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REGULAR GENERAL CONNECTIONS AND LORENTZIAN TWISTED PRODUCTS

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Abstract. Regular general connections of conformal type provided for space-times are investigated. Using the Hessian-type tensors of time function, certain isometric decomposition theorems of space-times can be obtained. The null geodesics concerning the general connection of conformal type are defined and a criterion for its incompleteness is researched.

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§0. Introduction

The concept of general connections was introduced by Tominoske Otsuki [14] and the fundamental theories on general connections were constructed in [15],[16],[19]. Recently the general connection is also called the Otsuki connection outside of Japan. It seems to be remarkable that the theories of general connections were applied to the geometrical investigations of space-times by H.Nagayama [10],[11] and T.Otsuki [17], [18],[20],[21]. On the other hand, N.Abe [1],[2],[3] developed the theory of general connections on arbitrary vector bundles, in particular, H.Nemoto [12] discussed a differential geometry of submanifolds using the induced general connections on tangent subbundles.

In the paper, we investigate space-times provided with certain kind of general connections and define the Hessian-type tensors of time function with respect to such general connections.

This paper is organized as follows:In Sect.1, the pseudo-Riemannian twisted products are equipped with torsion-free, metrical, regular general connections and the induced general connections on the factor manifolds are researched. In

Sect.2, the quasi-Hessian tensor of a function relative to a torsion-free general connection is defined by using the notion of tangent bundle of order 2. And also the regular general connection of conformal type is defined. In Sect.3, the regular general connection of conformal type is given to a simply connected space-time admitting a time function such that every timelike geodesic is complete. On such a space-time, a certain foliated structure is assumed, which concerns with the level hypersurfaces of time function and the principal endomorphism of given general connection. By supposing that a certain Hessian-type tensor of time function is symmetric and a scalar multiple of Lorentzian metric, a globally-isometric decomposition theorem is established. In Sect.4, the null geodesic is defined in the sense of the regular general connection of conformal type and a criterion for its geodesic incompleteness can be obtained in the space-time decomposed as a certain Lorentzian warped product. As an application, the isometric decomposition theorem obtained in Sect.3 is applied to the space-time given a regular general connection of scalar type.

In case of defining the Hessian-type tensors, we require the notion of the tangent bundle of order 2 appeared in [14], for the reason, we shall review the definition of a general connection due to T.Otsuki.

The following exposition of general connections is owing to Otsuki's papers [15],[19]:

Let N be an n-dimensional differentiable manifold. Throughout the paper, we use the Einstein convention for summation. Now we choose any different coordinate neighborhood (U, u^i) and (V, v^j) covering a point p of N. Then we assign for the point p an $(n + n^2)$-dimensional vector space spanned by the tensor products $du^i \otimes du^j$ and the differentials d^2u^i of order 2, which are related with the ones of (V, v^i) as follows:

$$d^2v^j = \frac{\partial v^j}{\partial u^i} d^2u^i + \frac{\partial^2 v^j}{\partial u^i \partial u^k} du^i \otimes du^k.$$

Now we call this vector space the cotangent space of order 2 at p, denoting hy $D_p^2 N$. Collecting $D_p^2 N$ for every p of N, we get the cotangent vector bundle of order 2 over N, which is denoted by D^2N. The dual vector bundle of D^2N is called the tangent bundle of order 2 over N and denoted by T^2N. A smooth cross-section Γ of the vector bundle $TN \otimes D^2N$ over N is said to be a general connection on N, which is represented in local coordinates u^i as follows:

$$\frac{\partial}{\partial u^i} \otimes (P^i_j d^2u^j + \Gamma^i_{jk} du^j \otimes du^k),$$

where P^i_j, Γ^i_{jk} are called the components of Γ in (U, u^i). If P'^i_j, $\Gamma'^{\ i}_{jk}$ are the components of Γ in (V, v^i), then we have the following relations:

$$P'^i_j = \frac{\partial v^i}{\partial u^k} P^k_h \frac{\partial u^h}{\partial v^j}, \tag{0.1}$$

and

$$\Gamma'^i_{jk} = \frac{\partial v^i}{\partial u^h}\left(P^h_r \frac{\partial^2 u^r}{\partial v^j \partial v^k} + \Gamma^h_{rs} \frac{\partial u^r}{\partial v^j}\frac{\partial u^s}{\partial v^k}\right). \tag{0.2}$$

Thus P^i_j are the components of a tensor of type $(1, 1\ )$, which we denote by P. P is called the principal endomorphism of Γ.

$$\nabla_X Y := \left(X^i \frac{\partial Y^j}{\partial u^i} P^k_j + X^i Y^j \Gamma^k_{ij}\right)\frac{\partial}{\partial u^k}, \tag{0.3}$$

where $X = X^i \partial/\partial u^i$, $Y = Y^i \partial/\partial u^i$ and the right hand side of (0.3) is independent of the choice of coordinate system, since the formula (0.1) and (0.2) hold. Then we have the following rules:

$$\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z, \tag{0.4}$$

$$\nabla_X (fY + Z) = (Xf)PY + f\nabla_X Y + \nabla_X Z. \tag{0.5}$$

In the paper, we denote by $\Gamma = (P, \nabla)$ a general connection such that P is the principal endomorphism of Γ and ∇ is the covariant derivative with respect to Γ. A general connection $\Gamma = (P, \nabla)$ is said to be torsion-free if for all tangent vector fields X, Y on N,

$$P[X, Y] = \nabla_X Y - \nabla_Y X. \tag{0.6}$$

If $\det(P^i_j(p)) \neq 0$ at every point p of N, then $\Gamma = (P, \nabla)$ is said to be regular.

Suppose (N, g) is a pseudo-Riemannian manifold and $\Gamma = (P, \nabla)$ a general connection on N. Then we say that Γ is metrical with respect to g if

$$Zg(PX, PY) = g(\nabla_Z X, PY) + g(PX, \nabla_Z Y) \tag{0.7}$$

for all tangent vector fields X, Y and Z on N.

Throughout the paper, we use the following notation: TN, the tangent bundle over N; VN, the vector space of smooth cross sections of TN; T^2N, the tangent bund1e of order 2 over N; V^2N, the vector space of smooth cross sections of T^2N; FN, the vector space of smooth functions on N; $D\varphi$, the differential map of a smooth map φ.

§1. Induced general connections

Let (L, g) and (H, h) be a pseudo-Riemannian manifold and a Riemannian manifold, respectively. Let ψ be a positive-valued smooth function on the product manifold $L \times H$.

Definition 1.1. We define the pseudo-Riemannian twisted product $M = L \times_\psi H$ as follows: M is the product manifold equipped with the pseudo-Riemannian metric $\tilde{g}$ defined for v, $w \in T_pM$ by

$$\tilde{g}_p(v, w) := g_{\pi(p)}((D\pi)_p v, (D\pi)_p w) + \psi(p) h_{\eta(p)}((D\eta)_p v, (D\eta)_p w),$$

where $\pi\colon M \to L$ and $\eta\colon M \to H$ denote the projection maps.

Definition 1.2. TL (resp., TH) is said to be P-invariant if $(D\eta)X = 0$ implies $(D\eta)PX = 0$ (resp., $(D\pi)X = 0$ implies $(D\pi)PX = 0$) for $X \in VM$.

Theorem 1.3. *If we give a general connection $\Gamma = (P, \nabla)$ to the twisted product $M = L \times_\psi H$ of (L, g) and (H, h), then there exist general connections $\Gamma^1 = (P^1, \nabla^1)$ and $\Gamma^2 = (P^2, \nabla^2)$ induced on $\pi^{-1}(TL)$ and $\eta^{-1}(TH)$, respectively. Furthermore the following (a), (b) and (c) are valid:*
(a) *If Γ is torsion-free, then Γ^1 and Γ^2 are torsion-free.*
(b) *Suppose that Γ is metrical with respect to $\tilde{g}$. Then*
(b1) *Γ^1 is metrical with respect to g, if TL is P-invariant.*
(b2) *Γ^2 is metrical with respect to h, if TH is P-invariant and ψ depends only on L.*
(c) *Suppose that Γ is regular. If TL and TH are P-invariant, then Γ^1 and Γ^2 are regular.*

Proof. Let $X, Y \in V(\pi^{-1}(TL))$. For a general connection (P, ∇) on $M = L \times_\psi H$, we define the maps $\nabla^1\colon V(\pi^{-1}(TL)) \times V(\pi^{-1}(TL)) \to V(\pi^{-1}(TL))$ (resp., $\nabla^2\colon V(\eta^{-1}(TH)) \times V(\eta^{-1}(TH)) \to V(\eta^{-1}(TH))$ and $P^1\colon V(\pi^{-1}(TL)) \to V(\pi^{-1}(TL))$ (resp., $P^2\colon V(\eta^{-1}(TH)) \to V(\eta^{-1}(TH))$) by

$$\nabla^1_X Y := (D\pi)\nabla_X Y \quad \textit{and} \quad P^1 X := (D\pi)PX$$

(resp., $\nabla^2_U W := (D\eta)\nabla_U W$ and $P^2 U := (D\eta)PU$ for $U, W \in V(\eta^{-1}(TH))$), where we are identifying X and U with $(X, 0) \in V(L \times H)$ and $(0, U) \in V(L \times H)$, respectively. Then we have for $f \in FL$ and $Z \in V(\pi^{-1}(TL))$,

$$\begin{aligned}\nabla^1_{fX+Y} Z &= (D\pi)(\nabla_{fX+Y} Z) \\ &= (D\pi)(f\nabla_X Z) + (D\pi)(\nabla_Y Z) = f\nabla^1_X Z + \nabla^1_Y Z.\end{aligned}$$

Similarly we have

$$\nabla^1_X(Y+Z) = \nabla^1_X Y + \nabla^1_X Z.$$

Also we obtain

$$\begin{aligned}\nabla^1_X(fY) &= (D\pi)\nabla_X(fY) = (D\pi)((Xf)PY + f\nabla_X Y)\\ &= (Xf)P^1Y + f\nabla^1_X Y.\end{aligned}$$

Consequently, (P^1, ∇^1) defines a general connection Γ^1 on $\pi^{-1}(TL)$ such that its principal endomorphism is P^1. Similarly we obtain the induced general connection $\Gamma^2 = (P^2, \nabla^2)$ on $\eta^{-1}(TH)$. Let Γ be torsion-free and X, $Y \in V(\pi^{-1}(TL))$. Then we have

$$\begin{aligned}\nabla^1_X Y - \nabla^1_Y X &= (D\pi)(\nabla_X Y - \nabla_Y X)\\ &= (D\pi)(P[X,Y]) = P^1[X,Y],\end{aligned}$$

which implies Γ^1 is torsion-free. Similarly we see Γ^2 is torsion-free.

Let Γ be metrical with respect to $\tilde{g}$ and let X, Y, $Z \in V(\pi^{-1}(TL))$ and U, V, $W \in V(\eta^{-1}(TH))$. From the direct computations, we obtain the following

$$\begin{aligned}&Zg(P^1X, P^1Y)\\ &\quad = g(\nabla^1_Z X, P^1Y) + g(P^1X, \nabla^1_Z Y) + \psi h((D\eta)\nabla_z X, (D\eta)PY)\\ &\qquad + \psi h((D\eta)PX, (D\eta)\nabla_Z Y) - Z(\psi h(D\eta)PX, (D\eta)PY)),\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}&Wh(P^2U, P^2V)\\ &\quad = h(\nabla^2_W U, P^2V) + h(P^2U, \nabla^2_w V) - \psi^{-1}Wg((D\pi)PU, (D\pi)PV)\\ &\qquad + \psi^{-1}g((D\pi)\nabla_W U, (D\pi)PV) + \psi^{-1}g((D\pi)PU, (D\pi)\nabla_W V)\\ &\qquad - \psi^{-2}(W\psi)\{\tilde{g}(PU, PV) - g((D\pi)PU, (D\pi)PV)\}.\end{aligned} \tag{1.2}$$

If TL be P-invariant, then the formula (1.1) implies

$$Zg(P^1X, P^1Y) = g(\nabla^1_Z X, P^1Y) + g(P^1X, \nabla^1_Z Y),$$

that is, Γ^1 is metrical with respect to g. If TH is P-invariant and ψ depends only on L, then from the formula (1.2), we have

$$Wh(P^2U, P^2V) = h(\nabla^2_W U, P^2V) + h(P^2U, \nabla^2_W V),$$

that is, Γ^2 is metrical with respect to h. Thus we have (b1) and (b2).

Let $x \in M$ and $\det P(x)$ the determinant of linear transformation $P(x) = (P^i_j(x))$ on T_xM induced by the principal endomorphism P of Γ. Then since TL and TH are P-invariant, we have for every point x of M

$$\det P(x) = (\det P^1(x))(\det P^2(x)),$$

where $P^1(x)$ (resp., $P^2(x)$) denotes the linear transformation induced by the principal endomorphism of Γ^1 (resp., Γ^2). Thus we obtain (c). □

Definition 1.4. $\Gamma^1 = (P^1, \nabla^1)$ and $\Gamma^2 = (P^2, \nabla^2)$ are called the induced general connections on $\pi^{-1}(TL)$ and $\eta^{-1}(TH)$, respectively.

Definition 1.5. A pseudo-Riemannian twisted product $M = L \times_\psi H$ is said to be a pseudo-Riemannian warped product if ψ depends only on L, and ψ is called a warping function. In particular we say M a Lorentzian warped product if (L, g) is a Lorentzian manifold.

In the following theorem, we lead an explicit formula with respect to the induced connections defined in Theorem 1.3.

Theorem 1.6. *Let $M = L \times_\psi H$ be a pseudo-Riemannian warped product $(L \times H, \tilde{g})$ given a regular general connection $\Gamma = (P, \nabla)$ that is torsion-free and metrical with respect to $\tilde{g}$. Let $\Gamma^1 = (P^1, \nabla^1)$ and $\Gamma^2 = (P^2, \nabla^2)$ be the induced general connections on $\pi^{-1}(TL)$ and $\eta^{-1}(TH)$, respectively. If TL and TH are P-invariant, moreover, P is symmetric with respect to $\tilde{g}$, i.e. $\tilde{g}(PX, Y) = \tilde{g}(X, PY)$ for any $X, Y \in VM$, then the following formula holds.*

$$\begin{aligned}\nabla_X Y \ &= \ \nabla^1_{X^1} Y^1 + \nabla^2_{X^2} Y^2 + \frac{1}{2}\{X^1(\log\psi) PY^2 + Y^1(\log\psi) PX^2 \\ &\qquad - \tilde{g}(PX^2, PY^2) Q(\mathrm{grad}(\log\psi))\},\end{aligned}$$

where Q denotes the inverse endomorphism of P and X^1 (resp., X^2) denotes $(D\pi)X$ (resp., $(D\eta)X$), respectively.

Proof. Let X, Y, $Z \in VM$. Throughout the proof, we shall identify $(D\pi)X \in V(\pi^{-1}(TL))$ with $((D\pi)X, 0) \in VM$. Since Γ is torsion-free and metrical with respect to $\tilde{g}$, we have

$$\begin{aligned}(1.3)\qquad & 2\tilde{g}(\nabla_X Y, PZ) \\ &= X\tilde{g}(PY, PZ) + Y\tilde{g}(PZ, PX) - Z\tilde{g}(PX, PY) \\ &\quad + \tilde{g}(P[Z, X], PY) + \tilde{g}(P[X, Y], PZ) - \tilde{g}(P[Y, Z], PX).\end{aligned}$$

$$(1.4)\qquad (D\pi)(PX) = (D\pi)(PX^1) = P^1 X^1,$$

$$(D\eta)(PX) = (D\eta)(PX^2) = P^2 X^2.$$

The identifications such as (1.4) are everywhere used in the proof. Similarly as (1.3), using Theorem 1.3, we have

$$\begin{aligned}(1.5)\qquad & 2\{g(\nabla^1_{X^1} Y^1, P^1 Z^1) + \psi h(\nabla^2_{X^2} Y^2, P^2 Z^2)\} \\ &= X^1 g(P^1 Y^1, P^1 Z^1) + Y^1 g(P^1 Z^1, P^1 X^1) - Z^1 g(P^1 X^1, P^1 Y^1)\end{aligned}$$

$$
\begin{aligned}
&+ \psi\{X^2 h(P^2Y^2, P^2Z^2) + Y^2 h(P^2Z^2, P^2X^2)\\
&\quad - Z^2 h(P^2X^2, P^2Y^2)\}\\
&+ g(P^1[Z^1, X^1], P^1Y^1) + g(P^1[X^1, Y^1], P^1Z^1)\\
&- g(P^1[Y^1, Z^1], P^1X^1)\\
&+ \psi\{h(P^2[Z^2, X^2], P^2Y^2) + h(P^2[X^2, Y^2], P^2Z^2)\\
&\quad - h(P^2[Y^2, Z^2], P^2X^2)\}.
\end{aligned}
$$

Since ψ depends only on L, we see

$$
\begin{aligned}
(1.6)\quad & X^1 g(P^1Y^1, P^1Z^1) + \psi X^2 h(P^2Y^2, P^2Z^2)\\
&= X\{g(P^1Y^1, P^1Z^1) + \psi h(P^2Y^2, P^2Z^2)\} - (X^1\psi)h(P^2Y^2, P^2Z^2)\\
&= X\tilde{g}(PY, PZ) - (X^1\psi)h(P^2Y^2, P^2Z^2).
\end{aligned}
$$

Using (1.3), (1.5), (1.6) and the P-invariance of TL and TH, we have

$$
\begin{aligned}
&2\{g(\nabla^1_{X^1}Y^1, P^1Z^1) + \psi h(\nabla^2_{X^2}Y^2, P^2Z^2)\}\\
&\quad = X\tilde{g}(PY, PZ) + Y\tilde{g}(PZ, PX) - Z\tilde{g}(PX, PY) + \tilde{g}(P[Z, X], PY)\\
&\qquad + \tilde{g}(P[X, Y], PZ) - \tilde{g}(P[Y, Z], PX) - (X^1\psi)h(P^2Y^2, P^2Z^2)\\
&\qquad - (Y^1\psi)h(P^2Z^2, P^2X^2) + (Z^1\psi)h(P^2X^2, P^2Y^2)\\
&\quad = 2\tilde{g}(\nabla_X Y, PZ) - (X^1\psi)h(P^2Y^2, P^2Z^2) - (Y^1\psi)h(P^2Z^2, P^2X^2)\\
&\qquad + (Z^1\psi)h(P^2X^2, P^2Y^2).
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
&2\{g(\nabla^1_{X^1}Y^1, P^1Z^1) + \psi h(\nabla^2_{X^2}Y^2, P^2Z^2)\}\\
&\quad = 2\tilde{g}(\nabla_X Y, PZ) - (X^1(\log\psi))\tilde{g}(P^2Y^2, P^2Z^2)\\
&\qquad - (Y^1(\log\psi))\tilde{g}(P^2Z^2, P^2X^2) + (Z^1(\log\psi))\tilde{g}(P^2X^2, P^2Y^2).
\end{aligned}
$$

Thus, from the identifications such as (1.4) and $PX^1 = P^1X^1$, we have

$$
\begin{aligned}
&2\tilde{g}(\nabla_X Y - \nabla^1_{X^1}Y^1 - \nabla^2_{X^2}Y^2, PZ)\\
&\quad = (X^1(\log\psi))\tilde{g}(P^2Y^2, PZ) + (Y^1(\log\psi))\tilde{g}(P^2X^2, PZ)\\
&\qquad - \tilde{g}(PX^2, PY^2)\tilde{g}(\mathrm{grad}(\log\psi), Z),
\end{aligned}
$$

from which, we see easily

$$
\begin{aligned}
(1.7)\qquad & 2\tilde{g}(\nabla_X Y - \nabla^1_{X^1}Y^1 - \nabla^2_{X^2}Y^2, Z)\\
&\quad = (X^1(\log\psi))\tilde{g}(P^2Y^2, Z) + (Y^1(\log\psi))\tilde{g}(P^2X^2, Z)\\
&\qquad - \tilde{g}(PX^2, PY^2)\tilde{g}(\mathrm{grad}(\log\psi), QZ).
\end{aligned}
$$

Since Q is symmetric with respect to $\tilde{g}$, (1.7) leads to the formula of Theorem 1.6. □

Remark 1.7. N.Abe [2] has developed a standard theory of induced general connections on vector subbundles.

§2. Regular general connections of conformal type

Let N be a differential manifold. We denote by T^2N and V^2N the tangent bundle of order 2 over N and the space of its smooth cross-sections, respectively. First we shall define the Hessian-type tensor of a function relative to a torsion-free general connection of N.

Definition 2.1. A bilinear map $E\colon VN \otimes VN \to V^2N$ is said to be left-sided F-linear if $E(fX, Y) = fE(X, Y)$ for every X, $Y \in VN$ and every $f \in FN$. Let $\Gamma = (P, \nabla)$ be a torsion-free general connection of N. We say a left-sided F-linear map E is associated with Γ if $E(X, fY) = fE(X, Y) + (Xf)PY$ for every X, $Y \in VN$ and every $f \in FN$.

Definition 2.2. Let $\Gamma = (P, \nabla)$ be a torsion-free general connection of N and $E\colon VN \otimes VN \to V^2N$ a left-sided F-linear map associated with Γ. Then, for $\nu \in FN$, we can define the tensor field H_ν on N of type $(0, 2)$ as follows:

$$H_\nu(X, Y) := E(X, Y)\nu - (\nabla_X Y)\nu$$

for every X, $Y \in VN$. We say H_ν the quasi-Hessian tensor field with respect to Γ of ν.

Note. In the definition above, H_ν is not always symmetric as a tensor of type $(0, 2)$. In fact, $H_\nu(X, Y) = H_\nu(Y, X)$ if and only if $E(X, Y)\nu - E(Y, X)\nu = (P[X, Y])\nu$.

Now we shall give the examples of quasi-Hessian tensor fields with respect to torsion-free general connections:

Examples 2.3. Let X, $Y \in VN$ and $\nu \in FN$. (a) Let D be a torsion-free linear connection of N, which we regard as a special one of torsion-free general connections. Since it is easy to verify that the left-sided F-linear map $E_1\colon (X, Y) \to XY \in V^2N$ is associated with D, we can see that the quasi-Hessian tensor h_ν of ν relative to D can be defined as follows:

$$h_\nu(X, Y) := E_1(X, Y)\nu - (D_X Y)\nu.$$

Then h_ν is consistent with the (usual) Hessian tensor of ν.

(b) Let $\lambda \in FN$ be positive-valued and $\Gamma_\lambda = (P, \nabla)$ a torsion-free, regular general connection of N such that $P = \lambda I$, where I denotes the identity endomorphism of TN. Then the left-sided F-linear map $E_\lambda \colon VN \otimes VN \to V^2N$ defined by $(X, Y) \to \lambda XY$ is associated with Γ_λ. Hence we can give the quasi-Hessian tensor H^λ_ν of ν by

$$H^\lambda_\nu(X, Y) := E_\lambda(X, Y)\nu - (\nabla_X Y)\nu = \lambda X(Y\nu) - (\nabla_X Y)\nu,$$

which is symmetric, since Γ_λ is torsion-free.

Next we shall define regular general connections of conformal type and the associated quasi-Hessian tensors of a function. Throughout the paragraph below, (M, g) denotes a pseudo-Riemannian manifold.

Definition 2.4. Let R be a regular tensor field of type $(1, 1)$ on (M, g). We say R is ω-conformal if $g(RX, RY) = \omega g(X, Y)$ for some positive function $\omega \in FM$ and for every X, $Y \in VM$.

On the other hand, the following is due to T.Otsuki:

Proposition 2.5. *Let R and g be a regular tensor field type $(1, 1)$ and a nondegenerate symmetric tensor field of type $(0, 2)$ on a differential manifold, respectively. Then there exists a uniquely determined general connection Γ satisfying the following:*

(a) *The principal endomorphism of Γ is R,*

(b) *Γ is torsion-free,*

and (c) *Γ is metrical with respect to g.*

According to the above proposition, we can state the following:

Definition 2.6. Let R be an ω-conformal regular tensor field of type $(1, 1)$ on (M, g) and let $\Gamma(\omega) = (R, \nabla)$ a uniquely determined general connection of (M, g) such that

(a) The principal endomorphism of $\Gamma(\omega)$ is R,

(b) $\Gamma(\omega)$ is torsion-free,

and (c) $\Gamma(\omega)$ is metrical with respect to g.

Then $\Gamma(\omega)$ is said to be an ω-conformal genenal connection of (M, g).

Proposition 2.7. *Let $\Gamma(\omega) = (R, \nabla)$ be an ω-conformal general connection of $(M,\ g)$ and $\nu \in FM$. The bilinear map $E^\omega \colon VM \otimes VM \to V^2M$ defined by*

$$E^\omega(X, Y)\nu = X((RY)\nu) - (X(\log \sqrt{\omega}))(RY)\nu$$

is a left-sided F-linear map associated with $\Gamma(\omega)$.

Proof. Let X, $Y \in VM$ and $f, h \in FM$. Noticing that

$$(X(f(RY)))h = (Xf)((RY)h) + fX((RY)h),$$

we have

$$\begin{aligned} E^\omega(X, fY)\nu &= X(f(RY)\nu) - f(X(\log\sqrt{\omega}))(RY)\nu \\ &= (Xf)((RY)\nu) + fX((RY)\nu) - f(X(\log\sqrt{\omega}))(RY)\nu \\ &= (Xf)((RY)\nu) + fE^\omega(X, Y)\nu \end{aligned}$$

and $E^\omega(fX, Y) = fE^\omega(X, Y)$, from which we see that E^ω is a left-sided F-linear map associated with $\Gamma(\omega)$. □

Using the proposition above we can state the following:

Definition 2.8. Let $\Gamma(\omega)$ be an ω-conformal general connection of (M, g) and $\nu \in FM$. The tensor H^ω_ν of type $(0, 2)$ defined by

$$H^\omega_\nu(X, Y) = E^\omega(X, Y)\nu - (\nabla_X Y)\nu \qquad (X, Y \in VM)$$

is called the quasi-Hessian tensor with respect to $\Gamma(\omega)$ of ν. We say H^ω_ν is proportional to g, if H^ω_ν is symmetric and $H^\omega_\nu = \mu g$ for some $\mu \in FM$.

The following is a standard formulation of the quasi-Hessian tensor:

Lemma 2.9.

$$H^\omega_\nu(X, Y) = g(\nabla_X S(\operatorname{grad}\nu), RY) - (X(\log\sqrt{\omega}))g(\operatorname{grad}\nu, RY),$$

where S denotes the inverse endomorphism of R.

Proof. Since $\Gamma(\omega)$ is metrical with respect to g, we obtain

$$\begin{aligned} H^\omega_\nu(X, Y) &= X((RY)\nu) - (X(\log\sqrt{\omega}))(RY)\nu - (\nabla_X Y)\nu \\ &= Xg(R(S(\operatorname{grad}\nu)), RY) - (\nabla_X Y)\nu - (X(\log\sqrt{\omega}))(RY)\nu \\ &= g(\nabla_X S(\operatorname{grad}\nu), RY) - (X(\log\sqrt{\omega}))(RY)\nu, \end{aligned}$$

which yields the formula of Lemma 2.9. □

§3. Isometric decompositions of space-times

First we shall prepare the definition of space-times. Throughout the paper below, we denote by (M, g) a Lorentzian manifold together with Lorentzian metric g of signature $(+, \cdots, +, -)$. A nonzero vector $v \in T_pM$ is said to be timelike (resp., null, spacelike) if $g(v, v) < 0$ (resp., $= 0$, > 0). A tangent vector field V on M is said to be timelike if $g(V_p, V_p) < 0$ for every point p of M. If (M, g) does admit a globally defined timelike vector field V, then we say that (M, g) is time-oriented by V. A time-oriented, noncompact, Lorentzian manifold is called a space-time. More precisely,

Definition 3.1. A space-time is a noncompact, connected, smooth Hausdorff manifold of dimension ≥ 2 which has a countable basis, a Lorentzian metric of signature $(+, \cdots, +, -)$ and a time orientation.

Definition 3.2. $\Phi \in FM$ is called a time function if the gradient vector field $\operatorname{grad} \Phi$ of Φ is timelike everywhere. If (M, g) admits a time function Φ, then (M, g) becomes a space-time by $\operatorname{grad} \Phi$. In this case, we say (M, g) is a space-time admitting a time function Φ. Moreover, $\beta := (-g(\operatorname{grad} \Phi, \operatorname{grad} \Phi))^{-1/2}$ is called the lapse function of Φ.

Remark 3.3. We make no assumptions about the causal topology [5], except requiring that (M, g) has a time function [4].

Let (M, g) be an $(n + 1)$-dimensional space-time admitting a time function Φ. Since (M, g) has a foliated structure by level hypersurfaces of Φ, we can describe the following

Definition 3.4. Let R be a regular endomorphism on TM. the assignment:

$$p \in M \quad \rightarrow \quad R(ker(D\Phi)_p) \subset T_pM$$

defines an n-dimensional distribution on M. This is called the $R\Phi$-disribution on (M, g).

Now we prove the following theorem with respect to locally isometric decompositions of space-times.

Theorem 3.5. *Let (M, g) be an $(n+1)$-dimensional space-time admitting a time function Φ, $\Gamma(\omega) = (R, \nabla)$ an ω-conformal general connection of (M, g) and S the inverse endomorphism of R. Suppose the $S\Phi$-distribution is involutive and assume the quasi-Hessian H^{ω}_{Φ} of Φ is proportional to g. Then (M, g) is locally isometric to the following:*

$$ds^2 = -dt^2 + (\omega\beta)^{-2} du^2,$$

where du^2 is a certain n-dimensional Riemannian metric and β is the lapse function of Φ.

Proof. Let $p \in M$. Since the $S\Phi$-distribution on (M, g) is involutive, we can choose a coordinate neighborhood $(U; y^1, \cdots, y^{n+1})$ at p such that $y^1, \cdots, y^n$ are local coordinates in the leaf of $S\Phi$-foliation through p. □

Lemma 3.6.

$$\frac{\partial}{\partial y^i} g(S(\operatorname{grad}\Phi), S(\operatorname{grad}\Phi)) = 0, \qquad 1 \leq i \leq n.$$

Proof. Using the covariant derivative with respect to $\Gamma(\omega)$ and Lemma 2.9, we have

$$\begin{aligned}
(3.1)\quad \frac{\partial}{\partial y^i} g(S(\operatorname{grad}\Phi), S(\operatorname{grad}\Phi)) &= \frac{\partial}{\partial y^i}(\omega^{-1} g(\operatorname{grad}\Phi, \operatorname{grad}\Phi)) \\
&= -\omega^{-2}\frac{\partial \omega}{\partial y^i} g(\operatorname{grad}\Phi, \operatorname{grad}\Phi) + 2\omega^{-1} g(\nabla_{\partial/\partial y^i} S(\operatorname{grad}\Phi), \operatorname{grad}\Phi) \\
&= 2\omega^{-1}\{g(\nabla_{\partial/\partial y^i} S(\operatorname{grad}\Phi), \operatorname{grad}\Phi) - 2^{-1}\omega^{-1}\frac{\partial \omega}{\partial y^i} g(\operatorname{grad}\Phi, \operatorname{grad}\Phi)\} \\
&= 2\omega^{-1} H^{\omega}_{\Phi}(\frac{\partial}{\partial y^i}, S(\operatorname{grad}\Phi)).
\end{aligned}$$

Let $(u^1, \cdots, u^n)$ be a suitable local coordinate system at p in Φ-level hypersurface. Then we obtain

$$g(S(\frac{\partial}{\partial u^i}), S(\operatorname{grad}\Phi)) = 0,$$

and $$span\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\} = span\{S(\frac{\partial}{\partial u^1}), \cdots, S(\frac{\partial}{\partial u^n})\}.$$

Thus, from the assumption of H^{ω}_{Φ}, we obtain

$$(3.2)\qquad H^{\omega}_{\Phi}(\frac{\partial}{\partial y^i}, S(\operatorname{grad}\Phi)) = 0, \qquad 1 \leq i \leq n.$$

From (3.1) and (3.2), we obtain the formula of Lemma 3.6. □

Using Lemma 3.6, we have a coordinate neighborhood $(W; y^1, \cdots, y^n, t)$ at p such that

$$g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1 \quad \text{and} \quad g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}) = 0.$$

Then $(y^1, \cdots, y^n, t)$ is a geodesically parallel coordinate system at p in the sense of the Levi-Civita connection. Let α and $F^i_j (i, j = 1, \cdots, n)$ be smooth functions on W such that

$$S(\operatorname{grad}\Phi) = \alpha \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial y^j} = F^i_j S(\frac{\partial}{\partial u^i}).$$

Now remark that α is a nonzero-valued function depending only on t in $(W; y^1, \cdots, y^n, t)$ (See Lemma 3.6). We argue on $(W; y^1, \cdots, y^n, t)$ in the proof below:

Lemma 3.7. *Let* $h_{ij} := g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ *and* $Z_j := F^i_j \frac{\partial}{\partial u^i}$ *(* $i, j = 1, \cdots, n$ *). Then* $\frac{\partial}{\partial t} g(Z_i, Z_j) = 2\alpha^{-1}\mu h_{ij}$*, where* $H^\omega_\Phi = \mu g$*.*

Proof. By Lemma 2.9, we have

$$\begin{aligned} H^\omega_\Phi(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) &= H^\omega_\Phi(\frac{\partial}{\partial y^i}, SZ_j) \\ &= g(\nabla_{\partial/\partial y^i} S(\operatorname{grad}\Phi), Z_j) - 2^{-1}\omega^{-1}\frac{\partial \omega}{\partial y^i} g(\operatorname{grad}\Phi, Z_j), \end{aligned}$$

furthermore, using $g(\operatorname{grad}\Phi,\, Z_j) = 0$ and the assumption of H^ω_Φ, we see

$$\begin{aligned} \mu h_{ij} &= g(\nabla_{\partial/\partial y^i} S(\operatorname{grad}\Phi), Z_j) \\ &= \alpha g(\nabla_{\partial/\partial y^i} \frac{\partial}{\partial t}, Z_j). \end{aligned} \tag{3.3}$$

$$\frac{\partial}{\partial t} g(Z_i, Z_j) = g(\nabla_{\partial/\partial t} \frac{\partial}{\partial y^i}, Z_j) + g(Z_i, \nabla_{\partial/\partial t} \frac{\partial}{\partial y^j}). \tag{3.4}$$

(3.4), we obtain

$$\frac{\partial}{\partial t} g(Z_i, Z_j) = \alpha^{-1}\mu(h_{ij} + h_{ji}) = 2\alpha^{-1}\mu h_{ij}.$$

□

Lemma 3.8.

$$\mu = \omega \frac{\partial \alpha}{\partial t}.$$

Proof. Now we can easily obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \frac{\partial}{\partial t}(\omega^{-1} g(R(\frac{\partial}{\partial t}), R(\frac{\partial}{\partial t}))) \\ &= \omega^{-1}\frac{\partial \omega}{\partial t} + 2\omega^{-1} g(\nabla_{\partial/\partial t}\frac{\partial}{\partial t}, \alpha^{-1}\operatorname{grad}\Phi), \end{aligned}$$

from which, we have

$$g(\nabla_{\partial/\partial t}\partial/\partial t, \operatorname{grad}\Phi) = -2^{-1}\alpha\frac{\partial \omega}{\partial t}. \tag{3.5}$$

Thus using $\operatorname{grad}\Phi = \alpha R(\frac{\partial}{\partial t})$ and (3.5), we obtain

$$\begin{aligned} -\mu &= H^{\omega}_{\Phi}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \\ &= g(\nabla_{\partial/\partial t} S(\operatorname{grad}\Phi), R(\frac{\partial}{\partial t})) - 2^{-1}\omega^{-1}\frac{\partial \omega}{\partial t} g(\operatorname{grad}\Phi, R(\frac{\partial}{\partial t})) \\ &= \frac{\partial \alpha}{\partial t} g(R(\frac{\partial}{\partial t}), R(\frac{\partial}{\partial t})) \\ &\quad + \alpha g(\nabla_{\partial/\partial t}\frac{\partial}{\partial t}, R(\frac{\partial}{\partial t})) - 2^{-1}\omega^{-1}\frac{\partial \omega}{\partial t} g(\operatorname{grad}\Phi, \alpha^{-1}\operatorname{grad}\Phi) \\ &= -\omega\frac{\partial \alpha}{\partial t} + g(\nabla_{\partial/\partial t}\frac{\partial}{\partial t}, \operatorname{grad}\Phi) - 2^{-1}\omega^{-1}\frac{\partial \omega}{\partial t}\alpha^{-1}(-\alpha^2\omega) \\ &= -\omega\frac{\partial \alpha}{\partial t}, \end{aligned}$$

hence, we get the formula of Lemma 3.8. □

Applying the lemma above, let us prove Theorem 3.5: From Lemma 3.7 and Lemma 3.8, we have

$$\frac{\partial}{\partial t} g(R(\frac{\partial}{\partial y^i}), R(\frac{\partial}{\partial y^j})) = 2\omega\alpha^{-1}\frac{\partial \alpha}{\partial t} h_{ij}. \tag{3.6}$$

Using $g(R(\frac{\partial}{\partial y^i}), R(\frac{\partial}{\partial y^j})) = \omega h_{ij}$ and (3.6), we obtain $\frac{\partial}{\partial t}(\omega\alpha^{-2}h_{ij}) = 0$. Consequently we see that $du^2 := \omega\alpha^{-2}h_{ij}dy^i dy^j$ becomes an n-dimensional Riemannian metric on the leaf of $S\Phi$-foliation through p in W. Then we can express the Lorentzian metric g on M as follows :

$$ds^2 = -dt^2 + h_{ij}dy^i dy^j = -dt^2 + \omega^{-1}\alpha^2 du^2.$$

Moreover, since $\alpha^2 = -\omega^{-1} g(\operatorname{grad}\Phi, \operatorname{grad}\Phi) = \omega^{-1}\beta^{-2}$, we have

$$ds^2 = -dt^2 + (\omega\beta)^{-2}du^2.$$

Thus we complete the proof of Theorem 3.5 because our argument holds for each point p of M.

By applying a theory of complementary orthogonal foliations to the Lorentzian locally twisted product structure in Theorem 3.5, we shall establish the following globally isometric decomposition theorem:

Theorem 3.9. *Let (M, g) be as in Theorem 3.5. Under the assumptions of Theorem 3.5, furthermore, suppose that (M, g) is simply connected and every timelike geodesic is complete in the sense of the Levi-Civita connection. Then (M, g) is globally isometric to the Lorentzian twisted product $L \times_\eta H$ defined by*

$$(L \times H, -dt^2 + \eta du^2) \quad \text{and} \quad \eta = (\omega\beta)^{-2},$$

where $L := (-\infty, +\infty)$ is given the negative definite metric $-dt^2$ and (H, du^2) is a certain n-dimensional Riemannian manifold.

Proof. Let F_1 and F_2 be foliations on M given by $S(\operatorname{grad}\Phi)$ and the $S\Phi$ -foliation, respectively. From the proof of Theorem 3.5, the leaves of F_1 are complete timelike geodesics and F_2 is a totally umblic foliation, respectively. Let $P(F_1)$ be $\cup\{P(L_1); L_1 \text{ is a leaf of } F_1\}$, where $P(L_1)$ denotes a linear frame bundle over L_1. Then $P(F_1)$ is a principal bundle over M with structure group $GL(1, \boldsymbol{R})$ and by π we denote its projection map. Then we have a foliation F_1^* on $P(F_1)$ defined by $(F_1^*)_u := (D\pi)^{-1}((F_1)_{\pi(u)})$ for every $u \in P(F_1)$. Let P^* be the reduced bundle of $P(F_1)$ obtained by taking normal frames of the leaves of F_1, noticing that each leaf of F_1 is a complete timelike geodesic in the sense of Levi-Civita connection. As we use the technique of R.A.Blumenthal and J.J.Hebda, we first prepare some concepts with respect to the theory of foliations described in [6], [7] and [9]. Throughout this section, a vertical (resp., horizontal) curve on M means a piecewise smooth curve whose tangent vector field lies in F_1 (resp., F_2). A tangent vector field X on M is said to be vertical (resp., horizontal) if X lies in F_1 (resp., F_2). Hereafter by $L_1(p)$ (resp., $L_2(p)$) we denote the leaf of F_1 (resp., F_2) through a point p of M. □

Definition 3.10([6][9]). For a horizontal curve $c: [0, 1] \to M$ there exists a family of diffeomorphisms $\sigma_t: U_0 \to U_t \quad (0 \le t \le 1)$ such that

(a) U_t is a neighborhood of $c(t)$ in $L_1(c(t))$ for each t,

(b) $\sigma_t(c(0)) = c(t)$,

(c) the curve $t \to \sigma_t(q)$ $(\sigma_0(q) = q)$ for each $q \in U_0$ is a horizontal curve, and

(d) σ_0 is the identity map of U_0.

This family of diffeomorphisms σ_t is said to be an element of holonomy along a horizontal curve c.

Definition 3.11([6][7]). A rectangle is a piecewise smooth map $\delta: [0, 1] \times [0, 1] \to M$ such that for each fixed $t_0 \in [0, 1]$ the curve $\delta(t_0, s)$ is vertical and for each fixed $s_0 \in [0, 1]$ the curve $\delta(t, s_0)$ horizontal. The curves $\delta(0, s)$ and $\delta(1, s)$ ($\delta(t, 0)$ and $\delta(t, 1)$) are called the initial vertical (horizontal) edge and the terminal vertical (horizontal) edge, respectively. In general, a complementary distribution D to a foliation F on a differentiable manifold is called an Ehresmann connection for F if, for each vertical curve c_1 and horizontal curve c_2 with the same starting point, there exists a rectangle whose initial edges are c_1 and c_2.

Definition 3.12([7]). Let $\dim F_1^* = m$. A complete parallelism for F_1^* is a family $\{V_i; 1 \leq i \leq m\}$ of complete vector fields on P^* that are linearly independent and tangent to F_1^* everywhere.

Lemma 3.13. *A natural lift of F_2 can be defined as a complementary distribution D^* to F_1^* on P^* .*

Proof. First we see that the elements of holonomy along horizontal curves on M are local isometries. In fact, since the leaves of F_1 can be considered as complete timelike normal geodesics, for a horizontal vector field Z parallel along the leaves of F_1, we have

$$(\mathcal{L}_Z g)(\dot{c}(t), \dot{c}(t)) = 0.$$

where D and $\mathcal{L}_Z$ denote the Levi-Civita connection and the Lie derivative relative to Z respectively and $\dot{c}(t)$ is the velocity vector field of a timelike normal geodesic $c(t)$, a leaf of F_1. Next we note that there exists a complementary distribution D^* to F_1^* on $P(F_1)$ as follows: Let $u_0 \in P(F_1)$ and $p_0 \in \pi(u_0)$. We now choose a neighborhood U at p_0 in M and a horizontal vector field Y in U which is parallel along the leaves of F_1 in the sense of the Levi-Civita connection. Since the local 1-parameter group ρ_t of transformations generated by Y sends the leaves of F_1 to the leaves of F_1, $D\rho_t$ is a local 1-parameter group of transformations in a certain neighborhood of u_0. Let Y^* be the vector field on $P(F_1)$ induced by $D\rho_t$ and we define $D^*(u_0) := \{Y^*; Y$ is a horizontal vector field on a neighborhood of p_0 and is parallel along the leaves of $F_1\}$. From this construction we have $(D\pi)Y^* = Y$. Thus D^* is a complementary distribution to F_1^* on $P(F_1)$. Furthermore, we note that a complete parallelism $\{V_1, \cdots, V_m\}$ for F_1^* is preserved by D^*, that is, each V_i is invariant under the elements of holonomy along D^* -curves, since the elements of holonomy along horizontal curves on M are local isometries. Therefore D^* is a complementary distribution to F_1^* on P^*. Thus we complete the proof of the lemma 3.13. □

Moreover the following lemmata can be proved as in [7][9]:

Lemma 3.14([7]). *If D^* preserves a complete parallelism for F_1^*, then D^* is an Ehresmann connection for F_1^*, and furthermore, F_2 is an Ehresmann connection for F_1.*

Lemma 3.15([9]). *If F_2 is an Ehresmann connection for F_1, then for each leaf L_1 of F_1 and each leaf L_2 of F_2, there exists a diffeomorphism: $\tilde{L}_1(p) \times \tilde{L}_2(p) \to \tilde{M}(p)$, where $\tilde{M}(p)$ (resp., $\tilde{L}_1(p), \tilde{L}_2(p)$) denote the universal covering manifold of M (resp., L_1, L_2) identfied with the set of all homotopy classes of curves in M (resp., L_1, L_2) starting at p ($p \in L_1 \cap L_2$).*

From Lemma 3.13, Lemma 3.14 and Lemma 3.15, we have a covering map $\Pi\colon L_1(p) \times L_2(p) \to M$, since M is simply connected. On the other hand, from the proof of Theorem 3.5, we see that (M, g) is locally isometric to $-dt^2 + (\omega\beta)^{-2}du^2$. Hence, Π is a globally isometric map such that (M, g) is isometric to the Lorentzian twisted product $(L_1(p) \times L_2(p), -dt^2 + (\omega\beta)^{-2}du^2)$. Since each leaf of F_1 is considered as a complete (timelike) normal geodesic and $L_1(p)$ is simply connected, we may put $L_1(p)$ and $L_2(p)$ as $L = (-\infty, +\infty)$ and H, respectively, thus we obtain the assertion of Theorem 3.9.

§4. Null geodesically incomplete theorems

Let $M^* = L \times_\psi H$ be a Lorentzian warped product equipped with Lorentzian metric $g^* = -dt^2 + \psi h$, where $L = (-\infty, +\infty)$ and (H, h) is an arbitrary Riemannian manifold. All nonspacelike (i.e., timelike or null) tangent vectors of M^* are divided into two separate classes, called future and past directed, by the timelike vector field $\partial/\partial t$.

Definition 4.1. A nonspacelike tangent vector $v \in T_pM^*$ is said to be future (resp., past) directed if $g((\partial/\partial t)(p), v) < 0$ (resp., $g((\partial/\partial t)(p), v) > 0$). A smooth curve is said to be timelike (resp., null, spacelike) if its tangent vector is always timelike (resp., null, spacelike).

In the following we recall the notion of geodesics with respect to general connections (Otsuki [19]):

Definition 4.2. Let N be a differentiable manifold and $\Gamma = (P, \nabla)$ a regular general connection of N. A smooth (regular) curve $c\colon (a, b) \to N$ with parameter t is said to be a Γ-geodesic if $\nabla\dot{c}(t) = \gamma(t)P(\dot{c}(t))$, where $\dot{c}(t)$ is the velocity vector field of $c(t)$ and γ is a suitable smooth function along c. The parameter s of c such that $\nabla c'(s) = 0$ is called an affine parameter of c, where $c'(s)$ denotes the velocity vector field of $c(s)$. A Γ-geodesic c with affine parameter s is said to be complete if c can be defined for $-\infty < s < +\infty$.

Proposition 4.3. *Let (M, g) be a space-time given an ω-conformal general connection $\Gamma(\omega) = (R, \nabla)$. Then null $\Gamma(\omega)$-geodesics can be defined.*

Proof. Let c be $\Gamma(\omega)$-geodesic with affine parameter s. Then we have

$$\frac{d}{ds} g(R(c'(s)), R(c'(s))) = 2g(\nabla c'(s), R(c'(s))) = 0.$$

Let now $g(c'(s_0), c'(s_0)) = 0$ for some value s_0. Since

$$g(R(c'(s_0), R(c'(s_0))) = \omega(s_0) g(c'(s_0), c'(s_0)),$$

$g(R(c'(s)), R(c'(s))) = 0$ along c. This implies $g(c'(s), c'(s)) = 0$, since ω is positive-valued. Thus c is null for all values of its affine parameter if c is null for some value of its affine parameter, hence we obtain the assertion. □

Definition 4.4. A future-directed Γ-geodesic in M^* is said to be future-incomplete (resp., past-incomplete) if it can not be extended to arbitrarily large positive (resp., negative) values of an affine parameter.

We now prove the following criterion for null $\Gamma(\omega)$-geodesic incompleteness of M^* equipped with an ω-conformal general connection $\Gamma(\omega)$. Throughout the rest of this section, let τ be an interior point of $(-\infty, +\infty)$.

Theorem 4.5. *Let M^* be given an ω-conformal general connection $\Gamma(\omega) = (R, \nabla)$ such that R is symmetric with respect to g^*. Assume that both TL and TH are R-invariant and ω depends only on L. If*

$$\lim_{\theta \to -\infty} \int_\theta^\tau \sqrt{\psi\omega} dt$$

is finite, then every future-directed null $\Gamma(\omega)$-geodesic in M^* is past-incomplete. Similarly, if

$$\lim_{\theta \to +\infty} \int_\tau^\theta \sqrt{\psi\omega} dt$$

is finite, then every future-directed null $\Gamma(\omega)$-geodesic in M^* is future-incomplete.

Proof. Let σ be a future-directed null $\Gamma(\omega)$-geodesic and we may put as follows:

$$\sigma(t) = (t, c(t)) \in L \times H, \quad \dot\sigma(t) = \frac{\partial}{\partial t} + \dot c(t),$$

$$\text{and} \quad g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1.$$

Using the formula of Theorem 1.6 (§2) and $g(R(\dot{c}(t)), R(\dot{c}(t))) = \omega g(\dot{c}(t), \dot{c}(t)) = \omega$, we have

$$\nabla\dot{\sigma}(t) = \nabla^1_{\partial/\partial t}\frac{\partial}{\partial t} + \nabla^2\dot{c}(t) + \frac{\partial \log\psi}{\partial t}R(\dot{c}(t)) - 2^{-1}\omega S(\mathrm{grad}(\log\psi)), \tag{4.1}$$

$$\nabla\dot{c}(t) = \nabla^2\dot{c}(t) - 2^{-1}\omega S(\mathrm{grad}(\log\psi)), \tag{4.2}$$

where S denotes the inverse endomorphism of R.

On the other hand, since σ is $\Gamma(\omega)$-geodesic, there exists a smooth function ζ along σ such that

$$\nabla\dot{\sigma}(t) = \zeta R(\dot{\sigma}(t)) = \zeta R(\frac{\partial}{\partial t}) + \zeta R(\dot{c}(t)). \tag{4.3}$$

Noticing that R is symmetric with respect to g^*, we have

$$R = \omega S. \tag{4.4}$$

Thus from (4.1)-(4.4), we obtain

$$\nabla^1_{\partial/\partial t}\frac{\partial}{\partial t} - 2^{-1}R(\mathrm{grad}(\log\psi)) = \zeta R(\frac{\partial}{\partial t}), \tag{4.5}$$

and
$$\nabla^2\dot{c}(t) + \frac{\partial\log\psi}{\partial t}R(\dot{c}(t)) = \zeta R(\dot{c}(t)).$$

Now we also obtain

$$g(\nabla_{\partial/\partial t}\frac{\partial}{\partial t}, R(\frac{\partial}{\partial t})) = -2^{-1}\frac{\partial\omega}{\partial t}, \tag{4.6}$$

since
$$g(R(\frac{\partial}{\partial t}), R(\frac{\partial}{\partial t})) = -\omega.$$

And also we have

$$\mathrm{grad}(\log\psi) = -\psi^{-1}\frac{\partial\psi}{\partial t}\frac{\partial}{\partial t}, \tag{4.7}$$

because ψ depends only on $(L, -dt^2)$. Thus using (4.5) and (4.6), we obtain

$$\begin{aligned}
-\zeta\omega &= g(\zeta R(\frac{\partial}{\partial t}), R(\frac{\partial}{\partial t})) \\
&= g(\nabla^1_{\partial/\partial t}\frac{\partial}{\partial t}, R(\frac{\partial}{\partial t})) - 2^{-1}g(R(\mathrm{grad}(\log\psi)), R(\frac{\partial}{\partial t})) \\
&= -2^{-1}\frac{\partial\omega}{\partial t} - 2^{-1}\omega g(\mathrm{grad}(\log\psi), \frac{\partial}{\partial t}).
\end{aligned}$$

From this and (4.7), it follows that

$$\zeta = 2^{-1}\frac{\partial \log(\omega\psi)}{\partial t}. \tag{4.8}$$

Let s be the following integral:

$$s := \int_t^{\tau} \sqrt{\omega\psi}dt.$$

Noticing that ω and ψ are positive-valued smooth functions, which depends only on t, we can easily obtain

$$\frac{d^2t}{dt\,ds} = -\zeta\frac{dt}{ds}. \tag{4.9}$$

On the other hand, a direct calculation implies

$$\nabla_{\sigma'}\sigma'(s) = \frac{dt}{ds}(\frac{d^2t}{dt\,ds} + \zeta\frac{dt}{ds})R(\dot{\sigma}(s)), \tag{4.10}$$

where $\sigma'(s)$ denotes the velocity vector field of $\sigma(s)$.

From (4.9) and (4.10), we see that s is an affine parameter of σ, consequently, if $\lim_{t\to-\infty} s(t)$ is finite, then σ is past-incomplete. This completes the proof of the first assertion. Similarly it is easy to see that the second assertion is also valid. □

Remark 4.6. J.K.Beem, P.E.Ehrlich and T.G.Powell have obtained the criterion for null geodesic incompleteness of M^* in the case of the Levi-Civita connection (Theorem 2.57; [5]).

As applications of Theorem 4.5, we states null $\Gamma(\omega)$-geodesic incompleteness theorems relative to space-times admitting time functions. The following is an immediate consequence of Theorem 3.9 and Theorem 4.5.

Theorem 4.7. *Let (M, g) be a simply connected space-time admitting a time function Φ of dimension $(n+1)$ and $\Gamma(\omega) = (R, \nabla)$ an ω-conformal general connection of (M, g). Now suppose that* (a) *the quasi-Hessian tensor H^{ω}_{Φ} of Φ is proportional to g,* (b) *the $S\Phi$-distribution is involutive, where S denotes the inverse endomorphism of R and* (c) *every timelike geodesic in (M, g) is complete in the sense of the Levi-Civita connection.*

Under (M, g) isometrically decomposed as the Lorentzian twisted product $(L \times H, -dt^2 + (\omega\beta)^{-2}du^2)$ in Theorem 3.9, assume that (d) *TL and TH be*

R-invariant, (e) *ω and β depend only on L and* (f) *R is symmetric with respect to g, i.e. $g(RX,Y) = g(X,RY)(X,Y \in VM)$. If*

$$\lim_{\theta\to-\infty}\int_\theta^\tau \omega^{-1/2}\beta^{-1}dt \quad (\text{resp.,} \lim_{\theta\to+\infty}\int_\tau^\theta \omega^{-1/2}\beta^{-1}dt)$$

is finite, then every future-directed null $\Gamma(\omega)$-geodesic is past incomplete (resp., future incomplete).

In particular, we have the following theorem if the principal endomorphism is a scalar-multiple of identity endomorphism:

Theorem 4.8. *Let (M,g) be an $(n+1)$-dimensional, simply connected space-time admitting a time function Φ and (R,∇) an ω-conformal general connection such that $R = \lambda I$ and $\omega = \lambda^2$, where $\lambda \in FM$ is positive-valued and I denotes the identity endomorphism of TM. Suppose H^ω_Φ is proportional to g and every timelike geodesic in (M,g) is complete in the sense of the Levi-Civita connection. Then (M,g) is globally isometric to the Lorentzian warped product $(L \times H, -dt^2 + \lambda^{-4}\beta^{-2}du^2)$, and if*

$$\lim_{\theta\to-\infty}\int_\theta^\tau \lambda^{-1}\left|\frac{\partial\Phi}{\partial t}\right| dt \quad (\text{resp.,} \lim_{\theta\to+\infty}\int_\tau^\theta \lambda^{-1}\left|\frac{\partial\Phi}{\partial t}\right| dt)$$

is finite, then every future-directed null $\Gamma(\omega)$-geodesic is past incomplete (resp., future incomplete).

Proof. Let H_Φ denote the quasi-Hessian tensor of Φ with respect to $\Gamma(\omega) = (R,\nabla)$, $R = \lambda I$, $\omega = \lambda^2$. Using Lemma 2.9 (§2.), we have for every $X, Y \in VM$,

$$\begin{aligned} H_\Phi(X,Y) &= g(\nabla_X(\lambda^{-1}\operatorname{grad}\Phi), \lambda Y) - (X(\log\lambda))g(\operatorname{grad}\Phi, \lambda Y) \\ &= Xg(\operatorname{grad}\Phi, \lambda Y) - g(\operatorname{grad}\Phi, \nabla_X Y) - (X\lambda)(Y\Phi) \\ &= X(\lambda(Y\Phi)) - (X\lambda)(Y\Phi) - (\nabla_X Y)\Phi \\ &= \lambda X(Y\Phi) - (\nabla_X Y)\Phi, \end{aligned}$$

while $\nabla_X Y - \nabla_Y X = \lambda[X,Y]$, therefore H_Φ is symmetric. Now let S be the inverse endomorphism of R. Then it is easy to verify that $S\Phi$-distribution is involutive. Thus, from Theorem 3.9 (and also the proof of Theorem 3.5), it turns out that (M,g) is globally isometric to the Lorentzian warped product $(-\infty,+\infty) \times H$ with the metric $-dt^2 + \lambda^{-4}\beta^{-2}du^2$. In fact, from the proof of Theorem 3.5, we see that Φ and λ depend only on $t \in (-\infty,+\infty)$, hence, $\lambda^{-4}\beta^{-2}$ is a warping function. Noticing that (d) and (e) in Theorem 4.7 are clearly satisfied and $\operatorname{grad}\Phi = -\frac{\partial\Phi}{\partial t}\partial/\partial t$, from Theorem 4.7, we also obtain the last assertion. □

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References

[1] N.Abe, *General connections on vector bundles*, Kodai Math. J., **8** (1985), 322-329.

[2] N.Abe, *Geometry of certain first order differential operators and its applications to general connections*, Kodai Math. J., **11** (1988), 205-223.

[3] N.Abe, *Metric general connections and torsion forms*, SUT Journal of Math. **26** (1990), 155-167.

[4] R.Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Commun. Math. Phys. **94** (1984), 155-175.

[5] J.K.Beem and P.E.Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, New York, **1981**.

[6] R.A.Blumenthal and J.J.Hebda, *Complementary distributions which preserve the leaf geometry and applications to totally geodesic foliations*, Quarterly J. Math. **35** (1985), 383-392.

[7] R.A.Blumenthal and J.J.Hebda, *An analogue of the holonomy bundle for a foliated manifold*, Tohoku Math. J. **40** (1988), 189-197.

[8] B.Y.Chen, *Geometry of Submanifolds and its Applications*, Science University of Tokyo, **1981**.

[9] N.Koike, *Totally umblic foliations and decomposition theorems*, Saitama Math. J. **8** (1990), 1-18.

[10] H.Nagayama, *A theory of general relativity by general connections* I, TRU Mathematics **20** (1984), 173-187.

[11] H.Nagayama, *A theory of general relativity by general connections* II, TRU Mathematics **21** (1985), 287-317.

[12] N.Nemoto, *On differential geometry of general connections*, TRU Mathematics **21** (1985), 67-94.

[13] B.O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, **1983**.

[14] T.Otsuki, *Tangent bundles of order 2 and general connections*, Math. J. Okayama Univ. **8** (1958), 143-180.

[15] T.Otsuki, *On general connections I*, Math. J. Okayama Univ. **9** (1960), 99-164.

[16] T.Otsuki, *On general connections II* , Math. J. Okayama Univ. **10** (1961), 113-124.

[17] T.Otsuki, *A certain space-time metric and smooth general connections*, Kodai Math. J. **8** (1985), 307-316.

[18] T.Otsuki, *Singular point sets of a general connection and black holes*, Math. J. Okayama Univ. **30** (1988), 199-211.

[19] T.Otsuki, *General connections*, Math. J. Okayama Univ. **32** (1990), 227-242.

[20] T.Otsuki, *Behaviour of geodesics around the singular set of a general connection*, SUT Journal of Math. **27** (1991), 169-228.

[21] T.Otsuki, *A family of Minkowski-type spaces with general connections*, SUT Journal of Math. **28** (1992), 61-103.

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