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WEIGHTED INEQUALITIES FOR DISCRETE AND INTEGRAL HARDY OPERATORS

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Abstract. We prove that results for discrete weighted Hardy inequality can be obtained from a two-weight inequality for the classical Hardy operator.

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1. Introduction.

Let $1 < p < \infty$ and $0 < q < \infty$. It is well-known in [Op-Ku] that the weighted Hardy inequality

$$(1) \qquad \left(\int_a^b \left[\int_a^x f(y)dy\right]^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \quad \text{for all } f \geq 0$$

holds for $p \leq q$ if and only if

$$(1.1) \qquad \sup_{R;\ a<R<b} \left(\int_R^b u(x)dx\right)^{\frac{1}{q}} \left(\int_a^R v^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty$$

and for $q < p$ if and only if

$$(1.2) \qquad \int_a^b \left[\left(\int_x^b u(z)dz\right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(y)dy\right)^{\frac{1}{q'}}\right]^r v^{1-p'}(x)dx < \infty.$$

Here $-\infty \leq a < b \leq \infty$ and throughout the paper $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$ and $r = \frac{pq}{p-q}$. The positive constant C does not depend on a, b and f.

The discrete weighted Hardy inequality

$$(2) \qquad \left(\sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \mathcal{F}(k)\right]^q \mathcal{U}(n)\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty} \mathcal{F}^p(n)\mathcal{V}(n)\right)^{\frac{1}{p}}$$

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plays as well as (1) an important role in analysis. Here $\mathcal{F}(n)$, $\mathcal{U}(n)$ and $\mathcal{V}(n)$ are sequences of non-negative numbers.

A characterization of $\mathcal{U}(n)$ and $\mathcal{V}(n)$, for which (2) holds, can be found in the work of K. Andersen and H. Heinig [An-He], [He], G. Bennet [Be] and M. Braverman and V. Stepanov [Br-St]. In [An-He], [He] and [Be], the inequality (2) for $p \le q$ and $1 \le q < p$ was treated by adapting the proofs for the integral inequality (1). The case $q < 1 < p$ is more difficult to handle. Indeed in [Be] a Maurey's factorization result was needed, and in [Br-St] the key was to discretize a Halperin's result on level functions (which is used in [Si] to solve the integral inequality (1) for the same range of p and q). However the level function of Halperin is a complicated object to construct. For this reason, G. Sinnamon and V. Stepanov [Si-St] have recently given a new proof of (1) for $q < 1 < p$.

All of these situations raise the question of obtaining direct proofs for the discrete inequality (2) from the integral inequality (1) without doing any adaptation of the methods used in the proofs of (1).

The purpose of this note is to show that inequality (2) can be immediately deduced from the integral inequality (1).

It is not difficult to see, as for the case of (1), that a necessary condition for (2) is

$$(2.1)\qquad \left(\sum_{n=N}^{\infty}\mathcal{U}(n)\right)^{\frac{1}{q}}\left(\sum_{n=1}^{N}\mathcal{V}^{1-p'}(n)\right)^{\frac{1}{p'}} \le A \quad \text{for all integers } N \ge 1.$$

The positive constant A does only depend on p, q, $\mathcal{U}$ and $\mathcal{V}$. For the case $q < p$ (see for instance [Br-St] for $q < 1$) one can show that a necessary condition for (2) is

$$(2.2)\qquad \sum_{N=1}^{\infty}\left[\Big(\sum_{n=N}^{\infty}\mathcal{U}(n)\Big)^{\frac{1}{q}}\Big(\sum_{m=1}^{N}\mathcal{V}^{1-p'}(m)\Big)^{\frac{1}{q'}}\right]^{r}\mathcal{V}^{1-p'}(N) \le A^{r}.$$

Now, we are going to prove the converse.

2. Theorem.

For $p \le q$, condition (2.1) *implies the discrete Hardy inequality* (2). *And for $q < p$, condition* (2.2) *is sufficient to ensure* (2).

In each case the inequality (2) holds with $C = cA$ where the positive constant c depends only on p and q. The constant C we get for (2) is in general greater than that obtained from a direct method as in [Be] and [Br-St].

The idea used in this paper can be exploited to treat weighted inequalities for more general (discrete) operators as $(T\mathcal{F})(n) = \sum_{m=1}^{n}\mathcal{K}(n,m)\mathcal{F}(m)$, and will be developed by the second author in a forthcoming paper.

3. Proof of the Theorem.

First we deal with the case $p \le q$. To benefit from the integral inequality (1) it is convenient to introduce the weight functions

$$u(x) = \sum_{n=1}^{\infty} \mathcal{U}(n)\chi_{[n,n+1)}(x) \qquad \text{and} \qquad v(x) = \sum_{n=1}^{\infty} \mathcal{V}(n)\chi_{[n,n+1)}(x). \tag{3.0}$$

Here χ_E denotes the characteristic function of the measurable set E.

By the definition of u and v and condition (2.1)

$$\Big(\int_n^{n+1} u(z)dz\Big)\Big(\int_n^{n+1} v^{1-p'}(y)dy\Big)^{\frac{q}{p'}} = \mathcal{U}(n)\Big(\mathcal{V}^{1-p'}(n)\Big)^{\frac{q}{p'}} \le A^q \tag{3.1}$$

for each integer $n \ge 1$. Similarly, for each $R > 1$ we obtain

$$\Big(\int_R^{\infty} u(x)dx\Big)^{\frac{1}{q}}\Big(\int_1^{R} v^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \le \Big(\int_N^{\infty} u(x)dx\Big)^{\frac{1}{q}}\Big(\int_1^{N+1} v^{1-p'}(y)dy\Big)^{\frac{1}{p'}}$$
$$=\Big(\sum_{n=N}^{\infty} \mathcal{U}(n)\Big)^{\frac{1}{q}}\Big(\sum_{n=1}^{N} \mathcal{V}^{1-p'}(n)\Big)^{\frac{1}{p'}} \le A.$$

Here N is the positive integer such that $N \le R < N+1$. Invoking (1.1) (with $a = 1,\ b = \infty$ and $p \le q$) it appears that

$$\Big(\int_1^{\infty}\Big[\int_1^{x} f(y)dy\Big]^q u(x)dx\Big)^{\frac{1}{q}} \le CA\Big(\int_1^{\infty} f^p(x)v(x)dx\Big)^{\frac{1}{p}} \tag{3.2}$$

for all $f \ge 0$. The constant C does only depend on p and q as $C = (1+\frac{q}{p'})^{\frac{1}{q}}(1+\frac{p'}{q})^{\frac{1}{p'}}$ (see Theorem 1.14 in [Op-Ku]).

Let $\mathcal{F}(n)$ be a non-negative sequence. Define

$$f(x) := \sum_{k=1}^{\infty} \mathcal{F}(k)\chi_{[k,k+1)}(x).$$

Then

$$\int_1^{n+1} f(y)dy = \sum_{k=1}^{n} \mathcal{F}(k) \quad \text{and} \quad \int_1^{\infty} f^p(x)v(x)dx = \sum_{n=1}^{\infty} \mathcal{F}^p(n)\mathcal{V}(n). \tag{3.3}$$

Now we are in position to get the discrete inequality (2). Indeed

$$\sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\mathcal{F}(k)\Big]^q\mathcal{U}(n) = \sum_{n=1}^{\infty}\Big[\int_1^{n+1} f(y)dy\Big]^q\Big(\int_n^{n+1} u(x)dx\Big) \quad \text{by (3.3)}$$

$$
\begin{aligned}
&\leq 2^{q-1}\sum_{n=1}^{\infty}\Big[\int_1^n f(y)dy\Big]^q\Big(\int_n^{n+1}u(x)dx\Big)\\
&\quad + 2^{q-1}\sum_{n=1}^{\infty}\Big[\int_n^{n+1} f(y)dy\Big]^q\Big(\int_n^{n+1}u(x)dx\Big)\\
&\leq 2^{q-1}\sum_{n=1}^{\infty}\int_n^{n+1}\Big[\int_1^x f(y)dy\Big]^q u(x)dx\\
&\quad + 2^{q-1}\sum_{n=1}^{\infty}\Big[\int_n^{n+1} f^p(x)v(x)dx\Big]^{\frac{q}{p}}\Big(\int_n^{n+1}v^{1-p'}(y)dy\Big)^{\frac{q}{p'}}\Big(\int_n^{n+1}u(z)dz\Big)\\
&\leq 2^{q-1}\int_1^{\infty}\Big[\int_1^x f(y)dy\Big]^q u(x)dx\\
&\quad + 2^{q-1}A^q\sum_{n=1}^{\infty}\Big[\int_n^{n+1} f^p(x)v(x)dx\Big]^{\frac{q}{p}} \quad \text{by (3.1)}\\
&\leq 2^{q-1}(CA)^q\Big[\int_1^{\infty} f^p(x)v(x)dx\Big]^{\frac{q}{p}} + 2^{q-1}A^q\Big[\sum_{n=1}^{\infty}\int_n^{n+1} f^p(x)v(x)dx\Big]^{\frac{q}{p}}
\end{aligned}
$$

by (3.2) and since $\frac{q}{p} \geq 1$

$$
\begin{aligned}
&\leq 2^{q-1}A^q(C^q+1)\Big[\int_1^{\infty} f^p(x)v(x)dx\Big]^{\frac{q}{p}}\\
&\quad = (2A)^q(1+c^q)\Big[\sum_{n=1}^{\infty}\mathcal{F}^p(n)\mathcal{V}(n)\Big]^{\frac{q}{p}}
\end{aligned}
$$

by (3.3).

Next we consider the case $q < p$. For u and v as in (3.0), the preceding estimate leads to

$$
\sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\mathcal{F}(k)\Big]^q\mathcal{U}(n) \leq c_1(S_1+S_2), \tag{3.4}
$$

where

$$
S_1 = \int_1^{\infty}\Big[\int_1^x f(y)dy\Big]^q u(x)dx \quad \text{and} \quad S_2 = \sum_{n=1}^{\infty}\Big[\int_n^{n+1} f^p(x)v(x)dx\Big]^{\frac{q}{p}}A(n)
$$

with $A(n) = \Big(\int_n^{n+1}v^{1-p'}(y)dy\Big)^{\frac{q}{p'}}\Big(\int_n^{n+1}u(z)dz\Big)$. Here $c_1 = 2^{q-1}$ for $q \geq 1$ and $c_1 = 1$ for $0 < q < 1$.

To estimate S_2, observe that $\frac{p}{q} > 1$ and $(\frac{p}{q})' = \frac{p}{p-q}$. By (2.2) we obtain

(3.5)
$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big[A(n)\Big]^{(\frac{p}{q})'}\\
&=\sum_{n=1}^{\infty}\Big[\Big(\int_n^{n+1}u(z)dz\Big)^{\frac{1}{q}}\Big(\int_n^{n+1}v^{1-p'}(y)dy\Big)^{\frac{1}{q'}}\Big]^r\Big(\int_n^{n+1}v^{1-p'}(x)dx\Big)\\
&=\sum_{n=1}^{\infty}\Big[\Big(\mathcal{U}(n)\Big)^{\frac{1}{q}}\Big(\mathcal{V}^{1-p'}(n)\Big)^{\frac{1}{q'}}\Big]^r\mathcal{V}^{1-p'}(n)\le A^r.
\end{aligned}$$

The Hölder inequality together with (3.5) yield

$$\begin{aligned}
S_2 &\le \Big[\sum_{m=1}^{\infty}\int_m^{m+1}f^p(x)v(x)dx\Big]^{\frac{q}{p}}\Big[\sum_{n=1}^{\infty}\Big[A(n)\Big]^{(\frac{p}{q})'}\Big]^{1-\frac{q}{p}}\\
&\le A^q\Big[\int_1^{\infty}f^p(x)v(x)dx\Big]^{\frac{q}{p}}.
\end{aligned}\tag{3.6}$$

In order to get an upper bound for S_1, we use again condition (2.2) as

$$\begin{aligned}
&\int_1^{\infty}\Big[\Big(\int_x^{\infty}u(z)dz\Big)^{\frac{1}{q}}\Big(\int_1^{x}v^{1-p'}(y)dy\Big)^{\frac{1}{q'}}\Big]^r v^{1-p'}(x)dx\\
&\le\sum_{n=1}^{\infty}\Big[\Big(\int_n^{\infty}u(z)dz\Big)^{\frac{1}{q}}\Big(\int_1^{n+1}v^{1-p'}(y)dy\Big)^{\frac{1}{q'}}\Big]^r\Big(\int_n^{n+1}v^{1-p'}(x)dx\Big)\\
&\le\sum_{n=1}^{\infty}\Big[\Big(\sum_{k=n}^{\infty}\mathcal{U}(k)\Big)^{\frac{1}{q}}\Big(\sum_{k=1}^{n}\mathcal{V}^{1-p'}(k)\Big)^{\frac{1}{q'}}\Big]^r\mathcal{V}^{1-p'}(n)\le A^r.
\end{aligned}$$

So, by (1.2) (with $a=1,\ b=\infty$ and $q<p$) we have

$$S_1 \le (C_1A)^q\Big[\int_1^{\infty}f^p(x)v(x)dx\Big]^{\frac{q}{p}},\tag{3.7}$$

for some positive constant C_1 which only depends on p and q like $C_1 = q^{\frac{1}{q}}p'^{\frac{1}{q'}}$ for $1<q<p$ (see Theorem 1.15 in [Op-Ku]). Consequently (3.4), (3.6), (3.7) and (3.3) lead to the discrete inequality

$$\sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\mathcal{F}(k)\Big]^q\mathcal{U}(n)\le c_1A^q(1+C_1^q)\Big(\sum_{n=1}^{\infty}\mathcal{F}^p(n)\mathcal{V}(n)\Big)^{\frac{q}{p}}.$$

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