and $t_1, t_2, \cdots, t_n$ by

$$res_{W;t_1,t_2,\cdots,t_n}(\omega) = \alpha_{-1} \in D_\wp.$$

F. Elzein [1, Theorem 1] proved that if ω is closed, then this value α_{-1} depends only on W and not depends on an affine open U such that $U \cap W \neq \phi$, nor on a coefficient field C of $\widehat{A}_\wp$. This results lead us to the following.

Definition 1.2. For a closed differential ω, the *residue* of ω at W, $res_W(\omega)$, is defined by

$$res_W(\omega) = \alpha_{-1}.$$

Definition 1.3. A closed differential ω is called *residue free* if $res_W(\omega) = 0$ for any prime divisor W of V.

Let $\omega = \sum_{i=0}^{n} f_i dt_i$, and let each f_i^* be expressed as follows:

$$f_i^* = \sum_j \alpha_{i,j} t_1^j,$$

where the number of the terms of negative powers is finite. Then the following equality holds:

$$\nu_\wp(f_i) = \min\{j | \alpha_{i,j} \neq 0\}.$$

Since $\nu_\wp(f_i)$ is independent from the choice of $t_1, \ldots, t_n$, we define

$$\nu_\wp(\omega) = \min_i \nu_\wp(f_i).$$

Definition 1.4. A differential in $\Omega^1_{K/k}$ is called of the *second kind* if, for any prime divisor W, there exists an element θ_W of K such that

$$\nu_\wp(\omega - d\theta_W) \geq 0$$

(cf. M. Rosenlicht [4]).

The purpose of this paper is to prove the following theorem:

Theorem 1.5. *Let ω be a closed element of $\Omega^1_{K/k}$. Then ω is residue free if and only if ω is of second kind.*

This theorem is well known in the case of one variable. Our result above is a first step for the several variables case.

§2. Preliminaries.

For the proof of Theorem, we prepare some lemmas in this section.

Lemma 2.1. *If a differential* $\omega = f'_1 dt_1 + f'_2 dt_2 + \cdots + f'_n dt_n$ *is closed and* $\nu_\wp(\omega) < 0$, *then we have*

$$\nu_\wp(f'_1) < \nu_\wp(f'_2), \nu_\wp(f'_3), \cdots, \nu_\wp(f'_n).$$

Proof. Setting $\nu = -\nu_\wp(\omega) > 0$ and $f_i = f'_i t_1^{\nu}$ ($\in A_\wp$) for each i, we have that $\omega = f_1 t_1^{-\nu} dt_1 + f_2 t_1^{-\nu} dt_2 + \cdots + f_n t_1^{-\nu} dt_n$.

From the defintion of $d\omega$, we get

$$\begin{aligned} (2.1) \qquad d\omega &= d(f_1 t_1^{-\nu}) \wedge dt_1 + d(f_2 t_1^{-\nu}) \wedge dt_2 + \cdots + d(f_n t_1^{-\nu}) \wedge dt_n \\ &= t_1^{-\nu} df_1 \wedge dt_1 + (t_1^{-\nu} df_2 \wedge dt_2) + (-\nu) f_2 t_1^{-\nu-1} dt_1 \wedge dt_2 \\ &\quad + (t_1^{-\nu} df_3 \wedge dt_3 + (-\nu) f_3 t_1^{-\nu-1} dt_1 \wedge dt_3) + \cdots \\ &\quad + (t_1^{-\nu} df_n \wedge dt_n + (-\nu) f_n t_1^{-\nu-1} dt_1 \wedge dt_n). \end{aligned}$$

By the equality (1.1), we can represent df_i's as follows:

$$(2.2) \qquad df_i = g_{i,1} dt_1 + g_{i,2} dt_2 + \cdots + g_{i,n} dt_n \quad \text{for each } i.$$

From (2.1) and (2.2), we have

$$d\omega = \sum_{i<j} (-g_{i,j} + g_{j.i}) t_1^{-\nu} dt_i \wedge dt_j + \sum_{j=2}^{n} (-\nu) t_1^{-\nu-1} f_j dt_1 \wedge dt_j.$$

Since ω is closed, we get that $d\omega = 0$ and

$$t_1(-g_{1,j} + g_{j.1}) + (-\nu) f_j = 0 \quad \text{for } 1 < j.$$

Since $-g_{1,j} + g_{j,1}$ belongs to $A_\wp$, we obtain the following inequality:

$$\nu_\wp(f_j t_1^{-\nu}) \geq -\nu + 1 \quad \text{for } j \geq 2.$$

So we get $\quad \nu_\wp(f'_i) = \nu_\wp(f_i t_1^{-\nu}) \geq -\nu + 1$ for each $j \geq 2$.

Since $-\nu = \min_i \nu_\wp(f'_i)$, it follows that $\nu_\wp(f'_1) = -\nu$. □

Lemma 2.2. *Let* f *be an element of* K. *Assume that* df *is represented as* $df = g dt_1 + g_2 dt_2 + \cdots + g_n dt_n$, *where* $g, g_j \in K$ $(j \geq 2)$. *Then we have*

$$g^* = \frac{d}{dt_1} f^*,$$

where the right hand side is the formal differentiation of the power series f^* *by* t_1.

Proof. Let $\psi : \widehat{A}_\wp \to \widehat{A}_\wp/\wp\widehat{A}_\wp \simeq D_\wp$ be the natural surjection.

For $F = \sum_i a_i t_1^i \in D_\wp((t_1))$, we define

$$DF = \sum_i a_i i t_1^{i-1} \in D_\wp((t_1)).$$

Then $D(= \frac{d}{dt_1})$ is a $D_\wp$-derivation of $D_\wp((t_1))$ into itself.

Since the field extension $K/k(t_1, t_2, \cdots, t_n)$ is separable, there exists an irreducible polynomial $H(X) = \sum_i q_i X^i$ $(q_i \in k(t_1, t_2, \cdots, t_n))$ such that $H'(f) \neq 0$ and $H(f) = 0$.

Here, by multiplying a suitable polynomial, we may assume that each coefficient belongs to $k[t_1, \cdots, t_n]$, i.e.,

$$q_i \in k[t_1, \cdots, t_n] \quad \text{for all } i.$$

Since $q_i \in k(t_2, \cdots, t_n)[t_1]$, we can identify q_i^* with q_i, in other words, $q_i^* = q_i$. Putting $F = f^*$, we obtain that $H(F) = \{H(f)\}^* = 0$.

Let D operate on both sides of $H(F) = 0$. Then we have

$$\sum_i (Dq_i)F^i + \sum_i q_i i F^{i-1} DF = 0.$$

Since

$$\sum_i q_i i F^{i-1} = H'(F), \quad H'(F) \neq 0,$$

we have

$$DF = -\frac{1}{H'(F)} \sum_i (Dq_i)F^i. \tag{2.3}$$

Operating d on both sides of $H(f) = 0$, we obtain

$$\sum_i (dq_i)f^i + \sum_i q_i i f^{i-1} df = 0,$$

from which we have

$$df = -\frac{1}{H'(f)} \sum_i (dq_i)f^i. \tag{2.4}$$

We compute the right hand side of this equality. Put

$$q_i\ (= q_i^*) = \sum a_{e_{i1}e_{i2}\cdots e_{in}} t_1^{e_{i1}} t_2^{e_{i2}} \cdots t_n^{e_{in}}.$$

Then we find

$$(2.5)\quad \begin{cases} dq_i = \sum\{a_{e_{i1}e_{i2}\cdots e_{in}} e_{i1} t_1^{e_{i1}-1} t_2^{e_{i2}} \cdots t_n^{e_{in}} dt_1 \\ \qquad + a_{e_{i1}e_{i2}\cdots e_{in}} e_{i2} t_1^{e_{i1}} t_2^{e_{i2}-1} \cdots t_n^{e_{in}} dt_2 \\ \qquad + \cdots + a_{e_{i1}e_{i2}\cdots e_{in}} e_{in} t_1^{e_{i1}} t_2^{e_{i2}} \cdots t_n^{e_{in}-1} dt_n\}, \\ Dq_i = \sum a_{e_{i1}e_{i2}\cdots e_{in}} e_{i1} t_1^{e_{i1}-1} t_2^{e_{i2}} \cdots t_n^{e_{in}}. \end{cases}$$

From (2.3) and (2.5), we have

$$DF = -\frac{1}{H'(F)} \sum_i (a_{e_{i1}e_{i2}\cdots e_{in}} e_{i1} t_1^{e_{i1}-1} t_2^{e_{i2}} \cdots t_n^{e_{in}}) F^i.$$

From (2.3) and (2.4), we obtain

$$\begin{aligned} (2.6)\quad df \quad = \quad & -\frac{1}{H'(F)}\{\sum_i (a_{e_{i1}e_{i2}\cdots e_{in}} e_{i1} t_1^{e_{i1}-1} t_2^{e_{i2}} \cdots t_n^{e_{in}}) f^i\} dt_1 \\ & - \frac{1}{H'(F)}\{\sum_i (a_{e_{i1}e_{i2}\cdots e_{in}} e_{i2} t_1^{e_{i1}} t_2^{e_{i2}-1} \cdots t_n^{e_{in}}) f^i\} dt_2 - \cdots \\ & - \frac{1}{H'(F)}\{\sum_i (a_{e_{i1}e_{i2}\cdots e_{in}} e_{in} t_1^{e_{i1}} t_2^{e_{i2}} \cdots t_n^{e_{in}-1}) f^i\} dt_n. \end{aligned}$$

Let G be the coefficient of dt_1 in (2.6). Then we have $G^* = DF$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *For a closed differential ω, if $\nu_\wp(\omega) = -\nu < -1$, then there exists an element $\eta \in K$ such that $\nu_\wp(\omega - d\eta) > -\nu$.*

Proof. We can assume that ω is represented as

$$\omega = f t_1^{-\nu} dt_1 + g_2 dt_2 + \cdots + g_n dt_n,$$

where $f \in A_\wp$. Put

$$\eta := \frac{1}{-\nu+1} f t_1^{-\nu+1} \quad (\in K).$$

Since f belongs to $A_\wp$, df is written as

$$df = f_1 dt_1 + f_2 dt_2 + \cdots + f_n dt_n,$$

where each $f_i \in A_\wp$. Then

$$\begin{aligned} \omega - d\eta \quad = \quad & -\frac{1}{-\nu+1} t_1^{-\nu+1} f_1 dt_1 + (g_2 - \frac{1}{-\nu+1} t_1^{-\nu+1} f_2) dt_2 \\ & + (g_3 - \frac{1}{-\nu+1} t_1^{-\nu+1} f_3) dt_3 \\ & + \cdots + (g_n - \frac{1}{-\nu+1} t_1^{-\nu+1} f_n) dt_n. \end{aligned}$$

To prove that $\nu_\wp(\omega - d\eta) > -\nu$, we may assume that $\nu_\wp(\omega - d\eta) < 0$. Then, by Lemma 2.1,

$$\nu_\wp(\omega - d\eta) = \nu_\wp(t_1^{-\nu+1} f_1) = -\nu + 1 + \nu_\wp(f_1) > -\nu,$$

since $f_i \in A_\wp$. Therefore Lemma 2.3 is proved. □

§3. Result.

We are now ready to prove our Theorem.

Theorem 3.1. *Let ω be a cosed element of $\Omega^1_{K/k}$. Then ω is residue free if and only if ω is of second kind.*

Proof. Let W be a prime divisor of V, $U =$ Spec (A) an affine open subset of V such that $U \cap W = V(\wp)$, where $\wp \in$ Spec (A).

(1) Assume that ω is a closed differential of the second kind represented as $\omega = f_1 dt_1 + f_2 dt_2 + \cdots + f_n dt_n$, where $f_i \in K$. Then by the definition, there exists an element θ of K such that $\nu_\wp(\omega - d\theta) \geq 0$.

Let θ be expanded as

$$\theta^* = \beta_{-m} t_1^{-m} + \beta_{-m+1} t_1^{-m+1} + \cdots + \beta_{-1} t_1^{-1} + \beta_0 + \beta_1 t_1 + \beta_2 t_1^2 + \cdots$$

and let

$$d\theta = g_1 dt_1 + g_2 dt_2 + \cdots + g_n dt_n,$$

$$g_1^* = \gamma_{-m} t_1^{-m} + \cdots + \gamma_{-1} t_1^{-1} + \gamma_0 + \gamma_1 t_1 + \gamma_2 t_1^2 + \cdots.$$

By Lemma 2.2, we have

$$g_1^* = \frac{d}{dt_1}\theta^*.$$

Since the coefficient of t_1^{-1} of the right hand side is zero, we get $\gamma_{-1} = 0$.

On the other hand,

$$\omega - d\theta = (f_1 - g_1)dt_1 + (f_2 - g_2)dt_2 + \cdots + (f_n - g_n)dt_n$$

is closed, so the residue is defined and it follows from $\nu_\wp(\omega - d\theta) \geq 0$ that $res_W(\omega - d\theta) = 0$.

Putting

$$f_1^* = \alpha_{-p} t_1^{-p} + \cdots + \alpha_{-1} t_1^{-1} + \alpha_0 + \alpha_1 t_1^1 + \alpha_2 t_1^2 + \cdots,$$

we have

$$(f_1 - g_1)^* = \cdots + (\alpha_{-1} - \gamma_{-1})t_1^{-1} + \cdots .$$

Hence we get $res_W(\omega - d\theta) = \alpha_{-1} - \gamma_{-1} = \alpha_{-1} = 0$. Thus $res_W(\omega) = 0$. This implies ω is residue free.

(2) Let ω be residue free. By repeating the procedure of Lemma 2.3, for the differential ω, we can find an element θ of K such that $\nu_\wp(\omega - d\theta) \geq -1$. We set

$$\omega - d\theta = f_1 dt_1 + f_2 dt_2 + \cdots + f_n dt_n,$$

and we assume that f_1^* is expressed as

$$f_1^* = \alpha_{-1}t_1^{-1} + \alpha_0 + \alpha_1 t_1^1 + \alpha_2 t_1^2 + \cdots .$$

Since $\omega - d\theta$ is closed, the residue is defined for $\omega - d\theta$. As we have just seen that $res_W(d\theta) = 0$, and $res_W(\omega) = 0$ by the hypothesis, we have

$$\alpha_{-1} = res_W(\omega - d\theta) = res_W(\omega) - res_W(d\theta) = 0.$$

Therefore we find $\nu_\wp(f_1) \geq 0$.

Furthermore, since $\omega - d\theta$ is closed, the inequality

$$\nu_\wp(\omega - d\theta) \geq 0$$

holds by Lemma 2.1. Hence ω is a differential of the second kind, and this completes the proof of the theorem. □

References

[1] F. Elzein, Residues en Géométrie Algébrique, Compositio Mathematica, Vol. 23, Fasc. 4, 1971, p. 379-405.

[2] K. Iwasawa, Algebraic Functions, A. M. S. Mathematical Monographs, Vol. 118.

[3] E. Kunz, Kähler Differentials, Vieweg Advanced Lectures in Mathematics, 1986.

[4] M. Rosenlicht, Differentials of the second kind for algebraic function fields of one variable, Annals of Mathematics, Vol. 57, 1953.

Takeo OHI
Faculty of Science, Science University of Tokyo
26 Wakamiya-cho, Shinjuku-ku, Tokyo, 162 Japan

Hirokazu NAGAI
Faculty of Science ,Science University of Tokyo
26 Wakamiya-cho, Shinjuku-ku, Tokyo, 162 Japan