SUT Journal of Mathematics
Vol. 32, No. 1 (1996), 1–33

A NONLINEAR PARTIAL DIFFERENTIAL EQUATION RELATED WITH CERTAIN SPACES WITH GENERAL CONNECTIONS(II)

Tominosuke OTSUKI

(Received May 9, 1995)

Abstract. From a Minkowski-type metric on R^n_+ satisfying the Einstein condition, we derive a nonlinear partial differential equation. In order to know the property of its solutions, we obtain an approximate solution with certain boundary conditions numerically by the finite element method, which will give us some clues to get theoretical solutions.

AMS 1991 *Mathematics Subject Classification.* Primary 53C05, 53C22.

Key words and phrases. General Connections, Minkowski Type, Einstein Condition.

§ 0. Introduction.

This work is a continuation of the previous paper [18] with the same title, in which we tried to get a numerical approximate solution of the nonlinear partial differential equations (5.16) and (5.17) in [18] under certain boundary conditions by the difference method, but failed due to the limitations of our personal computer. In the beginning, we give some preliminaries in this section which will be used in the studies of this subject.

We consider a Minkowski-type pseudo-Riemannian metric on $R^n = R^{n-1} \times R$ with the canonical coordinates $(x_1, \dots, x_{n-1}, x_n)$:

$$(0.1) \qquad ds^2 = \frac{1}{Q} dr dr + r^2 \sum_{\alpha,\beta=2}^{n-1} h_{\alpha\beta} du^\alpha du^\beta - P dx_n dx_n ,$$

where Q and P are positive functions on $R^n - \{0\}$, $r^2 = {x_1}^2 + \cdots + {x_{n-1}}^2$ and let

$$d\sigma^2 = \sum_{\alpha,\beta=2}^{n-1} h_{\alpha\beta}(u)du^\alpha du^\beta$$

be the standard metric of the unit sphere S^{n-2}: $r^2 = 1$ in R^{n-1} in its local coordinates $(u^2,\dots,u^{n-1})$. Then, we consider the following conformal change of ds^2 :

$$d\bar{s}^2 = \frac{1}{x_n^2} ds^2 = \sum_{i,j=1}^{n} \bar{g}_{ij} dx_i dx_j \tag{0.2}$$

on $R^n_+(x_n > 0)$, where we set anew

$$x_1 = r,\ x_2 = u^2,\ \cdots,\ x_{n-1} = u^{n-1},\ x_n = x_n\ .$$

We consider the Einstein condition for the metric $d\bar{s}^2$:

$$\bar{R}_{ij} = \frac{\bar{R}}{n} \bar{g}_{ij} \tag{0.3}$$

where $\bar{R}_{ij}$ and $\bar{R}$ are the components of the Ricci tensor and the scalar curvature of (0.2) respectively.

Theorem 1 of [18] says that in order that the metric (0.2) for $n > 3$ with $Q = P$ satisfies the Einstein condition, it is necessary and sufficient that Q is a function of $x = r / x_n$ only and satisfies the ordinary differential equation of $F(x)$:

$$\left(1+\frac{x^2}{F^2}\right)F'' - \frac{2x^2}{F^3}(F')^2 + \left(\frac{nx}{F^2}+\frac{n-4}{x}\right)F' + \frac{2(n-3)(1-F)}{x^2} = 0\ . \tag{E}$$

The non-constant solutions $F(x)$ of (E) with $F(m)$=1 at $m > 0$ is given by

$$F(x) = \frac{1}{2}\left\{-\frac{(b+1)m^{n-1}}{x^{n-3}} + 1 + bx^2 + \sqrt{\left(-\frac{(b+1)m^{n-1}}{x^{n-3}} + 1 + bx^2\right)^2 + 4x^2}\right\}, \tag{0.4}$$

where $b \neq -1$ is constant, and we have

$$F'(m) = \frac{m}{2}\left\{(n-1)b + n - 3 + \frac{(1-m^2)((n-1)b+n-3)+4}{1+m^2}\right\} \tag{0.5}$$

by (4.6) and (4.7) of [18].

We consider the metric (0.2) for $n > 3$ which satisfies the Einstein condition (0.3) under the restrictions :

$$Q_\alpha = \frac{\partial Q}{\partial x_\alpha} = 0,\ P_\alpha = \frac{\partial P}{\partial x_\alpha} = 0,\ \alpha = 2,\ 3,\ \cdots,\ n-1\ .$$

Theorem 3 of [18] says that for such metric it is necessary and sufficient that Q and P as functions of x_1 and x_n satisfy the conditions :

$$\frac{x_n Q_n}{Q} + \frac{x_1 P_1}{P} = 0 \,, \tag{0.6}$$

$$\frac{Q}{x_1}\left(\frac{Q_1}{Q} - \frac{P_1}{P}\right) - \frac{1}{x_n P}\left(\frac{Q_n}{Q} - \frac{P_n}{P}\right) = 0 \,, \tag{0.7}$$

$$\begin{aligned} \Phi(Q,P) := {} & \frac{Q}{P}P_{11} + \frac{1}{QP}Q_{nn} + \frac{QP_1}{2P}\left(\frac{Q_1}{Q} - \frac{P_1}{P}\right) - \frac{3Q_nQ_n}{2Q^2P} - \frac{Q_nP_n}{2QP^2} \\ & + \frac{1}{x_1}\left((n-2)Q_1 - \frac{2QP_1}{P}\right) - \frac{1}{x_nP}\left(\frac{n-1}{Q}Q_n - \frac{P_n}{P}\right) + \frac{2(n-3)(1-Q)}{x_1^{\,2}} = 0 \,, \end{aligned} \tag{0.8}$$

where $Q_{ij} = \partial^2 Q / \partial x_i \partial x_j$, $P_{ij} = \partial^2 P / \partial x_i \partial x_j$, $i, j = 1, \cdots, n$.

Setting $x_1 = e^u$ and $x_n = e^v$, since $x_1 > 0$ and $x_n > 0$, we can put by (0.6)

$$Q = \exp\left(-\frac{\partial W}{\partial u}\right), \quad P = \exp\left(\frac{\partial W}{\partial v}\right) \,, \tag{0.9}$$

where $W = W(u,v)$ is a suitable function of u, v. Then, the equation (0.7) can be written as

$$\exp\left(\frac{\partial W}{\partial v} + 2v\right)\left(\frac{\partial^2 W}{\partial u \partial u} + \frac{\partial^2 W}{\partial u \partial v}\right) = \exp\left(\frac{\partial W}{\partial u} + 2u\right)\left(\frac{\partial^2 W}{\partial v \partial u} + \frac{\partial^2 W}{\partial v \partial v}\right) . \tag{0.10}$$

Using these relations, the equation (0.8) is reduced to

$$\begin{aligned} e^{2u}\Phi(Q,P) = {} & \exp\left(-\frac{\partial W}{\partial u}\right)\left[\frac{\partial^3 W}{\partial u \partial u \partial v} - 3\frac{\partial^2 W}{\partial u \partial v}\right. \\ & \left. + \frac{1}{2}\left(\frac{\partial^2 W}{\partial u \partial v}\right)^2 - \frac{1}{2}\frac{\partial^2 W}{\partial u \partial u}\frac{\partial^2 W}{\partial u \partial v} - (n-2)\frac{\partial^2 W}{\partial u \partial u}\right] \\ & - \exp\left(-\frac{\partial W}{\partial v} + 2(u-v)\right)\left[\frac{\partial^3 W}{\partial u \partial v \partial v} - n\frac{\partial^2 W}{\partial u \partial v} + \frac{1}{2}\left(\frac{\partial^2 W}{\partial u \partial v}\right)^2\right. \\ & \left. - \frac{1}{2}\frac{\partial^2 W}{\partial u \partial v}\frac{\partial^2 W}{\partial v \partial v} - \frac{\partial^2 W}{\partial v \partial v}\right] + 2(n-3)(1-Q) = 0 \,. \end{aligned} \tag{0.11}$$

Since we see from (0.10) that

$$\exp\left(-\frac{\partial W}{\partial u}\right) - \exp\left(-\frac{\partial W}{\partial v} + 2u - 2v\right) = Q - \frac{x^2}{P}$$

depends only on $x = x_1 / x_n$, we can put

$$Q - \frac{1}{P}x^2 = \varphi(x) \,. \tag{0.12}$$

Using $x = x_1 / x_n$ and $t = x_n$ as independent variables, and using notations

$$W_x = \frac{\partial W}{\partial x},\ W_{xt} = \frac{\partial^2 W}{\partial x \partial t},\ etc.,$$

the equation (0.11) can be written ((5.16) in [18]) as

$$\begin{aligned} x^2t^2\Phi(Q,P) = {} & \exp(-xW_x)[-x^3W_{xxx} + x^2tW_{xxt} \\ & - (n-2)x^2W_{xx} - 2xtW_{xt} - (n-4)xW_x - 2(n-3) \\ & + \frac{1}{2}(x^2W_{xx} - xtW_{xt} + xW_x)(2x^2W_{xx} - xtW_{xt} + 2xW_x)] \\ & - x^2\exp(xW_x - tW_t)[x^3W_{xxx} - 2x^2tW_{xxt} + xt^2W_{xtt} \\ & + (n+2)x^2W_{xx} - (n-1)xtW_{xt} - t^2W_{tt} + nxW_x - tW_t \\ & + \frac{1}{2}(x^2W_{xx} - xtW_{xt} + xW_x)(2x^2W_{xx} - 3xtW_{xt} + t^2W_{tt} + 2xW_x + tW_t)] \\ & + 2(n-3) = 0\ . \end{aligned} \tag{0.13}$$

§ 1. The fundamental partial differential equation.

First we express (0.13) as a partial differential equation of Q. From $Q = \exp(-xW_x)$ we obtain

$$\begin{aligned} & xW_{xx} + W_x = -Q_x / Q, \\ & xW_{xt} = -Q_t / Q, \\ & xW_{xxx} + 2W_x = (Q_x / Q)^2 - Q_{xx} / Q, \\ & xW_{xxt} + W_{xt} = Q_xQ_t / Q^2 - Q_{xt} / Q, \\ & xW_{xtt} = (Q_t / Q)^2 - Q_{tt} / Q\ . \end{aligned}$$

From (0.12) we have

$$-x^2\exp(xW_x - tW_t) = \varphi(x) - Q. \tag{1.1}$$

Using these relations, regarding (0.13) we obtain

$$\begin{aligned} & \exp(-xW_x)\Big[-x^3W_{xxx} + x^2tW_{xxt} - (n-2)x^2W_{xx} - 2xtW_{xt} - (n-4)xW_x \\ & -2(n-3) + \frac{1}{2}\left(x^2W_{xx} - xtW_{xt} + xW_x\right)\left(2x^2W_{xx} - xtW_{xt} + 2xW_x\right)\Big] \\ & = Q\Bigg[-x^2\left(-2W_{xx} + \left(\frac{Q_x}{Q}\right)^2 - \frac{Q_{xx}}{Q}\right) + xt\left(-W_{xt} + \frac{Q_xQ_t}{Q^2} - \frac{Q_{xt}}{Q}\right) \\ & \quad - (n-2)x^2W_{xx} - 2xtW_{xt} - (n-4)xW_x - 2(n-3) \\ & \quad + \frac{1}{2}\left(-\frac{xQ_x}{Q} + \frac{tQ_t}{Q}\right)\left(-\frac{2xQ_x}{Q} + \frac{tQ_t}{Q}\right)\Bigg] \end{aligned}$$

$$= Q\Bigg[-(n-4)x^2W_{xx} - 3xtW_{xt} - (n-4)xW_x - x^2\left(\left(\frac{Q_x}{Q}\right)^2 - \frac{Q_{xx}}{Q}\right)$$

$$+ xt\left(\frac{Q_xQ_t}{Q^2} - \frac{Q_{xt}}{Q}\right) - 2(n-3) + \frac{1}{2Q^2}(xQ_x - tQ_t)(2xQ_x - tQ_t)\Bigg]$$

$$= x^2Q_{xx} - xtQ_{xt} + (n-4)xQ_x + 3tQ_t - \frac{tQ_t}{2Q}(xQ_x - tQ_t) - 2(n-3)Q$$

and

$$-x^2\exp(xW_x - tW_t)\big[x^3W_{xxx} - 2x^2tW_{xxt} + xt^2W_{xtt}$$

$$+(n+2)x^2W_{xx} - (n-1)xtW_{xt} - t^2W_{tt} + nxW_x - tW_t$$

$$+\frac{1}{2}(x^2W_{xx} - xtW_{xt} + xW_x)(2x^2W_{xx} - 3xtW_{xt} + t^2W_{tt} + 2xW_x + tW_t)\big]$$

$$= (\varphi - Q)\Bigg[x^2\left(-2W_{xx} + \left(\frac{Q_x}{Q}\right)^2 - \frac{Q_{xx}}{Q}\right) - 2xt\left(-W_{xt} + \frac{Q_xQ_t}{Q^2} - \frac{Q_{xt}}{Q}\right)$$

$$+ t^2\left(\left(\frac{Q_t}{Q}\right)^2 - \frac{Q_{tt}}{Q}\right) + (n+2)x^2W_{xx} - (n-1)xtW_{xt} - t^2W_{tt} + nxW_x$$

$$- tW_t + \frac{1}{2}\left(-\frac{xQ_x}{Q} + \frac{tQ_t}{Q}\right)\left(-2x\frac{Q_x}{Q} + 3t\frac{Q_t}{Q} + t(tW_{tt} + W_t)\right)\Bigg]$$

$$= (\varphi - Q)\Big[nx^2W_{xx} - (n-3)xtW_{xt} - t(tW_{tt} + W_t) + nxW_x$$

$$+ x^2\left(\left(\frac{Q_x}{Q}\right)^2 - \frac{Q_{xx}}{Q}\right) - 2xt\left(\frac{Q_xQ_t}{Q^2} - \frac{Q_{xt}}{Q}\right) + t^2\left(\left(\frac{Q_t}{Q}\right)^2 - \frac{Q_{tt}}{Q}\right)$$

$$+ \frac{1}{2Q^2}(xQ_x - tQ_t)(2xQ_x - 3tQ_t - tQ(tW_{tt} + W_t))\Big]$$

$$= (\varphi - Q)\Bigg[-\frac{nxQ_x}{Q} + (n-3)t\frac{Q_t}{Q} - t(tW_{tt} + W_t)$$

$$+ \frac{(xQ_x - tQ_t)^2}{Q^2} - \frac{1}{Q}(x^2Q_{xx} - 2xtQ_{xt} + t^2Q_{tt})$$

$$+ \frac{1}{2Q^2}(xQ_x - tQ_t)(2xQ_x - 3tQ_t) - \frac{t}{2Q}(xQ_x - tQ_t)(tW_{tt} + W_t)\Bigg]$$

$$= (\varphi - Q)\Bigg[-\frac{1}{Q}(x^2Q_{xx} - 2xtQ_{xt} + t^2Q_{tt}) - \frac{nxQ_x}{Q} + \frac{(n-3)tQ_t}{Q}$$

$$+ \frac{1}{2Q^2}(xQ_x - tQ_t)(4xQ_x - 5tQ_t) - t\left(1 + \frac{xQ_x - tQ_t}{2Q}\right)(tW_{tt} + W_t)\Bigg].$$

On the other hand, differentiating (1.1) with respect to t we obtain

$$-x^2\exp(xW_x - tW_t)(xW_{xt} - tW_{tt} - W_t) = -Q_t,$$

i.e.

$$(\varphi - Q)\left(-\frac{Q_t}{Q} - tW_{tt} - W_t\right) = -Q_t,$$

hence

$$tW_{tt} + W_t = -\frac{2Q-\varphi}{Q-\varphi}\,\frac{Q_t}{Q}. \tag{1.2}$$

From the above arguments, the equation (0.13) can be written as

$$\begin{aligned} x^2t^2\Phi(Q,P) &= \left(2-\frac{\varphi}{Q}\right)x^2Q_{xx} - \left(3-\frac{2\varphi}{Q}\right)xtQ_{xt} + \left(1-\frac{\varphi}{Q}\right)t^2Q_{tt} \\ &+\left(2n-4-\frac{n\varphi}{Q}\right)xQ_x - \left(n-6-(n-3)\frac{\varphi}{Q}\right)tQ_t \\ &+\frac{1}{2Q}(xQ_x - tQ_t)\left(4\left(\frac{\varphi}{Q}-1\right)xQ_x - \left(\frac{5\varphi}{Q}-4\right)tQ_t\right) \\ &+2(n-3)(1-Q) + t(\varphi - Q)\left(1+\frac{xQ_x - tQ_t}{2Q}\right)\frac{2Q-\varphi}{Q-\varphi}\,\frac{Q_t}{Q} \\ &= \left(2-\frac{\varphi}{Q}\right)x^2Q_{xx} - \left(3-\frac{2\varphi}{Q}\right)xtQ_{xt} + \left(1-\frac{\varphi}{Q}\right)t^2Q_{tt} \\ &+\left(2n-4-\frac{n\varphi}{Q}\right)xQ_x - \left(n-4-(n-2)\frac{\varphi}{Q}\right)tQ_t \\ &-\frac{1}{Q}(xQ_x - tQ_t)\left(2\left(1-\frac{\varphi}{Q}\right)xQ_x - \left(1-\frac{2\varphi}{Q}\right)tQ_t\right) \\ &+2(n-3)(1-Q) = 0\,. \end{aligned}$$

Theorem 1. In order that the n-dimensional pseudo-Riemannian metric (0.2) with (0.1) satisfies the Einstein condition (0.3) for $n > 3$ under the restrictions

$$\frac{\partial Q}{\partial x_\alpha} = \frac{\partial P}{\partial x_\alpha} = 0,\ \alpha = 2,\ 3,\ \cdots,\ n-1,$$

it is necessary and sufficient that Q satisfies the partial differential equation

$$
(1.3)\qquad \begin{aligned}
&(2Q-\varphi)x^2\frac{\partial^2 Q}{\partial x^2}-(3Q-2\varphi)xt\frac{\partial^2 Q}{\partial x\partial t}+(Q-\varphi)t^2\frac{\partial^2 Q}{\partial t^2}\\
&+\{(2n-4)Q-n\varphi\}x\frac{\partial Q}{\partial x}-\{(n-4)Q-(n-2)\varphi\}t\frac{\partial Q}{\partial t}\\
&-\frac{1}{Q}\left(x\frac{\partial Q}{\partial x}-t\frac{\partial Q}{\partial t}\right)\left\{2(Q-\varphi)x\frac{\partial Q}{\partial x}-(Q-2\varphi)t\frac{\partial Q}{\partial t}\right\}\\
&+2(n-3)Q(1-Q)=0,
\end{aligned}
$$

where $\varphi=\varphi(x)$, $x=x_1/x_n$, $t=x_n$, is an auxiliary function of x, satisfying (0.12);

$$
(1.4)\qquad P=\frac{x^2}{Q-\varphi}.
$$

Remark 1. If we consider the case ; $\partial W/\partial t=0$ in the above arguments, we have

$$
\frac{\partial W}{\partial u}=\frac{\partial W}{\partial x}x,\ \frac{\partial W}{\partial v}=-\frac{\partial W}{\partial x}x+\frac{\partial W}{\partial t}t=-\frac{\partial W}{\partial u},
$$

hence we have $P=Q$ by (0.9) and so

$$
Q-\frac{x^2}{Q}=\varphi,
$$

from which we obtain

$$
1-\frac{\varphi}{Q}=\frac{x^2}{Q^2},\ 2-\frac{\varphi}{Q}=1+\frac{x^2}{Q^2},\ 2n-4-\frac{n\varphi}{Q}=n-4+\frac{nx^2}{Q^2}.
$$

Using these relations, the equation (1.3) in this case can be written as

$$
Qx^2\left[\left(1+\frac{x^2}{Q^2}\right)\frac{\partial^2 Q}{\partial x^2}-\frac{2x^3}{Q^3}\left(\frac{\partial Q}{\partial x}\right)^2+\left(\frac{nx}{Q^2}+\frac{n-4}{x}\right)\frac{\partial Q}{\partial x}+\frac{2(n-3)(1-Q)}{x^2}\right]=0,
$$

which says that Q satisfies (E) in § 0.

§ 2. Certain boundary conditions.

We consider the nonlinear partial differential equation (1.3) and its solution $Q(x,t)$ on the square

$$
\Omega\ :\ 0\le x\le 1\ \text{ and }\ 0\le t\le 1.
$$

We wish to connect it with the solution $Q\equiv 1$ outside of Ω in the right angle : $0\le x<\infty$ and $0\le t<\infty$, which is given by $b=1$ for the solution $F(x)$ of (0.4) and corresponds to the metric

$$d\bar{s}^2 = \frac{1}{{x_n}^2}\left(\sum_{a=1}^{n-1} dx_a dx_a - dx_n dx_n\right)$$

The integral free function $\varphi(x)$ in Theorem 1 corresponding to $Q = P \equiv 1$ is $1-x^2$ by (0.12). We set

(2.1) $\qquad n = 4$ and $\varphi(x) = 1 - x^2$.

Then, the equation (1.3) becomes

$$(2.2)\qquad \begin{aligned} &(2Q-1+x^2)x^2\frac{\partial^2 Q}{\partial x^2} - (3Q-2+2x^2)xt\frac{\partial^2 Q}{\partial x \partial t} + (Q-1+x^2)t^2\frac{\partial^2 Q}{\partial t^2} \\ &+4(Q-1+x^2)x\frac{\partial Q}{\partial x} + 2(1-x^2)t\frac{\partial Q}{\partial t} \\ &-\frac{1}{Q}\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)\left\{2(Q-1+x^2)x\frac{\partial Q}{\partial x} - (Q-2+2x^2)t\frac{\partial Q}{\partial t}\right\} \\ &+2Q(1-Q) = 0 . \end{aligned}$$

Now, we set the boundary condition

(2.3) $\qquad Q = 1$ along $t = 1$ and $x = 1$

for the above mentioned purpose. First, from the condition $Q = 1$ along $t = 1$ we have $\partial Q / \partial x = \partial^2 Q / \partial x^2 = 0$ and obtain from (2.2)

$$(2.4)\qquad -(1+2x^2)x\frac{\partial^2 Q}{\partial x \partial t} + x^2\frac{\partial^2 Q}{\partial t^2} + 2(1-x^2)\frac{\partial Q}{\partial t} + (1-2x^2)\left(\frac{\partial Q}{\partial t}\right)^2 = 0$$

along $t = 1$.

Second, from the condition $Q = 1$ along $x = 1$, we have $\partial Q / \partial t = \partial^2 Q / \partial t^2 = 0$ and obtain from (2.2)

$$(2.5)\qquad 2\frac{\partial^2 Q}{\partial x^2} - 3t\frac{\partial^2 Q}{\partial x \partial t} + 4\frac{\partial Q}{\partial x} - 2\left(\frac{\partial Q}{\partial x}\right)^2 = 0 \quad \text{along } x = 1.$$

As a reference function to Q, we consider F given by (0.4) with $n = 4$, $m = 1$ as

$$F(x) = \frac{1}{2}\left\{-\frac{(b+1)}{x} + 1 + bx^2 + \sqrt{\left(-\frac{(b+1)}{x} + 1 + bx^2\right)^2 + 4x^2}\right\}$$

and so we have

$$F'(1)=\frac{3}{2}(b+1).$$

Here, setting $b+1=(1-t)^2$ and substituting this into the right hand side of the above expression, we obtain

$$F(x,t):=\frac{1}{2}\left\{1-x^2-\frac{1}{x}\left(1-x^3\right)\left(1-t\right)^2+\sqrt{\left(1-x^2-\frac{1}{x}\left(1-x^3\right)\left(1-t\right)^2\right)^2+4x^2}\right\} \tag{2.6}$$

and

$$\frac{\partial F(x,t)}{\partial x}\Big|_{x=1}=\frac{3}{2}(1-t)^2 \quad\text{and}\quad \frac{\partial F(x,t)}{\partial t}\Big|_{t=1}=0. \tag{2.7}$$

Now, as a boundary condition for Q along $x=1$, we set the same as above for $F(x,t)$;

$$\frac{\partial Q(x,t)}{\partial x}\Big|_{x=1}=\frac{3}{2}(1-t)^2,$$

from which we have

$$\frac{\partial^2 Q}{\partial x\partial t}=-3(1-t).$$

Hence, we obtain by (2.5)

$$\begin{aligned}\frac{\partial^2 Q}{\partial x^2}&=\frac{3}{2}t\frac{\partial^2 Q}{\partial x\partial t}-2\frac{\partial Q}{\partial x}+\left(\frac{\partial Q}{\partial x}\right)^2\\&=-\frac{9}{2}t(1-t)-3(1-t)^2+\frac{9}{4}(1-t)^4\\&=(1-t)\left\{\frac{9}{4}(1-t)^3+\frac{3}{2}(1-t)-\frac{9}{2}\right\}.\end{aligned}$$

Collecting these we set

$$\begin{cases}Q=1,\ \ \dfrac{\partial Q}{\partial x}=\dfrac{3}{2}(1-t)^2,\\ \dfrac{\partial^2 Q}{\partial x^2}=(1-t)\left\{\dfrac{9}{4}(1-t)^3+\dfrac{3}{2}(1-t)-\dfrac{9}{2}\right\},\end{cases}\quad \text{along } x=1. \tag{2.8}$$

Using (2.8), we obtain the Taylor approximation of Q(9/10,t) by

$$\begin{aligned}Q\left(\frac{9}{10},t\right)&\approx 1-\frac{3}{20}(1-t)^2+\frac{1-t}{200}\left\{\frac{9}{4}(1-t)^3+\frac{3}{2}(1-t)-\frac{9}{2}\right\}\\&=1-\frac{9}{400}(1-t)-\frac{57}{400}(1-t)^2+\frac{9}{800}(1-t)^4,\end{aligned} \tag{2.9}$$

from which we obtain especially

$$(2.10)\qquad Q\left(\frac{9}{10},\frac{9}{10}\right)\approx 0.996326\ .$$

Next, as a boundary condition for Q along $t=1$, we set

$$\frac{\partial Q(x,t)}{\partial t}\Big|_{t=1}=a(1-x)$$

where a is a constant to be determined afterward. Then, we have

$$\frac{\partial^2 Q}{\partial x\partial t}=-a \quad \text{along } \ t=1,$$

and substituting these into (2.4) we obtain

$$ax(1+2x^2)+x^2\frac{\partial^2 Q}{\partial t^2}+2a(1-x)(1-x^2)+a^2(1-x^2)(1-2x^2)=0\ .$$

Collecting these we set

$$(2.11)\qquad \begin{cases} Q=1,\ \ \dfrac{\partial Q}{\partial t}=a(1-x), \\ \dfrac{\partial^2 Q}{\partial t^2}=-\dfrac{a}{x^2}\left\{x\left(1+2x^2\right)+(1-x)^2\left(2(1+x)+a\left(1-2x^2\right)\right)\right\} \end{cases}$$

along $\ t=1$,

from which we have at $x=9/10$

$$\frac{\partial Q}{\partial t}\left(\frac{9}{10},1\right)=\frac{a}{10},$$

$$\frac{\partial^2 Q}{\partial t^2}\left(\frac{9}{10},1\right)=-\frac{100a}{81}\left\{\frac{9\times 262}{10^3}+\frac{1}{10^2}\left(\frac{38}{10}-\frac{62a}{10^2}\right)\right\}$$

$$=-\frac{a}{81\cdot 10^2}(23960-62a)$$

and so approximately $\quad Q\left(\dfrac{9}{10},\dfrac{9}{10}\right)\approx 1-\dfrac{a}{10^2}-\dfrac{a(23960-62a)}{162\cdot 10^4}.$

Using (2.10), we put

$$1-\frac{a}{10^2}-\frac{a(23960-62a)}{162\cdot 10^4}=0.996326$$

i.e.

$$31a^2-20080a+\frac{3674\times 81}{10^2}=0,$$

from which we obtain a ≈ 0.148238 and 647.594. Since

$$\frac{4}{27} \approx 0.14815,$$

putting

$$a=\frac{4}{27}$$

we obtain an approximation of $Q\left(x,\frac{9}{10}\right)$ by (2.11) as

$$Q\left(x,\frac{9}{10}\right) \approx 1-\frac{4(1-x)}{27\cdot 10}$$

(2.12)
$$+\frac{2}{27\cdot 10^2x^2}\times\left\{x\left(1+2x^2\right)+(1-x)^2\left(2(1+x)+\frac{4}{27}\left(1-2x^2\right)\right)\right\}$$

$$=1-\frac{4(1-x)}{27\cdot 10}+\frac{2\left(58-35x-58x^2+124x^3-8x^4\right)}{27^2\cdot 10^2x^2}.$$

By means of (2.9) and (2.12), we obtain the following approximate values of $Q\left(\frac{9}{10},\frac{j}{10}\right)$, $Q\left(\frac{i}{10},\frac{9}{10}\right)$, $i,j=1,\ 2,\ \cdots,\ 9$

(2.13)

j	$Q\left(\frac{9}{10},\frac{j}{10}\right)$	i	$Q\left(\frac{i}{10},\frac{9}{10}\right)$	$F\left(\frac{i}{10},\frac{9}{10}\right)$
9	0.996326	9	0.996328	0.998338
8	0.989818	8	0.994761	0.996286
7	0.980516	7	0.992997	0.993714
6	0.968488	6	0.990883	0.990417
5	0.953828	5	0.988093	0.986040
4	0.936658	4	0.983832	0.979885
3	0.917126	3	0.975741	0.970320
2	0.895408	2	0.954088	0.952399
1	0.871706	1	0.838400	0.901196
0	0.846250			

and compare them with the corresponding F values of (2.6).

Remark 2. As a boundary condition for Q along $t=1$, if we put $a=0$, then (2.11) implies

(2.14)
$$Q=1,\ \frac{\partial Q}{\partial t}=0,\ \frac{\partial^2 Q}{\partial t^2}=0,\quad \text{along } t=1.$$

Regarding $F(x,t)$, we have

$$\frac{\partial F(x,t)}{\partial t}=(1-t)\frac{1-x^3}{x}\left[1+\frac{1-x^2-\dfrac{(1-x^3)(1-t)^2}{x}}{\sqrt{\left(1-x^2-\dfrac{(1-x^3)(1-t)^2}{x}\right)^2+4x^2}}\right],$$

from which we obtain

$$\text{(2.15)}\qquad \frac{\partial F(x,t)}{\partial t}\Big|_{t=1}=0,\ \frac{\partial^2 F(x,t)}{\partial t^2}\Big|_{t=1}=-\frac{1-x^3}{x}\cdot\frac{2}{1+x^2}.$$

§ 3. The weak form of (2.2) on the square Ω.

In this section, we shall derive the weak form of (2.2) for the finite element approximation.

First in place of (2.2) we consider the following expression:

$$\text{(3.1)}\qquad \begin{aligned}\Psi(Q):=\ &(2Q-1+x^2)x^2Q\frac{\partial^2 Q}{\partial x^2}-(3Q-2+2x^2)xtQ\frac{\partial^2 Q}{\partial x\partial t}\\ &+(Q-1+x^2)t^2Q\frac{\partial^2 Q}{\partial t^2}+4(Q-1+x^2)xQ\frac{\partial Q}{\partial x}+2(1-x^2)tQ\frac{\partial Q}{\partial t}\\ &-\left(x\frac{\partial Q}{\partial x}-t\frac{\partial Q}{\partial t}\right)\left\{2(Q-1+x^2)x\frac{\partial Q}{\partial x}-(Q-2+2x^2)t\frac{\partial Q}{\partial t}\right\}\\ &+2Q^2(1-Q)=0\,.\end{aligned}$$

Taking any smooth test function W(x,t) on Ω, we have the equality:

$$\int_0^1\int_0^1\Psi(Q)W(x,t)dxdt=0,$$

which we shall change by the integration by parts, under the boundary conditions (2.8) and (2.11). Using the notation $d\Omega=dxdt$, first we have

$$\begin{aligned}&\int\int_\Omega(2Q-1+x^2)x^2Q\frac{\partial^2 Q}{\partial x^2}Wd\Omega\\ &=\int_0^1\left[(2Q-1+x^2)x^2Q\frac{\partial Q}{\partial x}W\right]_{x=0}^{x=1}dt\\ &\quad-\int\int_\Omega\left\{\left(4x^2Q+x^2(x^2-1)\right)\frac{\partial Q}{\partial x}+4xQ^2+(4x^3-2x)Q\right\}\frac{\partial Q}{\partial x}Wd\Omega\\ &\quad-\int\int_\Omega(2Q-1+x^2)x^2Q\frac{\partial Q}{\partial x}\frac{\partial W}{\partial x}d\Omega\end{aligned}$$

$$
\begin{aligned}
&= 3\int_0^1 (1-t)^2 W(1,t)dt \\
&\quad - \iint_\Omega \left\{ \left(4x^2 Q + x^2(x^2-1)\right)\left(\frac{\partial Q}{\partial x}\right)^2 + 2xQ(2Q+2x^2-1)\frac{\partial Q}{\partial x} \right\} W d\Omega \\
&\quad - \iint_\Omega \left\{ (2Q-1+x^2)x^2 Q \frac{\partial Q}{\partial x} \right\} \frac{\partial W}{\partial x} d\Omega .
\end{aligned}
$$

Second, we have

$$
\begin{aligned}
&\iint_\Omega (3Q-2+2x^2) xtQ \frac{\partial^2 Q}{\partial x \partial t} W d\Omega \\
&= \int_0^1 \left[(3Q-2+2x^2) xtQ \frac{\partial Q}{\partial t} W \right]_{x=0}^{x=1} dt \\
&\quad - \iint_\Omega \left\{ \left(6xtQ + 2x(x^2-1)xt\right)\frac{\partial Q}{\partial x} + 3tQ^2 + (6x^2-2)tQ \right\} \frac{\partial Q}{\partial t} W d\Omega \\
&\quad - \iint_\Omega (3Q-2+2x^2) xtQ \frac{\partial Q}{\partial t}\frac{\partial W}{\partial x} d\Omega \\
&= -\iint_\Omega \left\{ 2(3Q-1+x^2) xt \frac{\partial Q}{\partial x}\frac{\partial Q}{\partial t} + \left(3Q + 2(3x^2-1)\right) tQ \frac{\partial Q}{\partial t} \right\} W d\Omega \\
&\quad - \iint_\Omega \left\{ (3Q-2+2x^2) xtQ \frac{\partial Q}{\partial t} \right\} \frac{\partial W}{\partial x} d\Omega .
\end{aligned}
$$

Third, we have

$$
\begin{aligned}
&\iint_\Omega (Q-1+x^2) t^2 Q \frac{\partial^2 Q}{\partial t^2} W d\Omega \\
&= \int_0^1 \left[(Q-1+x^2) t^2 Q \frac{\partial Q}{\partial t} W \right]_{t=0}^{t=1} dx \\
&\quad - \iint_\Omega \left\{ \left(2t^2 Q + (x^2-1)t^2\right)\left(\frac{\partial Q}{\partial t}\right)^2 + \left(2tQ^2 + 2t(x^2-1)Q\right)\frac{\partial Q}{\partial t} \right\} W d\Omega \\
&\quad - \iint_\Omega \left\{ (Q-1+x^2) t^2 Q \frac{\partial Q}{\partial t} \right\} \frac{\partial W}{\partial t} d\Omega \\
&= a\int_0^1 x^2 (1-x) W(x,1) dx \\
&\quad - \iint_\Omega \left\{ (2Q-1+x^2) t^2 \left(\frac{\partial Q}{\partial t}\right)^2 + 2Q(Q-1+x^2) t \frac{\partial Q}{\partial t} \right\} W d\Omega \\
&\quad - \iint_\Omega \left\{ (Q-1+x^2) t^2 Q \frac{\partial Q}{\partial t} \right\} \frac{\partial W}{\partial t} d\Omega .
\end{aligned}
$$

By means of these expressions, we obtain

$$\iint_\Omega \Psi(Q)W d\Omega = a\int_0^1 x^2(1-x)W(x,1)dx + 3\int_0^1 (1-t)^2 W(1,t)dt$$
$$+\iint_\Omega \left\{\left(3Q-2+2x^2\right)xtQ\frac{\partial Q}{\partial t} - \left(2Q-1+x^2\right)x^2 Q\frac{\partial Q}{\partial x}\right\}\frac{\partial W}{\partial x}d\Omega$$
$$-\iint_\Omega \left\{\left(Q-1+x^2\right)t^2 Q\frac{\partial Q}{\partial t}\right\}\frac{\partial W}{\partial t}d\Omega + \iint_\Omega \Xi W d\Omega = 0\ ,$$

where

$$\Xi = -\left\{\left(4x^2Q + x^2\left(x^2-1\right)\right)\left(\frac{\partial Q}{\partial x}\right)^2 + 2xQ\left(2Q+2x^2-1\right)\frac{\partial Q}{\partial x}\right\}$$
$$+\left\{2\left(3Q-1+x^2\right)xt\frac{\partial Q}{\partial x}\frac{\partial Q}{\partial t} + \left(3Q+2\left(3x^2-1\right)\right)tQ\frac{\partial Q}{\partial t}\right\}$$
$$-\left\{\left(2Q-1+x^2\right)t^2\left(\frac{\partial Q}{\partial t}\right)^2 + 2Q\left(Q-1+x^2\right)t\frac{\partial Q}{\partial t}\right\}$$
$$+4\left(Q-1+x^2\right)xQ\frac{\partial Q}{\partial x} + 2\left(1-x^2\right)tQ\frac{\partial Q}{\partial t}$$
$$-\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)\left\{2\left(Q-1+x^2\right)x\frac{\partial Q}{\partial x} - \left(Q-2+2x^2\right)t\frac{\partial Q}{\partial t}\right\}$$
$$+2Q^2(1-Q)$$
$$= -3Q\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)\left(2x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right) + 3\left(1-x^2\right)\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)^2$$
$$-2Q\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right) + \left(Q+2x^2\right)tQ\frac{\partial Q}{\partial t} + 2Q^2(1-Q)\ .$$

Thus, we obtain

Theorem 2. For any solution $Q(x,t)$ of (2.2) satisfying the boundary conditions (2.8) and (2.11), the following equality holds for any smooth test function $W(x,t)$:

$$\iint_\Omega \left\{-3Q\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)\left(2x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right) + 3\left(1-x^2\right)\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right)^2\right.$$
$$\left. -2Q\left(x\frac{\partial Q}{\partial x} - t\frac{\partial Q}{\partial t}\right) + \left(Q+2x^2\right)tQ\frac{\partial Q}{\partial t} + 2Q^2(1-Q)\right\}W d\Omega$$
$$(3.2)\qquad +\iint_\Omega \left\{xQ\left(\left(3Q-2+2x^2\right)t\frac{\partial Q}{\partial t} - \left(2Q-1+x^2\right)x\frac{\partial Q}{\partial x}\right)\right\}\frac{\partial W}{\partial x}d\Omega$$
$$-\iint_\Omega \left\{tQ\left(Q-1+x^2\right)t\frac{\partial Q}{\partial t}\right\}\frac{\partial W}{\partial t}d\Omega$$
$$= -a\int_0^1 x^2(1-x^2)W(x,1)dx - 3\int_0^1 (1-t)^2 W(1,t)dt\ .$$

In order to approximately analyze the above relation (3.2), we suppose that an approximate solution $\overline{Q}(x,t)$ is known. Then we linearize (3.2) for the required function $Q(x,t)$ in the form:

$$\begin{aligned}(3.3)\quad &\iint_\Omega\Bigg[\left\{-3\overline{Q}\left(2x\frac{\partial\overline{Q}}{\partial x}-t\frac{\partial\overline{Q}}{\partial t}\right)+3(1-x^2)\left(x\frac{\partial\overline{Q}}{\partial x}-t\frac{\partial\overline{Q}}{\partial t}\right)-2\overline{Q}\right\}\\ &\times\left(x\frac{\partial Q}{\partial x}-t\frac{\partial Q}{\partial t}\right)+\left(\overline{Q}+2x^2\right)t\overline{Q}\frac{\partial Q}{\partial t}+2\overline{Q}(1-\overline{Q})Q\Bigg]Wd\Omega\\ &+\iint_\Omega\left\{x\overline{Q}\left(\left(3\overline{Q}-2+2x^2\right)t\frac{\partial Q}{\partial t}-\left(2\overline{Q}-1+x^2\right)x\frac{\partial Q}{\partial x}\right)\right\}\frac{\partial W}{\partial x}d\Omega\\ &-\iint_\Omega\left\{t\overline{Q}\left(\overline{Q}-1+x^2\right)t\frac{\partial Q}{\partial t}\right\}\frac{\partial W}{\partial t}d\Omega\\ &=-a\int_0^1 x^2(1-x)W(x,1)dx-3\int_0^1(1-t)^2W(1,t)dt\,.\end{aligned}$$

§ 4. Computations of (3.3) on E_{ij} and $\overline{E}_{ij}$.

Now, we divide the square Ω into 200 small right triangles E_{ij} and $\overline{E}_{ij}$, $i,j=0,1,\cdots,9$, with the vertices :

$$\begin{aligned}E_{ij} &: \quad P[i,j],\quad P[i+1,j],\quad P[i+1,j+1]\\ \overline{E}_{ij} &: \quad P[i,j],\quad P[i,j+1],\quad P[i+1,j+1],\end{aligned}$$

where $P[i,j]=\left(\frac{i}{10},\frac{j}{10}\right)$, and suppose that $\overline{Q}(x,t)$, $Q(x,t)$ and $W(x,t)$ in (3.3) are linear on each triangle E_{ij} and $\overline{E}_{ij}$. We compute the integrals in (3.3) on E_{ij} and $\overline{E}_{ij}$. Denoting

$$\overline{Q}(P[i,j])=z_{i,j},\ Q(P[i,j])=y_{i,j},\ W(P[i,j])=v_{i,j},$$

and

$$(4.1)\qquad \begin{cases}\Delta^h_{i,j}[z]=z_{i+1,j}-z_{i,j},\ \Delta^v_{i,j}[z]=z_{i,j+1}-z_{i,j},\\ \Delta^h_{i,j}[y]=y_{i+1,j}-y_{i,j},\Delta^v_{i,j}[y]=y_{i,j+1}-y_{i,j},\\ \Delta^h_{i,j}[v]=v_{i+1,j}-v_{i,j},\ \Delta^v_{i,j}[v]=v_{i,j+1}-v_{i,j},\end{cases}$$

we have

$$(4.2)\qquad \mathrm{Q(x,t)}=\begin{cases}10\Delta^h_{i,j}[y]\left(x-\frac{i}{10}\right)+10\Delta^v_{i+1,j}[y]\left(t-\frac{j}{10}\right)+y_{i,j} & E_{ij}\\ & \text{on}\\ 10\Delta^h_{i,j+1}[y]\left(x-\frac{i}{10}\right)+10\Delta^v_{i,j}[y]\left(t-\frac{j}{10}\right)+y_{i,j} & \overline{E}_{ij}\end{cases}$$

and analogous formulas for $\overline{Q}(x,t)$ and $W(x,t)$.

We can prove easily the following formulas :

$$
(4.3)\qquad
\begin{aligned}
&\iint_{E_{ij}}\left(x-\frac{i}{10}\right)^{m}\left(t-\frac{j}{10}\right)^{n}dxdt=\frac{1}{10^{m+n+2}(m+n+2)(n+1)},\\
&\iint_{\overline{E}_{ij}}\left(x-\frac{i}{10}\right)^{m}\left(t-\frac{j}{10}\right)^{n}dxdt=\frac{1}{10^{m+n+2}(m+n+2)(m+1)}.
\end{aligned}
$$

Regarding the expressions in the integrand of (3.3), we obtain the following by (4.1) and (4.2), using the notation $X=x-\frac{i}{10}$ and $T=t-\frac{j}{10}$.

$$
(4.4)\qquad
\begin{aligned}
&-3\overline{Q}\left(2x\frac{\partial\overline{Q}}{\partial x}-t\frac{\partial\overline{Q}}{\partial t}\right)+3(1-x^{2})\left(x\frac{\partial\overline{Q}}{\partial x}-t\frac{\partial\overline{Q}}{\partial t}\right)-2\overline{Q}\\
&=-3\cdot10\Delta^{h}[z]X^{3}+3\cdot10\Delta^{v}[z]X^{2}T\\
&\quad-3\left(2\cdot10^{2}\left(\Delta^{h}[z]\right)^{2}+3i\Delta^{h}[z]-j\Delta^{v}[z]\right)X^{2}\\
&\quad-3\left(10^{2}\Delta^{h}[z]\Delta^{v}[z]-2i\Delta^{v}[z]\right)XT+3\cdot10^{2}\left(\Delta^{v}[z]\right)^{2}T^{2}\\
&\quad-\left(60i\left(\Delta^{h}[z]\right)^{2}-30j\Delta^{h}[z]\Delta^{v}[z]+10\Delta^{h}[z]\left(6z_{i,j}-1+\frac{9i^{2}}{10^{2}}\right)-\frac{6ij}{10}\Delta^{v}[z]\right)X\\
&\quad-\left(60i\Delta^{h}[z]\Delta^{v}[z]-30j\left(\Delta^{v}[z]\right)^{2}-10\Delta^{v}[z]\left(3z_{i,j}-5+\frac{3i^{2}}{10^{2}}\right)\right)T\\
&\quad-3i\Delta^{h}[z]\left(2z_{i,j}-1+\frac{i^{2}}{10^{2}}\right)+3j\Delta^{v}[z]\left(z_{i,j}-1+\frac{i^{2}}{10^{2}}\right)-2z_{i,j}\,,
\end{aligned}
$$

$$
(4.5)\qquad x\frac{\partial Q}{\partial x}-t\frac{\partial Q}{\partial t}=10\Delta^{h}[y]X-10\Delta^{v}[y]T+i\Delta^{h}[y]-j\Delta^{v}[y]\,,
$$

$$
(4.6)\qquad
\begin{aligned}
&\left(\overline{Q}+2x^{2}\right)t\overline{Q}\\
&=20\Delta^{h}[z]X^{3}T+20\Delta^{v}[z]X^{2}T^{2}+2j\Delta^{h}[z]X^{3}\\
&\quad+\left(10^{2}\left(\Delta^{h}[z]\right)^{2}+4i\Delta^{h}[z]+2j\Delta^{v}[z]+2z_{i,j}\right)X^{2}T\\
&\quad+2\left(10^{2}\Delta^{h}[z]\Delta^{v}[z]+2i\Delta^{v}[z]\right)XT^{2}+10^{2}\left(\Delta^{v}[z]\right)^{2}T^{3}\\
&\quad+\frac{j}{10}\left(10^{2}\left(\Delta^{h}[z]\right)^{2}+4i\Delta^{h}[z]+2z_{i,j}\right)X^{2}\\
&\quad+\frac{2}{10}\left(j\cdot10^{2}\Delta^{h}[z]\Delta^{v}[z]+10^{2}\Delta^{h}[z]\left(z_{i,j}+\frac{i^{2}}{10^{2}}\right)+2ij\Delta^{v}[z]+2iz_{i,j}\right)XT
\end{aligned}
$$

$$+\left(j\cdot 10\left(\Delta^v[z]\right)^2+20\Delta^v[z]\left(z_{i,j}+\frac{i^2}{10^2}\right)\right)T^2$$
$$+2\left(j\Delta^h[z]\left(z_{i,j}+\frac{i^2}{10^2}\right)+\frac{2ij}{10^2}z_{i,j}\right)X$$
$$+\left(2j\Delta^v[z]\left(z_{i,j}+\frac{i^2}{10^2}\right)+z_{i,j}\left(z_{i,j}+\frac{2i^2}{10^2}\right)\right)T+\frac{j}{10}z_{i,j}\left(z_{i,j}+\frac{2i^2}{10^2}\right),$$

(4.7)
$$\begin{aligned}&\overline{Q}(\overline{Q}-1)Q\\
&=10^3\left(\Delta^h[z]\right)^2\Delta^h[y]X^3+10^3\Delta^h[z]\left(\Delta^h[z]\Delta^v[y]+2\Delta^v[z]\Delta^h[y]\right)X^2T\\
&\quad+10^3\Delta^v[z]\left(2\Delta^h[z]\Delta^v[y]+\Delta^v[z]\Delta^h[y]\right)XT^2+10^3\left(\Delta^v[z]\right)^2\Delta^v[y]T^3\\
&\quad+10^2\left(\left(\Delta^h[z]\right)^2y_{i,j}+\Delta^h[z]\left(2z_{i,j}-1\right)\Delta^h[y]\right)X^2\\
&\quad+10^2\left(\Delta^h[z]\left(2z_{i,j}-1\right)\Delta^v[y]+\Delta^v[z]\left(2z_{i,j}-1\right)\Delta^h[y]+2\Delta^h[z]\Delta^v[z]y_{i,j}\right)XT\\
&\quad+10^2\left(\Delta^v[z]\left(2z_{i,j}-1\right)\Delta^v[y]+\left(\Delta^v[z]\right)^2y_{i,j}\right)T^2\\
&\quad+10\left(z_{i,j}\left(z_{i,j}-1\right)\Delta^h[y]+\Delta^h[z]\left(2z_{i,j}-1\right)y_{i,j}\right)X\\
&\quad+10\left(z_{i,j}\left(z_{i,j}-1\right)\Delta^v[y]+\Delta^v[z]\left(2z_{i,j}-1\right)y_{i,j}\right)T+z_{i,j}\left(z_{i,j}-1\right)y_{i,j},\end{aligned}$$

(4.8)
$$\begin{aligned}&x\overline{Q}(3\overline{Q}-2+2x^2)t\\
&=20\Delta^h[z]X^4T+20\Delta^v[z]X^3T^2+2j\Delta^h[z]X^4\\
&\quad+\left(3\cdot 10^2\left(\Delta^h[z]\right)^2+6i\Delta^h[z]+2j\Delta^v[z]+2z_{i,j}\right)X^3T\\
&\quad+6\left(10^2\Delta^h[z]\Delta^v[z]+i\Delta^v[z]\right)X^2T^2+3\cdot 10^2\left(\Delta^v[z]\right)^2XT^3\\
&\quad+\frac{j}{10}\left(3\cdot 10^2\left(\Delta^h[z]\right)^2+6i\Delta^h[z]+2z_{i,j}\right)X^3\\
&\quad+10\Bigg(3i\left(\Delta^h[z]\right)^2+6j\Delta^h[z]\Delta^v[z]\\
&\qquad+2\Delta^h[z]\left(3z_{i,j}+\frac{3i^2}{10^2}-1\right)+6\Delta^v[z]\frac{ij}{10^2}+\frac{6i}{10^2}z_{i,j}\Bigg)X^2T\\
&\quad+10\left(6i\Delta^h[z]\Delta^v[z]+3j\left(\Delta^v[z]\right)^2+2\Delta^v[z]\left(3z_{i,j}+\frac{3i^2}{10^2}-1\right)\right)XT^2\\
&\quad+30i\left(\Delta^v[z]\right)^2T^3+j\left(3i\left(\Delta^h[z]\right)^2+2\Delta^h[z]\left(3z_{i,j}+\frac{3i^2}{10^2}-1\right)\right.\end{aligned}$$

$$+\frac{6i}{10^2}z_{i,j}\Big)X^2+\Big(6ij\Delta^h[z]\Delta^v[z]+2i\Delta^h[z]\Big(3z_{i,j}+\frac{i^2}{10^2}-1\Big)$$

$$+2j\Delta^v[z]\Big(3z_{i,j}+\frac{3i^2}{10^2}-1\Big)+z_{i,j}\Big(3z_{i,j}+\frac{6i^2}{10^2}-1\Big)\Big)XT$$

$$+i\Big(3j\big(\Delta^v[z]\big)^2+2\Delta^v[z]\Big(3z_{i,j}+\frac{i^2}{10^2}-1\Big)\Big)T^2$$

$$+\frac{j}{10}\Big(2i\Delta^h[z]\Big(3z_{i,j}+\frac{i^2}{10^2}-1\Big)+z_{i,j}\Big(3z_{i,j}+\frac{6i^2}{10^2}-2\Big)\Big)X$$

$$+\frac{i}{10}\Big(2j\Delta^v[z]\Big(3z_{i,j}+\frac{i^2}{10^2}-1\Big)+z_{i,j}\Big(3z_{i,j}+\frac{2i^2}{10^2}-2\Big)\Big)T$$

$$+\frac{ij}{10^2}z_{i,j}\Big(3z_{i,j}+\frac{2i^2}{10^2}-2\Big),$$

$$x\overline{Q}(2\overline{Q}-1+x^2)x$$
$$=10\Delta^h[z]X^5+10\Delta^v[z]X^4T+\Big(2\cdot10^2\big(\Delta^h[z]\big)^2+4i\Delta^h[z]+z_{i,j}\Big)X^4$$
$$+\Big(4\cdot10^2\Delta^h[z]\Delta^v[z]+4i\Delta^v[z]\Big)X^3T+2\cdot10^2\big(\Delta^v[z]\big)^2X^2T^2$$
$$+\Big(4\cdot10i\big(\Delta^h[z]\big)^2+10\Delta^h[z]\Big(4z_{i,j}+\frac{6i^2}{10^2}-1\Big)+\frac{4i}{10}z_{i,j}\Big)X^3$$
$$+\Big(8\cdot10i\Delta^h[z]\Delta^v[z]+10\Delta^v[z]\Big(4z_{i,j}+\frac{6i^2}{10^2}-1\Big)\Big)X^2T$$
$$+4\cdot10i\big(\Delta^v[z]\big)^2XT^2 \tag{4.9}$$
$$+\Big(2\cdot i^2\big(\Delta^h[z]\big)^2+2i\Delta^h[z]\Big(4z_{i,j}+\frac{2i^2}{10^2}-1\Big)+z_{i,j}\Big(2z_{i,j}+\frac{6i^2}{10^2}-1\Big)\Big)X^2$$
$$+\Big(4\cdot i^2\Delta^h[z]\Delta^v[z]+2i\Delta^v[z]\Big(4z_{i,j}+\frac{2i^2}{10^2}-1\Big)\Big)XT+2\cdot i^2\big(\Delta^v[z]\big)^2T^2$$
$$+\Big(\frac{i^2}{10}\Delta^h[z]\Big(4z_{i,j}+\frac{i^2}{10^2}-1\Big)+\frac{2i}{10}z_{i,j}\Big(2z_{i,j}+\frac{2i^2}{10^2}-1\Big)\Big)X$$
$$+\frac{i^2}{10}\Delta^v[z]\Big(4z_{i,j}+\frac{i^2}{10^2}-1\Big)T+\frac{i^2}{10}z_{i,j}\Big(2z_{i,j}+\frac{i^2}{10^2}-1\Big)$$

and

$$
\begin{aligned}
&t\overline{Q}(\overline{Q}-1+x^2)t \\
&=10\Delta^h[z]X^3T^2+10\Delta^v[z]X^2T^3+2j\Delta^h[z]X^3T \\
&\quad+\left(10^2\left(\Delta^h[z]\right)^2+2i\Delta^h[z]+2j\Delta^v[z]+z_{i,j}\right)X^2T^2 \\
&\quad+2\left(10^2\Delta^h[z]\Delta^v[z]+i\Delta^v[z]\right)XT^3+10^2\left(\Delta^v[z]\right)^2T^4+\frac{j^2}{10}\Delta^h[z]X^3 \\
&\quad+\frac{j}{10}\left(2\cdot 10^2\left(\Delta^h[z]\right)^2+4i\Delta^h[z]+j\Delta^v[z]+2z_{i,j}\right)X^2T \\
&\quad+\left(4\cdot 10j\Delta^h[z]\Delta^v[z]+10\Delta^h[z]\left(2z_{i,j}+\frac{i^2}{10^2}-1\right)\right. \\
&\qquad\left.+4\cdot 10\Delta^v[z]\frac{ij}{10^2}+\frac{2i}{10^2}z_{i,j}\right)XT^2 \\
&\quad+10\left(2j\left(\Delta^v[z]\right)^2+\Delta^v[z]\left(2z_{i,j}+\frac{i^2}{10^2}-1\right)\right)T^3 \\
&\quad+\frac{j^2}{10}\left(10^2\left(\Delta^h[z]\right)^2+2i\Delta^h[z]+z_{i,j}\right)X^2 \\
&\quad+2j\left(j\Delta^h[z]\Delta^v[z]+\Delta^h[z]\left(2z_{i,j}+\frac{i^2}{10^2}-1\right)+\frac{ij}{10^2}\Delta^v[z]+\frac{2i}{10^2}z_{i,j}\right)XT \\
&\quad+\left(j^2\left(\Delta^v[z]\right)^2+2j\Delta^v[z]\left(2z_{i,j}+\frac{i^2}{10^2}-1\right)+z_{i,j}\left(z_{i,j}+\frac{i^2}{10^2}-1\right)\right)T^2 \\
&\quad+\frac{j^2}{10^2}\left(10\Delta^h[z]\left(2z_{i,j}+\frac{i^2}{10^2}-1\right)+\frac{2i}{10}z_{i,j}\right)X \\
&\quad+\frac{j}{10}\left(j\Delta^v[z]\left(2z_{i,j}+\frac{i^2}{10^2}-1\right)+2z_{i,j}\left(z_{i,j}+\frac{i^2}{10^2}-1\right)\right)T \\
&\quad+\frac{j^2}{10^2}z_{i,j}\left(z_{i,j}+\frac{i^2}{10^2}-1\right),
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{cases}
\Delta^h[z]=\Delta^h_{i,j}[z] & \Delta^v[z]=\Delta^v_{i+1,j}[z] \\
\Delta^h[z]=\Delta^h_{i,j+1}[z] & \Delta^v[z]=\Delta^v_{i,j}[z]
\end{cases}
\quad \text{on} \quad
\begin{matrix} E_{ij} \\ \overline{E}_{ij} \end{matrix}
\tag{4.11}
$$

Using the expressions (4.4)-(4.10), we obtain the integrand of (3.3) on E_{ij} or $\overline{E}_{ij}$

which is expanded in terms of $X^m T^n$.

Then, using (4.3), we obtain the integrals on the left hand side of (3.3) on both E_{ij} and $\overline{E}_{ij}$ written as polynomials of $\Delta^h[z]$, $\Delta^v[z]$, $\cdots$, $\Delta^v[v]$, $z_{i,j}$, $y_{i,j}$, $v_{i,j}$, which we denote by (4.12) and (4.13) respectively.

We compute the right hand side of (3.3) as follows: First, owing to piecewise linearity of W(x,t), we have the following expression on $\frac{j}{10} \leq t \leq \frac{j+1}{10}$:

$$(1-t)^2 W(1,t) = \left\{\left(t-\frac{j}{10}\right)^2 - 2\left(1-\frac{j}{10}\right)\left(t-\frac{j}{10}\right)+\left(1-\frac{j}{10}\right)^2\right\} \times \left\{10\Delta^v_{10,j}\left(t-\frac{j}{10}\right)+v_{10,j}\right\}$$

$$= 10\Delta^v_{10,j}\left(t-\frac{j}{10}\right)^3 + \left\{-20\Delta^v_{10,j}\left(1-\frac{j}{10}\right)+v_{10,j}\right\}\left(t-\frac{j}{10}\right)^2 + \left\{10\Delta^v_{10,j}\left(1-\frac{j}{10}\right)-2v_{10,j}\right\}\left(1-\frac{j}{10}\right)\left(t-\frac{j}{10}\right)+\left(1-\frac{j}{10}\right)^2 v_{10,j},$$

from which we obtain

$$\int_{j/10}^{(j+1)/10}(1-t)^2 W(1,t)dt = \frac{10\Delta^v_{10,j}}{4\cdot 10^4} + \frac{1}{3\cdot 10^3}\left\{-20\Delta^v_{10,j}\left(1-\frac{j}{10}\right)+v_{10,j}\right\} + \frac{1-j/10}{2\cdot 10^2}\left\{10\Delta^v_{10,j}\left(1-\frac{j}{10}\right)-2v_{10,j}\right\}+\frac{1}{10}\left(1-\frac{j}{10}\right)^2 v_{10,j}$$

$$= \frac{1}{10^3}\left[\left\{\frac{1}{4}-\frac{2}{3}(10-j)+\frac{1}{2}(10-j)^2\right\}v_{10,j+1} + \left\{\frac{1}{12}-\frac{1}{3}(10-j)+\frac{1}{2}(10-j)^2\right\}v_{10,j}\right]$$

and hence

$$-3\int_0^1(1-t)^2 W(1,t)dt = -\frac{561}{4\cdot 10^3}v_{10,0} - \frac{1}{10^3}\sum_{j=1}^{9}\left(\frac{1}{2}+3(10-j)^2\right)v_{10,j} - \frac{1}{4\cdot 10^3}v_{10,10}. \tag{4.14}$$

Next, we have on $\frac{i}{10} \le x \le \frac{i+1}{10}$

$$x^2(1-x)W(x,1) = x^2(1-x)\left\{(-10x+i+1)v_{i,10} + (10x-i)v_{i+1,10}\right\}$$
$$= \left\{10x^4 - (11+i)x^3 + (i+1)x^2\right\}v_{i,10} + \left\{-10x^4 + (10+i)x^3 - ix^2\right\}v_{i+1,10},$$

$$\int_{i/10}^{(i+1)/10} \left\{10x^4 - (11+i)x^3 + (i+1)x^2\right\}dx$$
$$= \frac{1}{5\cdot 10^4}\left(1+5i+10i^2+10i^3+5i^4\right) - \frac{11+i}{4\cdot 10^4}\left(1+4i+6i^2+4i^3\right)$$
$$+ \frac{1+i}{3\cdot 10^4}\left(1+3i+3i^2\right) = \frac{47}{6\cdot 10^5} + \frac{37i}{12\cdot 10^4} + \frac{9i^2}{2\cdot 10^4} - \frac{i^3}{2\cdot 10^4},$$

$$\int_{i/10}^{(i+1)/10} \left\{-10x^4 + (10+i)x^3 - ix^2\right\}dx$$
$$= -\frac{1}{5\cdot 10^4}\left(1+5i+10i^2+10i^3+5i^4\right) + \frac{10+i}{4\cdot 10^4}\left(1+4i+6i^2+4i^3\right)$$
$$- \frac{i}{3\cdot 10^4}\left(1+3i+3i^2\right) = \frac{23}{10^5} + \frac{71i}{12\cdot 10^4} + \frac{2i^2}{5\cdot 10^3} - \frac{i^3}{2\cdot 10^4},$$

from which we obtain

$$\int_{i/10}^{(i+1)/10} x^2(1-x)W(x,1)dx = \left(\frac{47}{6\cdot 10^5} + \frac{37i}{12\cdot 10^4} + \frac{9i^2}{2\cdot 10^4} - \frac{i^3}{2\cdot 10^4}\right)v_{i,10}$$
$$+ \left(\frac{23}{10^5} + \frac{71i}{12\cdot 10^4} + \frac{2i^2}{5\cdot 10^3} - \frac{i^3}{2\cdot 10^4}\right)v_{i+1,10}$$

and hence

$$\int_0^1 x^2(1-x)W(x,1)dx = \frac{47}{6\cdot 10^5}v_{0,10} + \sum_{i=1}^{9}\left\{\frac{47}{6\cdot 10^5} + \frac{37i}{12\cdot 10^4}\right.$$
$$\left. + \frac{9i^2}{2\cdot 10^4} - \frac{i^3}{2\cdot 10^4} + \frac{23}{10^5} + \frac{71(i-1)}{12\cdot 10^4} + \frac{2(i-1)^2}{5\cdot 10^3} - \frac{(i-1)^3}{2\cdot 10^4}\right\}v_{i,10}$$
$$+ \left(\frac{23}{10^5} + \frac{213}{4\cdot 10^4} + \frac{2\times 9^2}{5\cdot 10^3} - \frac{9^3}{2\cdot 10^4}\right)v_{10,10}$$
$$= \frac{47}{6\cdot 10^5}v_{0,10} + \sum_{i=1}^{9}\left(\frac{1}{6\cdot 10^3} - \frac{i}{2\cdot 10^4} + \frac{i^2}{10^3} - \frac{i^3}{10^4}\right)v_{i,10} + \frac{301}{2\cdot 10^5}v_{10,10}$$

i.e.

$$\int_0^1 x^2(1-x)W(x,1)dx$$
$$= \frac{47}{6\cdot 10^5}v_{0,10} + \frac{1}{10^4}\sum_{i=1}^{9}\left(\frac{5}{3} - \frac{i}{2} + 10i^2 - i^3\right)v_{i,10} + \frac{301}{2\cdot 10^5}v_{10,10}. \tag{4.15}$$

§ 5. Principle of computation of the approximate solution Q(x,t) of (3.3) such that $Q(x,t)=\overline{Q}(x,t)$.

For a fixed pair (i,j), $i,j=1,2,\cdots,9$, we consider the elements $E_{ij}(=E_{i,j})$, $\overline{E}_{ij}(=\overline{E}_{i,j})$, $E_{i-1,j}$, $\overline{E}_{i-1,j-1}$, $E_{i-1,j-1}$, $\overline{E}_{i,j-1}$ having a vertex at the point $P[i,j]$, only for which the expressions (4.12) and (4.13) depend on $v_{i,j}$ as a free variable.

Since we have

$$\Delta^h_{i,j}[v]=v_{i+1,j}-v_{i,j},\ \Delta^v_{i,j}[v]=v_{i,j+1}-v_{i,j},$$

we have

$$\begin{array}{lll}
\partial\Delta^h[v]/\partial v_{i,j}=-1, & \partial\Delta^v[v]/\partial v_{i,j}=0 & \text{for}\quad E_{ij};\\
\partial\Delta^h[v]/\partial v_{i,j}=0, & \partial\Delta^v[v]/\partial v_{i,j}=-1 & \text{for}\quad \overline{E}_{ij};\\
\partial\Delta^h[v]/\partial v_{i,j}=1, & \partial\Delta^v[v]/\partial v_{i,j}=-1 & \text{for}\quad E_{i-1,j};\\
\partial\Delta^h[v]/\partial v_{i,j}=1, & \partial\Delta^v[v]/\partial v_{i,j}=0 & \text{for}\quad \overline{E}_{i-1,j-1};\\
\partial\Delta^h[v]/\partial v_{i,j}=0, & \partial\Delta^v[v]/\partial v_{i,j}=1 & \text{for}\quad E_{i-1,j-1};\\
\partial\Delta^h[v]/\partial v_{i,j}=-1, & \partial\Delta^v[v]/\partial v_{i,j}=1 & \text{for}\quad \overline{E}_{i,j-1}
\end{array}$$

and for simplicity we set

$$z_{i,j}=z_0,\ z_{i+1,j}=z_1,\ z_{i+1,j+1}=z_2,\ z_{i,j+1}=z_3,\ z_{i-1,j+1}=z_4,$$
$$z_{i-1,j}=z_5,\ z_{i-1,j-1}=z_6,\ z_{i,j-1}=z_7,\ z_{i+1,j-1}=z_8$$

and use similar notations for $y_{i,j}$. Using (4.12) and (4.13), we denote the partial derivatives of the integrals of (3.3) on E_{ij}, $\overline{E}_{ij}$, $E_{i-1,j}$, $\overline{E}_{i-1,j-1}$, $E_{i-1,j-1}$ and $\overline{E}_{i,j-1}$ with respect to $v_{i,j}$ by (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) respectively.

If we assume that all $z_{i,j}=\overline{Q}\left(\frac{i}{10},\frac{j}{10}\right)$ are known, and put the sum of (5.1)-(5.6) being equal to zero, then we obtain the nodal linear equation of $y_0=y_{i,j}$, $y_1=y_{i+1,j}$, $y_2=y_{i+1,j+1}$, $y_3=y_{i,j+1}$, $y_5=y_{i-1,j}$, $y_6=y_{i-1,j-1}$, $y_7=y_{i,j-1}$. We assemble the nodal equations for $i,j=1,2,\cdots,9$ and we applied the boundary condition (2.8) and (2.11) with a=4/27. We solve the resulting system of linear equations for $y_{\alpha\beta}$, $\alpha,\beta=1,2,\cdots,8$, to obtain $Q(x,t)$ as an improved solution of $\overline{Q}(x,t)$. If we take $F(x,t)=\overline{Q}_0(x,t)$ for example as a starting (pedestal) function and repeat the above mentioned process we obtain a series of functions $\overline{Q}_m(x,t)$, $m=1,2,\cdots$, which will be expected to converge to an approximate function $Q(x,t)$, depending on the prescribed boundary conditions and on the mesh of elements $E_{i,j}$ and $\overline{E}_{i,j}$.

We shall compute directly a stable solution $Q(x,t)$ in the following way. In the arguments in the beginning of this section, for the fixed pair (i,j), we suppose at the Step $\{i,j\}$ that

(5.7) $$z_0 = y_0,\ z_1 = y_1,\ z_2 = y_2,\ z_3 = y_3,\ z_5 = y_5,\ z_7 = y_7$$

and consider the condition for $z_6 = z_{i-1,j-1}$ such that the remaining variable $y = y_6$ coincide with z_6, which will become a cubic equation. Using the condition (5.7), the expressions (5.1) - (5.6) can be rewritten respectively as follows:

(5.1) $\rightarrow$ $f_1(z,i,j,u,v)$, where $z = z_0,\ u = z_1 - z_0,\ v = z_2 - z_1$,

and

$$f_1 := \left(\frac{1}{15\cdot 10^2} + \frac{i}{5\cdot 10^2}\right)u^3 + \left(\frac{1}{2\cdot 10^3} + \frac{17i}{4\cdot 10^3} - \frac{j}{10^3} + \frac{i^2}{4\cdot 10^2}\right)u^2 v$$
$$+\left(-\frac{1}{5\cdot 10^2} + \frac{i}{6\cdot 10^3} - \frac{19j}{4\cdot 10^3} + \frac{i^2}{6\cdot 10^2} - \frac{3ij}{8\cdot 10^2} - \frac{j^2}{4\cdot 10^2}\right)uv^2$$
$$-\left(\frac{19}{12\cdot 10^3} + \frac{3i}{2\cdot 10^3} + \frac{17j}{6\cdot 10^3} + \frac{ij}{4\cdot 10^2} + \frac{j^2}{8\cdot 10^2}\right)v^3$$
$$+\left(-\frac{69}{14\cdot 10^4} - \frac{82i}{5\cdot 10^4} + \frac{509i^2}{3\cdot 10^5} - \frac{i^4}{6\cdot 10^4} + \frac{z}{100}\left(\frac{1}{5} + i + \frac{i^2}{3}\right)\right)u^2$$
$$+\left(\frac{211}{42\cdot 10^4} - \frac{497i}{15\cdot 10^4} + \frac{83j}{5\cdot 10^4} - \frac{97i^2}{6\cdot 10^4} - \frac{103ij}{3\cdot 10^5} + \frac{i^3}{2\cdot 10^4} + \frac{i^2 j}{6\cdot 10^4}\right.$$
$$\left.+\frac{i^4}{6\cdot 10^4} + \frac{i^3 j}{3\cdot 10^4} + \frac{z}{100}\left(-\frac{9}{40} + \frac{5i}{8} - \frac{7j}{12} + \frac{2i^2}{3} - \frac{ij}{2}\right)\right)uv$$
$$+\left(\frac{3137}{126\cdot 10^4} + \frac{734i}{45\cdot 10^4} + \frac{581j}{10^5} - \frac{i^2}{24\cdot 10^3} + \frac{488ij}{15\cdot 10^4} + \frac{997j^2}{2\cdot 10^5}\right.$$
$$-\frac{i^3}{6\cdot 10^4} - \frac{11i^2 j}{12\cdot 10^4} - \frac{ij^2}{2\cdot 10^4} - \frac{i^3 j}{3\cdot 10^4} - \frac{i^2 j^2}{2\cdot 10^4}$$
$$\left.-\frac{z}{100}\left(\frac{31}{60} + \frac{i}{2} + \frac{11j}{12} + ij + \frac{j^2}{2}\right)\right)v^2$$
$$+\frac{z}{100}\left(-\frac{49}{6\cdot 10^2} - \frac{496i}{5\cdot 10^2} - \frac{97i^2}{2\cdot 10^2} + \frac{4i^3}{3\cdot 10^2} + \frac{i^4}{2\cdot 10^2} + z\left(\frac{4i}{3} + i^2\right)\right)u$$
$$+\frac{z}{100}\left(\frac{748}{15\cdot 10^2} + \frac{197i}{6\cdot 10^2} + \frac{997j}{10^3} - \frac{2i^2}{3\cdot 10^2} + \frac{593ij}{6\cdot 10^2} - \frac{i^3}{3\cdot 10^2} - \frac{i^2 j}{6\cdot 10}\right.$$

$$-\frac{i^3 j}{10^2}-z\left(\frac{7}{12}+\frac{i}{2}+\frac{5j}{6}+\frac{3ij}{2}\right)\Bigg)v-\frac{1}{3\cdot 10^2}z^2(z-1);$$

(5.2) $\to$ $f_2(z,i,j,u,v)$, where $z=z_0$, $u=z_2-z_3$, $v=z_3-z_0$,

and

$$\begin{aligned}
f_2 := &-\left(\frac{1}{15\cdot 10^2}+\frac{i}{5\cdot 10^2}+\frac{i^2}{4\cdot 10^2}\right)u^3\\
&+\left(\frac{1}{3\cdot 10^3}-\frac{3i}{4\cdot 10^3}+\frac{3j}{10^3}-\frac{i^2}{2\cdot 10^2}+\frac{3ij}{8\cdot 10^2}+\frac{j^2}{12\cdot 10^2}\right)u^2 v\\
&+\left(\frac{2}{10^3}+\frac{9i}{2\cdot 10^3}+\frac{21j}{4\cdot 10^3}+\frac{3ij}{4\cdot 10^2}+\frac{j^2}{8\cdot 10^2}\right)uv^2\\
&+\left(\frac{1}{3\cdot 10^3}+\frac{3j}{2\cdot 10^3}\right)v^3\\
&+\left(\frac{349}{7\cdot 10^5}+\frac{497i}{3\cdot 10^5}+\frac{497i^2}{10^5}-\frac{i^3}{2\cdot 10^4}-\frac{i^4}{2\cdot 10^4}-\frac{z}{100}\left(\frac{1}{5}+\frac{i}{2}+i^2\right)\right)u^2\\
&+\Bigg(-\frac{179}{12\cdot 10^4}-\frac{2984i}{45\cdot 10^4}-\frac{2491j}{6\cdot 10^5}+\frac{3i^2}{5\cdot 10^4}-\frac{2981ij}{3\cdot 10^5}-\frac{997j}{6\cdot 10^5}+\frac{i^3}{2\cdot 10^4}\\
&\quad+\frac{13i^2 j}{12\cdot 10^4}+\frac{ij^2}{6\cdot 10^4}+\frac{i^3 j}{10^4}+\frac{i^2 j^2}{6\cdot 10^4}+\frac{z}{100}\left(\frac{7}{40}+\frac{3i}{4}+\frac{23j}{24}+\frac{3ij}{2}+\frac{j^2}{3}\right)\Bigg)uv\\
&+\Bigg(\frac{1895}{126\cdot 10^4}+\frac{2i}{15\cdot 10^4}+\frac{251j}{15\cdot 10^4}+\frac{3i^2}{2\cdot 10^5}+\frac{ij}{5\cdot 10^4}+\frac{1001j^2}{6\cdot 10^5}\\
&\quad+\frac{i^2 j}{6\cdot 10^4}-\frac{i^2 j^2}{6\cdot 10^4}+\frac{z}{100}\left(\frac{1}{20}+\frac{2j}{3}+\frac{j^2}{6}\right)\Bigg)v^2\\
&+\frac{z}{100}\left(\frac{1}{12}-\frac{i}{3}-\frac{z}{4}\right)u\\
&+\frac{z}{100}\Bigg(\frac{2257}{9\cdot 10^3}+\frac{3i}{10^3}-\frac{199j}{6\cdot 10^2}+\frac{5i^2}{12\cdot 10^2}+\frac{2ij}{3\cdot 10^2}-\frac{599j^2}{12\cdot 10^2}+\frac{i^2 j}{10^2}\\
&\quad+\frac{ij^2}{3\cdot 10^2}+\frac{i^2 j^2}{2\cdot 10^2}+z\left(-\frac{1}{6}+\frac{5j}{6}+\frac{j^2}{2}\right)\Bigg)v-\frac{1}{3\cdot 10^2}z^2(z-1);
\end{aligned}$$

(5.3) $\to$ $f_3(z,i,j,u,v)$, where $z=z_5$, $u=z_0-z_5$, $v=z_3-z_0$, and

$$
\begin{aligned}
f_3 := &-\left(\frac{1}{10^2}+\frac{i-1}{5\cdot 10}+\frac{(i-1)^2}{8\cdot 10}\right)u^3 \\
&+\left(-\frac{1}{12\cdot 10^2}-\frac{3(i-1)}{5\cdot 10^2}+\frac{9j}{5\cdot 10^2}-\frac{3(i-1)^2}{4\cdot 10^2}+\frac{15(i-1)j}{8\cdot 10^2}+\frac{j^2}{4\cdot 10^2}\right)u^2v \\
&+\left(\frac{1}{3\cdot 10^2}+\frac{17(i-1)}{6\cdot 10^3}+\frac{31j}{3\cdot 10^3}-\frac{(i-1)^2}{6\cdot 10^2}+\frac{9(i-1)j}{8\cdot 10^2}-\frac{j^2}{8\cdot 10^2}\right)uv^2 \\
&+\left(\frac{1}{8\cdot 10^2}+\frac{3(i-1)}{2\cdot 10^3}+\frac{13j}{6\cdot 10^3}+\frac{(i-1)j}{4\cdot 10^2}-\frac{j^2}{24\cdot 10^2}\right)v^3 \\
&+\left(\frac{695}{14\cdot 10^4}+\frac{295(i-1)}{3\cdot 10^4}+\frac{241(i-1)^2}{3\cdot 10^4}-\frac{5(i-1)^3}{2\cdot 10^4}-\frac{5(i-1)^4}{6\cdot 10^4}\right. \\
&\left.-\frac{z}{100}\left(2+\frac{7(i-1)}{2}+\frac{7(i-1)^2}{3}\right)\right)u^2 \\
&+\left(-\frac{967}{42\cdot 10^4}-\frac{146(i-1)}{45\cdot 10^3}-\frac{743j}{6\cdot 10^4}+\frac{174(i-1)^2}{10^5}-\frac{2443(i-1)j}{15\cdot 10^4}-\frac{497j^2}{15\cdot 10^4}\right. \\
&+\frac{17(i-1)^2j}{4\cdot 10^4}+\frac{(i-1)j^2}{2\cdot 10^4}-\frac{(i-1)^4}{6\cdot 10^4}+\frac{5(i-1)^3j}{3\cdot 10^3}+\frac{(i-1)^2j^2}{3\cdot 10^4} \\
&\left.+\frac{z}{100}\left(\frac{3}{10}+\frac{i-1}{8}+\frac{27j}{8}-\frac{2(i-1)^2}{3}+\frac{7(i-1)j}{2}+\frac{2j^2}{3}\right)\right)uv \\
&+\left(-\frac{275}{42\cdot 10^4}-\frac{293(i-1)}{18\cdot 10^4}-\frac{49j}{6\cdot 10^4}+\frac{13(i-1)^2}{3\cdot 10^5}-\frac{491(i-1)j}{15\cdot 10^4}\right. \\
&+\frac{497j^2}{15\cdot 10^4}+\frac{(i-1)^3}{6\cdot 10^4}+\frac{3(i-1)^2j}{4\cdot 10^4}-\frac{(i-1)j^2}{2\cdot 10^4}+\frac{(i-1)^3j}{3\cdot 10^4} \\
&\left.-\frac{(i-1)^2j^2}{3\cdot 10^4}+\frac{z}{100}\left(\frac{23}{60}+\frac{i-1}{2}+\frac{11j}{12}+(i-1)j-\frac{j^2}{6}\right)\right)v^2 \\
&+\frac{z}{100}\left(\frac{299}{6\cdot 10^2}+\frac{488(i-1)}{15\cdot 10^2}+\frac{97(i-1)^2}{2\cdot 10^2}-\frac{4(i-1)^3}{3\cdot 10^2}\right. \\
&\left.-\frac{(i-1)^4}{2\cdot 10^2}-z\left(\frac{5}{4}+\frac{4(i-1)}{3}+(i-1)^2\right)\right)u \\
&+\frac{z}{100}\left(-\frac{145}{18\cdot 10^2}-\frac{487(i-1)}{15\cdot 10^2}-\frac{988j}{15\cdot 10^2}+\frac{11(i-1)^2}{12\cdot 10^2}-\frac{39(i-1)j}{4\cdot 10^2}\right. \\
&-\frac{199j^2}{4\cdot 10^2}+\frac{(i-1)^3}{3\cdot 10^2}+\frac{8(i-1)^2j}{3\cdot 10^2}+\frac{2(i-1)j^2}{3\cdot 10^2}+\frac{(i-1)^3j}{10^2}
\end{aligned}
$$

$$+\frac{(i-1)^2 j^2}{2\cdot 10^2}+z\left(-\frac{1}{4}+\frac{i}{2}+\frac{3ij}{2}+\frac{j^2}{2}\right)\Bigg)v-\frac{1}{3\cdot 10^2}z^2(z-1);$$

(5.4) $\rightarrow\ f_4(y,i,j,u,v)$, where $u=z_0-z_5$, $v=z_5-y$,

and

$$\begin{aligned}
f_4:=&-\left(\frac{1}{3\cdot 10^2}+\frac{4(i-1)}{5\cdot 10^2}+\frac{2(i-1)^2}{3\cdot 10^2}\right)u^3\\
&-\left(\frac{1}{8\cdot 10^2}+\frac{8(i-1)}{15\cdot 10^2}-\frac{13(j-1)}{2\cdot 10^3}+\frac{(i-1)^2}{8\cdot 10}-\frac{(i-1)(j-1)}{10^2}\right)u^2v\\
&+\left(\frac{1}{3\cdot 10^2}+\frac{11(i-1)}{10^3}+\frac{11(j-1)}{15\cdot 10^2}-\frac{(i-1)^2}{2\cdot 10^2}+\frac{15(i-1)(j-1)}{8\cdot 10^2}-\frac{(j-1)^2}{4\cdot 10^2}\right)uv^2\\
&-\left(\frac{1}{12\cdot 10^2}-\frac{3(i-1)}{5\cdot 10^2}+\frac{j-1}{5\cdot 10^2}-\frac{3(i-1)(j-1)}{4\cdot 10^2}+\frac{3(j-1)^2}{8\cdot 10^2}\right)v^3\\
&+\left(\frac{418}{21\cdot 10^4}+\frac{742(i-1)}{15\cdot 10^4}+\frac{982(i-1)^2}{15\cdot 10^4}-\frac{4(i-1)^3}{3\cdot 10^4}-\frac{2(i-1)^4}{3\cdot 10^4}\right.\\
&\left.-\frac{y}{100}\left(\frac{4}{5}+\frac{5(i-1)}{3}+\frac{5(i-1)^2}{3}\right)\right)u^2\\
&-\left(\frac{691}{35\cdot 10^4}+\frac{99(i-1)}{10^4}+\frac{149(j-1)}{3\cdot 10^4}-\frac{104(i-1)^2}{3\cdot 10^4}+\frac{1979(i-1)(j-1)}{15\cdot 10^4}\right.\\
&-\frac{(i-1)^3}{2\cdot 10^4}-\frac{13(i-1)^2(j-1)}{6\cdot 10^4}+\frac{(i-1)^4}{3\cdot 10^4}-\frac{4(i-1)^3(j-1)}{3\cdot 10^4}\\
&\left.-\frac{y}{100}\left(\frac{1}{15}+\frac{7(i-1)}{8}+\frac{17(j-1)}{12}-\frac{4(i-1)^2}{3}+\frac{5(i-1)(j-1)}{2}\right)\right)uv\\
&+\left(\frac{1401}{28\cdot 10^4}-\frac{448(i-1)}{9\cdot 10^4}+\frac{899(j-1)}{12\cdot 10^4}+\frac{(i-1)^2}{2\cdot 10^4}-\frac{334(i-1)(j-1)}{5\cdot 10^4}\right.\\
&+\frac{997(j-1)^2}{2\cdot 10^5}+\frac{(i-1)^3}{2\cdot 10^4}+\frac{(i-1)^2(j-1)}{4\cdot 10^4}-\frac{(i-1)(j-1)^2}{2\cdot 10^4}+\frac{2(i-1)^3(j-1)}{3\cdot 10^4}\\
&\left.-\frac{(i-1)^2(j-1)^2}{2\cdot 10^4}+\frac{y}{100}\left(-\frac{1}{10}+\frac{3(i-1)}{2}+\frac{j-1}{4}+2(i-1)(j-1)-\frac{(j-1)^2}{2}\right)\right)v^2\\
&+\frac{y}{100}\left(\frac{749}{3\cdot 10^3}-\frac{i-1}{5\cdot 10^2}+\frac{99(i-1)^2}{2\cdot 10^2}-\frac{2(i-1)^3}{3\cdot 10^2}-\frac{(i-1)^4}{2\cdot 10^2}-y\left(\frac{2i}{3}+(i-1)^2\right)\right)u\\
&+\frac{y}{100}\left(\frac{301}{6\cdot 10^2}-\frac{33(i-1)}{5\cdot 10}+\frac{j-1}{5\cdot 10^2}+\frac{(i-1)^2}{10^2}-\frac{119(i-1)(j-1)}{12\cdot 10}\right.
\end{aligned}$$

$$+\frac{2(i-1)^3}{3\cdot 10^2}+\frac{4(i-1)^2(j-1)}{3\cdot 10^2}+\frac{(i-1)^3(j-1)}{10^2}$$
$$+y\left(-\frac{1}{4}+(i-1)+\frac{2(j-1)}{3}+\frac{3(i-1)(j-1)}{2}\right)\Bigg)v$$
$$-\frac{1}{3\cdot 10^2}y^2(y-1);$$

(5.5) $\rightarrow$ $f_5(y,i,j,u,v)$, where $u=z_7-y$, $v=z_0-z_7$,

and

$$f_5:=-\left(\frac{2}{3\cdot 10^2}+\frac{6(i-1)}{5\cdot 10^2}+\frac{3(i-1)^2}{4\cdot 10^2}\right)u^3$$
$$-\left(\frac{1}{6\cdot 10^2}+\frac{i-1}{5\cdot 10^2}-\frac{4(j-1)}{5\cdot 10^2}+\frac{(i-1)^2}{2\cdot 10^2}-\frac{9(i-1)(j-1)}{8\cdot 10^2}+\frac{(j-1)^2}{4\cdot 10^2}\right)u^2v$$
$$+\left(\frac{9(i-1)}{2\cdot 10^3}+\frac{j-1}{15\cdot 10^2}+\frac{3(i-1)(j-1)}{4\cdot 10^2}-\frac{5(j-1)^2}{8\cdot 10^2}\right)uv^2$$
$$-\left(\frac{1}{6\cdot 10^2}+\frac{7(j-1)}{2\cdot 10^3}+\frac{(j-1)^2}{3\cdot 10^2}\right)v^3$$
$$+\Bigg(\frac{417}{14\cdot 10^4}+\frac{49(i-1)}{10^4}+\frac{241(i-1)^2}{5\cdot 10^4}-\frac{3(i-1)^3}{2\cdot 10^4}-\frac{(i-1)^4}{2\cdot 10^4}$$
$$-\frac{y}{100}\left(-\frac{3}{10}+\frac{3i}{2}+(i-1)^2\right)\Bigg)u^2$$
$$+\Bigg(-\frac{133}{21\cdot 10^4}-\frac{59(i-1)}{9\cdot 10^3}-\frac{49(j-1)}{2\cdot 10^4}+\frac{19(i-1)^2}{15\cdot 10^4}$$
$$-\frac{491(i-1)(j-1)}{5\cdot 10^4}+\frac{497(j-1)^2}{15\cdot 10^4}+\frac{(i-1)^3}{2\cdot 10^4}+\frac{9(i-1)^2(j-1)}{4\cdot 10^4}$$
$$-\frac{(i-1)(j-1)^2}{2\cdot 10^4}+\frac{(i-1)^3(j-1)}{10^4}-\frac{(i-1)^2(j-1)^2}{3\cdot 10^4}$$
$$+\frac{y}{100}\left(-\frac{1}{5}+\frac{3(i-1)}{4}+\frac{7(j-1)}{8}+\frac{3(i-1)(j-1)}{2}-\frac{2(j-1)^2}{3}\right)\Bigg)uv$$
$$+\Bigg(\frac{559}{14\cdot 10^4}-\frac{i-1}{6\cdot 10^4}+\frac{249(j-1)}{3\cdot 10^4}-\frac{(i-1)^2}{10^5}-\frac{4(i-1)(j-1)}{5\cdot 10^4}$$
$$+\frac{994(j-1)^2}{15\cdot 10^4}-\frac{(i-1)^2(j-1)}{2\cdot 10^4}-\frac{(i-1)(j-1)^2}{10^4}$$

$$-\frac{2(i-1)^2(j-1)^2}{3\cdot 10^4}-\frac{y}{100}\left(\frac{9}{20}+\frac{2(j-1)}{3}+\frac{5(j-1)^2}{6}\right)\Bigg)v^2$$

$$+\frac{y}{100}\left(\frac{1}{4}-\frac{i-1}{3}-\frac{3y}{4}\right)u$$

$$+\frac{y}{100}\Bigg(\frac{1051}{18\cdot 10^2}+\frac{2(i-1)}{15\cdot 10^2}+\frac{2(j-1)}{3}+\frac{(i-1)^2}{12\cdot 10^2}+\frac{199(j-1)^2}{4\cdot 10^2}$$

$$-\frac{2(i-1)(j-1)^2}{3\cdot 10^2}-\frac{(i-1)^2(j-1)^2}{2\cdot 10^2}-y\left(\frac{1}{3}+\frac{j}{6}+\frac{(j-1)^2}{2}\right)\Bigg)v$$

$$-\frac{1}{3\cdot 10^2}y^2(y-1);$$

(5.6) $\to$ $f_6(z,i,j,u,v)$, where $z=z_7$, $u=z_1-z_0$, $v=z_0-z_7$,

and

$$f_6:=-\frac{i^2}{12\cdot 10^2}u^3$$

$$-\left(\frac{1}{24\cdot 10^2}-\frac{7i}{3\cdot 10^3}+\frac{7(j-1)}{6\cdot 10^3}+\frac{i^2}{4\cdot 10^2}-\frac{i(j-1)}{8\cdot 10^2}+\frac{(j-1)^2}{12\cdot 10^2}\right)u^2v$$

$$-\left(\frac{1}{3\cdot 10^2}-\frac{7i}{10^3}+\frac{19(j-1)}{3\cdot 10^3}-\frac{i^2}{2\cdot 10^2}-\frac{3i(j-1)}{8\cdot 10^2}+\frac{3(j-1)^2}{8\cdot 10^2}\right)uv^2$$

$$-\left(\frac{3}{4\cdot 10^2}+\frac{3i}{5\cdot 10^2}+\frac{6(j-1)}{5\cdot 10^2}+\frac{3i(j-1)}{4\cdot 10^2}+\frac{5(j-1)^2}{8\cdot 10^2}\right)v^3$$

$$+\left(\frac{1}{105\cdot 10^4}+\frac{i}{3\cdot 10^5}+\frac{i^2}{3\cdot 10^2}-\frac{i^3}{6\cdot 10^4}-\frac{i^4}{3\cdot 10^4}+\frac{z}{100}\left(\frac{i}{6}-\frac{i^2}{3}\right)\right)u^2$$

$$+\Bigg(\frac{1399}{14\cdot 10^5}-\frac{899i}{9\cdot 10^4}+\frac{299(j-1)}{12\cdot 10^4}-\frac{983i^2}{3\cdot 10^5}-\frac{1003i(j-1)}{15\cdot 10^4}+\frac{997(j-1)^2}{6\cdot 10^5}$$

$$+\frac{i^3}{10^4}+\frac{i^2(j-1)}{12\cdot 10^4}-\frac{i(j-1)^2}{6\cdot 10^4}+\frac{i^4}{3\cdot 10^4}+\frac{2i^3(j-1)}{3\cdot 10^4}-\frac{i^2(j-1)^2}{6\cdot 10^4}$$

$$+\frac{z}{100}\left(-\frac{11}{30}+\frac{11i}{8}-\frac{13(j-1)}{24}+\frac{4i^2}{3}+\frac{i(j-1)}{2}-\frac{(j-1)^2}{3}\right)\Bigg)uv$$

$$+\Bigg(\frac{9229}{84\cdot 10^4}+\frac{445i}{9\cdot 10^4}+\frac{699(j-1)}{4\cdot 10^4}-\frac{9i^2}{10^5}+\frac{328i(j-1)}{5\cdot 10^4}$$

$$+\frac{4993(j-1)^2}{6\cdot 10^5}-\frac{i^3}{2\cdot 10^4}-\frac{7i^2(j-1)}{4\cdot 10^4}-\frac{i(j-1)^2}{2\cdot 10^4}-\frac{2i^3(j-1)}{3\cdot 10^4}$$
$$-\frac{5i^2(j-1)^2}{6\cdot 10^4}-\frac{z}{100}\left(\frac{17}{10}+\frac{3i}{2}+\frac{9(j-1)}{4}+2i(j-1)+\frac{7(j-1)^2}{6}\right)\Bigg)v^2$$
$$+\frac{z}{100}\left(\frac{1}{3\cdot 10^3}-\frac{997i}{15\cdot 10^2}-\frac{99i^2}{2\cdot 10^2}+\frac{2i^3}{3\cdot 10^2}+\frac{i^4}{2\cdot 10^2}+z\left(-\frac{1}{12}+\frac{2i}{3}+i^2\right)\right)u$$
$$+\frac{z}{100}\left(\frac{281}{225}+\frac{331i}{5\cdot 10^2}+\frac{1997(j-1)}{15\cdot 10^2}-\frac{3i^2}{4\cdot 10^2}+\frac{119i(j-1)}{12\cdot 10}\right.$$
$$+\frac{599(j-1)^2}{12\cdot 10^2}-\frac{2i^3}{3\cdot 10^2}-\frac{4i^2(j-1)}{3\cdot 10^2}-\frac{i(j-1)^2}{3\cdot 10^2}-\frac{i^3(j-1)}{10^2}$$
$$\left.-\frac{i^2(j-1)^2}{2\cdot 10^2}-z\left(\frac{1}{4}+i+j+\frac{3i(j-1)}{2}+\frac{(j-1)^2}{2}\right)\right)v$$
$$-\frac{1}{3\cdot 10^2}z^2(z-1)\ .$$

We take the sum of $f_1, f_2, \dots, f_6$ with the above described substitutions for z, u, v, then we obtain a cubic polynomial $f(y)$. We take one of the real roots of the cubic equation:

$$f(y) = 0$$

as $z_6 = y_6$. For the Step $\{i, j\}$, the process taken in this section up to this place, $\overline{Q}(x,t)$ and $Q(x,t)$ will have the same values on the related 6 elements with the point $P[i,j]=\left(\frac{i}{10},\frac{j}{10}\right)$.

The values of $Q\left(\frac{i}{10},\frac{j}{10}\right)$, $i, j = 8,\ 7,\ \dots,\ 0$, are calculated in the following order:

Step $\{9, 9\}$ → Step $\{9, 8\}$ →···→ Step $\{9, 1\}$ →

Step $\{8, 9\}$ → Step $\{8, 8\}$ →···→ Step $\{8, 1\}$ →

....

Step $\{1, 9\}$ → Step $\{1, 8\}$ →···→ Step $\{1, 1\}$.

Calculated values $z_{i,j} = Q\left(\frac{i}{10}, \frac{j}{10}\right)$ are tabulated below:

$z_{i,j}$ j / i	9 / 4	8 / 3	7 / 2	6 / 1	5 / 0
9	0.996326	0.989818	0.980516	0.968488	0.953828
	0.936658	0.917126	0.895408	0.871706	0.846250
8	0.994761	0.995911	1.02276	1.11625	1.31953
	1.61412	1.89928	2.04518	-0.014844	-2.55916
7	0.992997	1.00158	1.03051	1.01185	-2.26671
	-4.54746	-5.40032	-6.36985	-3.49049	3.80730
6	0.990883	1.01334	1.04924	1.01481	1.23511
	9.95392	20.2110	23.7909	24.6980	-3.57460
5	0.988093	1.03341	1.06050	0.946635	1.15808
	-9.05000	-42.3420	-85.3574	-90.8515	-68.3357
4	0.983832	1.06160	1.03801	0.779018	1.13560
	6.88996	58.3011	197.794	373.008	267.978
3	0.975741	1.09129	0.918812	0.649263	1.39041
	-4.93982	-60.0707	-319.694	-953.121	-1568.00
2	0.954088	1.10070	0.641869	0.931546	1.88231
	3.90994	58.2717	412.286	1796.60	4822.75
1	0.838400	0.944782	0.710186	1.57548	0.761220
	-2.89701	-46.7383	-447.132	-2718.02	-11200.5

Using the list of the values $Q\left(\frac{i}{10}, \frac{j}{10}\right)$, we obtained the profiles of $Q\left(x, \frac{j}{10}\right)$, $j = 0, 1, 2, \ldots, 9$, as shown in Fig. 1 and the 3 - dimensional perspectives of the stable function $z = Q(x,t)$ as shown in Figs. 2 - 4, which will give us a hint how we find a theoretical solutions of the partial differential equation (2.2) which must have a singularity along $x = 0$.

Remark 3. If we start the above computation from Step $\{10,10\}$, we must use the formulas (4.14) and (4.15). Considering the errors, we used the approximate values of $Q\left(\frac{9}{10}, \frac{j}{10}\right)$ and $Q\left(\frac{i}{10}, \frac{9}{10}\right)$ given by (2.13), since the point $P[9,j]$ and $P[i,9]$ are located near the boundary of the square Ω and we can compute them directly from the boundary conditions by the method of calculus of finite differences.

Acknowledgments. The computations above by computers was programed and implemented by the author's son Shigeru Otsuki, Senior Researcher at Systems Development Laboratory of Hitachi, Ltd..

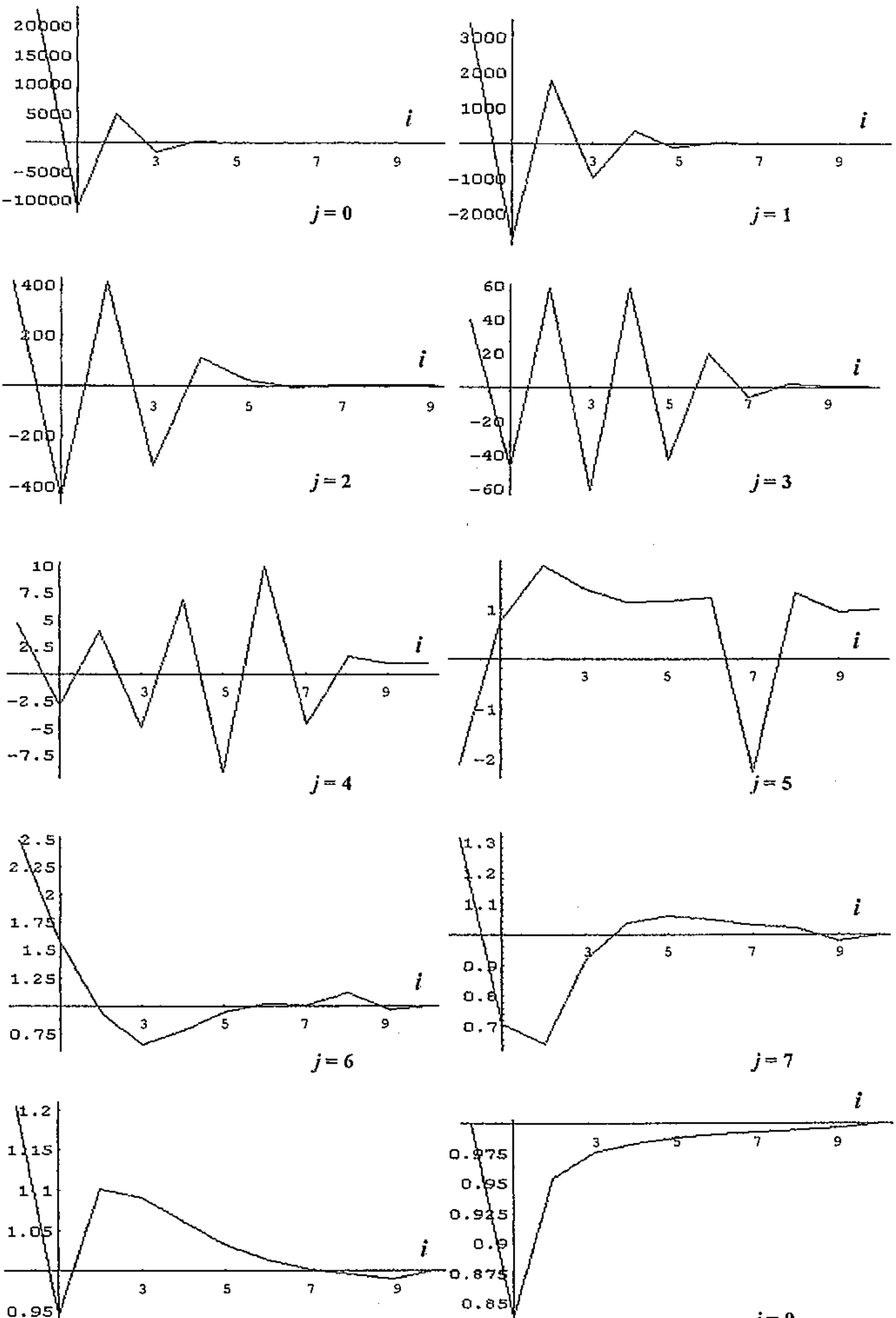

Fig.1 Profiles of $Q(x, j/10)$ $\quad x = \dfrac{i}{10}$

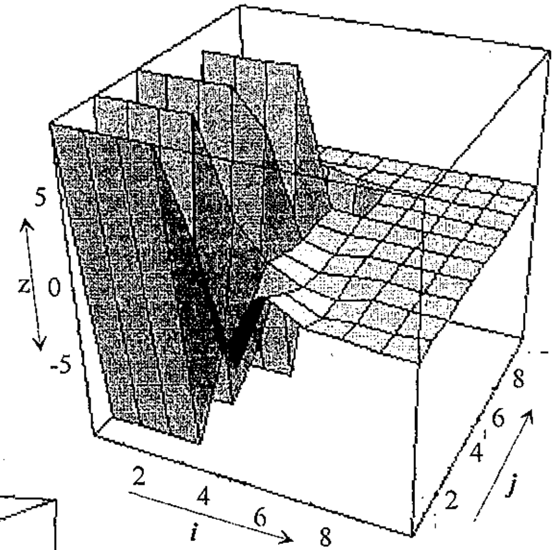

Fig.2 3-dimensional graph of $z = Q(x,t)$ (first view)

$$x = \frac{i}{10},\ t = \frac{j}{10}$$

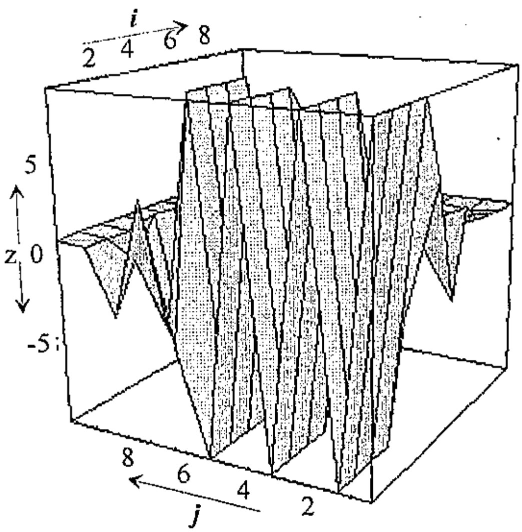

Fig.3 3-dimensional graph of $z = Q(x,t)$ (second view)

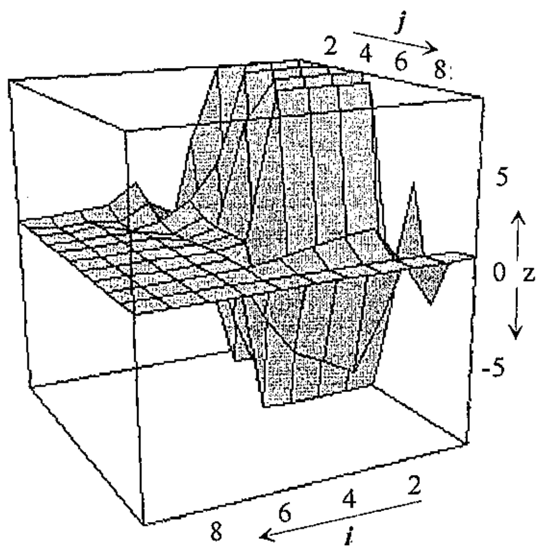

Fig.4 3-dimensional graph of $z = Q(x,t)$ (third view)

REFERENCES

[1] N. ABE, *General connections on vector bundles,* Kodai Math. J.**8** (1985), 322-329.
[2] H. NAGAYAMA, *A theory of general relativity by general connections* I, TRU Mathematics **20** (1984), 173-187.
[3] H. NAGAYAMA, *A theory of general relativity by general connections* II, TRU Mathematics **21** (1985), 287-317.
[4] H. NEMOTO, *On differential geometry of general connections,* TRU Mathematics **21**(1985), 67-94.
[5] T. OTSUKI, *On general connections* I, Math. J. Okayama Univ. **9**(1960), 99-164.
[6] T. OTSUKI, *On general connections* II, Math. J. Okayama Univ. **10**(1961), 113-124.
[7] T. OTSUKI, *On metric general connections,* Proc. Japan Acad.**37**(1961), 183-188.
[8] T. OTSUKI, *A Note on metric general connections,* Proc. Japan Acad. **38**(1962), 409-413.
[9] T. OTSUKI, *On normal general connections,* Kodai Math. Sem. Reports **13**(1961), 152-166.
[10] T. OTSUKI, *A construction of spaces which have points swallowing geodesics,* Math. J. Okayama Univ. **24**(1982), 157-165.
[11] T. OTSUKI, *A certain space-time metric and smooth general connections,* Kodai Math. J. **8**(1985), 307-316.
[12] A.BEJANCU and T. OTSUKI, *General Finsler connections on a Finsler vector bundle,* Kodai Math. J. **10**(1987), 143-152.
[13] T. OTSUKI, *Singular point sets of a general connection and black holes,* Math. J. Okayama Univ. **30**(1988), 199-211.
[14] T. OTSUKI, *General connections,* Math. J. Okayama Univ. **32**(1990), 227-242.
[15] T. OTSUKI, *A Cauchy type theorem on ordinary differential equations and its applications,* SUT Journal of Math. **26**(1990), 69-88.
[16] T. OTSUKI, *Behavior of geodesics around the singular set of a general connection,* SUT Journal of Math. **27**(1991), 169-228.
[17] T. OTSUKI, *A family of Minkowski-type spaces with general connections,* SUT Journal of Math. **28**(1992), 61-103.
[18] T. OTSUKI, *A nonlinear partial differential equation related with certain spaces with general connections,* SUT Journal of Math. **29**(1993), 167-192.

Tominosuke OTSUKI
Department of Mathematics
Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku,
Tokyo 162, JAPAN