SUT Journal of Mathematics
Vol. 31, No. 2(1995), 155–160

A CELLULAR SIMPLEX WITH PRESCRIBED NUMBERS OF POINTS IN REGIONS DETERMINED BY ITS FACETS

Shin-ichi TOKUNAGA

(Received November 9, 1995)

Abstract. Let P be a finite set of points in the 3-dimensional Euclidean space $\mathbb{R}^3$ in general position. For $x_0, x_1, x_2, x_3 \in P$, let $H^+(x_0; x_1, x_2, x_3)$ (resp. $H^-(x_0, x_1, x_2, x_3)$) denote the open half space containing x_0 (resp. not containing x_0) and bounded by the plane containing x_1, x_2, x_3. Further let

$$\begin{aligned} P(x_0; x_1, x_2, x_3) := P &\cap H^+(x_1; x_0, x_2, x_3) \\ &\cap H^+(x_2; x_0, x_1, x_3) \\ &\cap H^+(x_3; x_0, x_1, x_3). \end{aligned}$$

In this paper, we show the following statement: if $|P| \geq 4$, and if k_1, k_2, k_3, k_4 are integers with $k_1 + k_2 + k_3 + k_4 = |P| - 4$,$0 \leq k_1, k_2, k_3, k_4 \leq \frac{|P|-2}{2}$ and $k_1 + k_2 \leq \frac{|P|-2}{2}$, then for any $p_1, p_2 \in P\,(p_1 \neq p_2)$, there exist $q_1, q_2 \in P$ such that the convex hull of $\{p_1, p_2, q_1, q_2\}$ is a 3-simplex (tetrahedron) containing no point of P in its interior and such that

$$\begin{aligned} |P(p_1; p_2, q_1, q_2)| &\leq k_1 \leq P \cap H^-(p_1; p_2, q_1, q_2)|, \\ |P(p_2; p_1, q_1, q_2)| &\leq k_2 \leq P \cap H^-(p_2; p_1, q_1, q_2)|, \\ |P(q_1; q_2, p_1, p_2)| &\leq k_3 \leq P \cap H^-(q_1; q_2, p_1, p_2)|, \\ |P(q_2; q_1, p_1, p_2)| &\leq k_4 \leq P \cap H^-(q_2; q_1, p_1, p_2)|. \end{aligned}$$

AMS 1991 *Mathematics Subject Classification.* 52A37.

Key words and phrases. Finite set of points in the Euclidean space, simplex, affine flat.

§1. Introduction.

For a subset V of the d-dimensional Euclidean space $\mathbb{R}^d$, let $\mathrm{conv}(V)$ denote the convex hull of V, and let $\mathrm{aff}(V)$ denote the affine flat spanned by V. For $d+1$ points $x_0, x_2, \cdots, x_d$ not lying in the same (affine) $(d-1)$-flat in $\mathbb{R}^d$, let $H^+(x_0; x_1, \cdots, x_d)$ (resp. $H^-(x_0; x_1, \cdots, x_d)$) denote the open half-space which is bounded by $\mathrm{aff}(\{x_1, \cdots, x_d\})$ and contains x (resp. does not contain x). Now let P be a fixed set of points in $\mathbb{R}^d$. We say that P is in general position if no $d+1$ points of P lie in the same $(d-1)$-flat. For $d+1$ points $x_0, x_1, \cdots, x_d$ not lying in the same $(d-1)$-flat, let

$$P(x_0; x_1, \cdots, x_d) \;:=\; P \cap \bigcap_{1 \le i \le d} H^+(x_i; x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d).$$

If a subset V of $\mathbb{R}^3$ contains no point of P in its interior, V is said to be *vacuum*. Further, following Kupitz[2], we call a polyhedron D *cellular* if D is vacuum and all vertices of D are points of P. In this paper, we show the following theorem as a 3-dimensional version of Lemma 3 in [1]:

Theorem 1. *Let P be a finite set of points in $\mathbb{R}^3$ in general position. Suppose that $|P| \ge 4$, and let k_1, k_2, k_3, k_4 be integers such that $k_1+k_2+k_3+k_4 = |P|-4$, $0 \le k_1, k_2, k_3, k_4 \le \frac{|P|-2}{2}$ and $k_1 + k_2 \le \frac{|P|-2}{2}$. Further let p_1, p_2 be specified points of P with $p_1 \ne p_2$. Then there exist two points q_1, q_2 of P such that $\mathrm{conv}(\{p_1, p_2, q_1, q_2\})$ is a cellular 3-simplex and the following inequalities hold:*

$$|P(p_1; p_2, q_1, q_2)| \le k_1 \le |P \cap H^-(p_1; p_2, q_1, q_2)|, \tag{1.1}$$

$$|P(p_2; p_1, q_1, q_2)| \le k_2 \le |P \cap H^-(p_2; p_1, q_1, q_2)|, \tag{1.2}$$

$$|P(q_1; q_2, p_1, p_2)| \le k_3 \le |P \cap H^-(q_1; q_2, p_1, p_2)|, \tag{1.3}$$

$$|P(q_2; q_1, p_1, p_2)| \le k_4 \le |P \cap H^-(q_2; q_1, p_1, p_2)|. \tag{1.4}$$

§2. Proof of Theorem 1.

Let $P, k_1, k_2, k_3, k_4, p_1, p_2$ be as in Theorem 1. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection in the direction of $\overrightarrow{p_1p_2}$. We use the following result in the plane case (a slight modification of Claim 1 in [1]):

Proposition 1. *Let P' be a finite set of points in $\mathbb{R}^2$, and let r'_0 be a specified point of P'. Suppose that $P' \ge 3$ and any line passing through r'_0 contains at most one point of P' other than r'_0. Let k'_1, k'_2, k'_3 be integers satisfying $0 \le k'_1, k'_2, k'_3 \le \frac{|P'|-1}{2}$ and $k'_1+k'_2+k'_3 = |P'|-3$. Then there exist $x' \in \mathbb{R}^2 - P$ and $r'_1, r'_2 \in P' - \{r'_0\}$ such that*

$$P' \;=\; \{r'_0, r'_1, r'_2\} \cup P'(r'_0; x', r'_1) \cup P'(r'_0; r'_1, r'_2) \cup P'(r'_0; r'_2, x')$$

and

$$|P'(r'_0; x', r'_1)| = k'_1,\ \ |P'(r'_0; r'_1, r'_2)| = k'_2,\ \ |P'(r'_0; r'_2, x')| = k'_3.$$

Proposition 1 is essentially the same as Claim 1 in [1], so we omit the proof. Since $k_1 + k_2 \le \frac{|P|-2}{2}$, we can apply Proposition 1 to $\pi(P) = \{\pi(p) \mid p \in P\}$ with $r'_0 = \pi(p_1) = \pi(p_2)$ and $k'_1 = k_3$, $k'_2 = k_1 + k_2$, $k'_3 = k_4$. Let x', r'_1, r'_2 be as in the conclusion of the Proposition 1. We use the same technique as in the proof of Lemma 3 in [1]. Let l_0 be a line passing through r'_0 and x'. Take $s'_1, s'_2 \in P'(r'_0; r'_1, r'_2) \cup \{r'_1, r'_2\}$ so that for $i = 1, 2$, s'_i lies in the same side of l_0 as r'_i, and

$$\text{(2.5)} \quad \begin{array}{c} \text{the line segment } \overline{s'_1 s'_2} \text{ is an edge of } \mathrm{conv}(P'(r'_0; r'_1, r'_2) \cup \{r'_1, r'_2\}) \\ \text{satisfying } \mathrm{conv}(\{r'_0, s'_1, s'_2\}) \cap H^-(r'_0; s'_1, s'_2) \;=\; \emptyset. \end{array}$$

Now we return to $\mathbb{R}^3$. Let x, r_i, s_i $(i = 1, 2)$ be the points of P such that x, $\pi(r_i) = r'_i$, $\pi(s_i) = s'_i$, respectively. Let

$$\begin{aligned} K_1 &:= H^+(x; r_1, p_1, p_2) \cap H^+(r_1; x, p_1, p_2), \\ K_2 &:= H^+(r_1; r_2, p_1, p_2) \cap H^+(r_2; r_1, p_1, p_2), \\ K_3 &:= H^+(r_2; x, p_1, p_2) \cap H^+(x; r_2, p_1, p_2). \end{aligned}$$

Then the conclusion of Proposition 1 implies that $K_i \cap K_j = \emptyset$ if $i \ne j$, and

$$\text{(2.6)} \quad |P \cap K_1| = k'_1 \;=\; k_3,$$

$$\text{(2.7)} \quad |P \cap K_2| = k'_2 \;=\; k_1 + k_2,$$

$$\text{(2.8)} \quad |P \cap K_3| = k'_3 \;=\; k_4.$$

Let $H_0 := \pi^{-1}(l_0)$ and let $S = (P \cap K_2) \cup \{r_1, r_2\}$. By(2.5),

$$\Delta := H^+(r'_1; r'_0, r'_2) \cap H^+(r'_2; r'_0, r'_1) \cap H^+(r'_0; r'_1, r'_2)$$

is vacuum. Since $K_2 \cap H^+(p_1; p_2, s_1, s_2) \cap H^+(p_2; p_1, s_1, s_2) \subset \pi^{-1}(\Delta)$, this implies that $S \cap H^+(p_1; p_2, s_1, s_2) \cap H^+(p_2; p_1, s_1, s_2) = \emptyset$. Thus by (2.7), $|S \cap H^+(p_2; p_1, s_1, s_2)| \le k_1 + 2$ or $|S \cap H^+(p_1; p_2, s_1, s_2)| \le k_2 + 2$ holds. By symmetry, we may assume

$$\text{(2.9)} \quad |S \cap H^+(p_2; p_1, s_1, s_2)| \le k_1 + 2.$$

For a plane H and a point $x \notin H$, let $H^+(x)$ (resp. $\bar{H}^+(x)$) denote the open (resp. closed) half-space which is bounded by H and contains x, and let $H^-(x)$ (resp. $\bar{H}^-(x)$) denote the open (resp. closed) half-space which is bounded by H and does not contain x. Let H_1 be a plane containing p_1 such that

$$\text{(2.10)} \quad |S \cap \bar{H}_1{}^+(p_2)| = k_1 + 2,$$

$$\text{(2.11)} \quad S \cap \bar{H}_1{}^+(p_2) \cap H_0^+(r_i) \ne \emptyset \text{ for } i = 1, 2.$$

Note that by (2.9), there exists a plane satisfying (2.10) and (2.11). We choose H_1 so that the angle between $H_0 \cap H_1 \cap K_2$ and $\overrightarrow{p_1p_2}$ is as small as possible. Take q_1, q_2 so that

$$q_1 \in S \cap \bar{H}_1^{\,+}(p_2) \cap H_0^+(r_1), \tag{2.12}$$

$$q_2 \in S \cap \bar{H}_1^{\,+}(p_2) \cap H_0^+(r_2), \tag{2.13}$$

and

$$\begin{array}{c} \triangle p_2q_1q_2 \text{ is a facet of } \mathrm{conv}\Big((S \cup \{p_2\}) \cap \bar{H}_1^{\,+}(p_2)\Big) \text{ satisfying} \\ \mathrm{conv}(\{p_1, p_2, q_1, q_2\}) \cap H^-(p_1; p_2, q_1, q_2) = \emptyset. \end{array} \tag{2.14}$$

By (2.14), $\mathrm{conv}(\{p_1, p_2, q_1, q_2\})$ is vacuum. We now proceed to verify the inequalities in the conclusion of Theorem 1. By (2.12) and (2.13),

$$\begin{array}{lll} P(q_1; q_2, p_1, p_2) \subseteq & P \cap K_1 \subseteq & P \cap H^-(q_1; q_2, p_1, p_2) \text{ and} \\ P(q_2; q_1, p_1, p_2) \subseteq & P \cap K_3 \subseteq & P \cap H^-(q_2; q_1, p_1, p_2) \end{array}$$

hold, and hence (2.6), (2.8) imply (1.3),(1.4), respectively. Similarly by (2.14),

$$P(p_1; p_2, q_1, q_2) \ \subseteq\ S \cap \bar{H}_1^{\,+}(p_2) - \{q_1, q_2\} \ \subseteq\ P \cap H^-(p_1; p_2, q_1, q_2)$$

holds, and hence (2.10) implies (1.1). Further, it also follows from the choice of q_1, q_2 that

$$P(p_2; p_1, q_1, q_2) \ \subseteq\ S \cap H_1^-(p_2).$$

Since

$$|S \cap H_1^-(p_2)| \ =\ (k_1 + k_2 + 2) - (k_1 + 2) \ =\ k_2$$

by (2.7) and (2.10), this immediately implies the first inequality in (1.2).

We are now left with the verification of the second inequality in (1.2). Suppose

$$|P \cap H^-(p_2; p_1, q_1, q_2)| < k_2.$$

Then clearly

$$|S \cap H^-(p_2; p_1, q_1, q_2)| < k_2. \tag{2.15}$$

On the other hand, by (2.7) and (2.9),

$$|S \cap H^-(p_2; p_1, s_1, s_2)| \ \geq\ (k_1 + k_2 + 2) - (k_1 + 2) \ =\ k_2 \tag{2.16}$$

holds. Let y, z be the intersection points of the line passing through s_1, s_2 and $\mathrm{aff}(\{p_1, p_2, r_1\})$, $\mathrm{aff}(\{p_1, p_2, r_2\})$, respectively. Then (2.15) and (2.16) imply that $S \cap H^-(p_2; p_1, s_1, s_2) \not\subseteq S \cap H^-(p_2; p_1, q_1, q_2)$, which implies that at least one of y, z belongs to $H^+(p_2; p_1, q_1, q_2)$. We may assume

$$y \in H^+(p_2; p_1, q_1, q_2) \tag{2.17}$$

without loss of generality. We now show the existence of a plane containing p_0 which gives rise to a contradiction to the choice of H_1. Toward this end, we divide the situation into two cases according to the location of q_2.

Case 1 $q_2 = s_2$ or $q_2 \in H^+(p_2; p_1, s_1, s_2)$
In this case,

$$S \cap H^-(p_2; p_1, y, q_2) \supseteq S \cap H^-(p_2; p_1, s_1, s_2)$$

holds and hence by (2.16),

$$|S \cap H^-(p_2; p_1, y, q_2)| \geq |S \cap H^-(p_2; p_1, s_1, s_2)| \geq k_2. \tag{2.18}$$

Let l_1 be the line passing through p_1, q_2, and let H be a (movable) plane containing l_1. If we gradually rotate H with l_1 as the axis, the value of $|S \cap H^-(p_2)|$ changes by one at each moment when H hits a point of P. Therefore by (2.15) and (2.18), there exists $H_2 \in l_1 \cup H^+(p_2, p_1, q_1, q_2) \cup H^+(p_2, p_1, y, q_2)$ such that $l_1 \in H_2$ and $|S \cap H_2^-(p_2)| = k_2$, or equivalently, $|S \cap \bar{H_2}^+(p_2)| = k_1 + 2$. Now to get a contradiction, we let $K_2' := H^+(q_1; q_2, p_1, p_2) \cap H^+(q_2; q_1, p_1, p_2)$ (note that by (2.12) and (2.13), H_0 intersects with K_2'). Then by (2.17), it is easy to see that

$$\begin{aligned} H_0 \cap K_2' \cap H_2^+(p_2) &\subset H_0 \cap K_2' \cap H^+(p_2; p_1, q_1, q_2) \\ &\subseteq H_0 \cap K_2' \cap H_1^+(p_2), \end{aligned}$$

which yields a contradiction to the minimality of the angle between $H_0 \cap H_1 \cap K_2$ and $\overrightarrow{p_1 p_2}$.

Case 2 $q_2 \in H^-(p_2; p_1, s_1, s_2)$
If (2.18) holds, a contradiction can be derived in the same way as in Case 2. Thus we may assume

$$|S \cap H^-(p_2; p_1, y, q_2)| < k_2. \tag{2.19}$$

Let l_2 be the line passing through p_1, y. Then again as in Case 1, (2.16) and (2.19) imply that we can find a plane $H_3 \in l_2 \cup H^+(p_2, p_1, y, q_2) \cup H^+(p_2, p_1, s_1, s_2)$ such that $l_2 \in H_3$ and $|S \cap \bar{H_3}^+(p_2)| = k_1 + 2$ by considering the rotation of a plane containing l_2 with l_2 as the axis. Thus again it is easy to see that

$$\begin{aligned} H_0 \cap K_2' \cap H_3^+(p_2) &\subset H_0 \cap K_2' \cap H^+(p_2; p_1, q_1, q_2) \\ &\subseteq H_0 \cap K_2' \cap H_1^+(p_2), \end{aligned}$$

which yields a contradiction. □

Acknowledgment

The author is grateful to Professor Yoshimi Egawa for his valuable remarks and helpful advice.

References

[1] S. Tokunaga, On a straight-line embedding problem of graphs, to appear in *Discrete Mathematics*.

[2] Y. S. Kupitz, Separation of a finite set in $\mathbb{R}^d$ by spanned hyperplanes, *Combinatorica* 13(3) pp.249-258.

Shin-ichi Tokunaga
Department of Applied Mathematics, Science University of Tokyo
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162, Japan