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Differential subordination and superordination for multivalent functions

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Abstract. In the present paper, the authors derive differential sandwich theorems involving convolution product for certain subclasses of multivalent normalized analytic functions in the open unit disk. The results in this paper generalize many earlier results in the literature.

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§1. Introduction and Motivations

Let $\mathcal{H} = \mathcal{H}(\Delta)$ be the space of all analytic functions in the *open* unit disk $\Delta := \{z : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that f is *subordinate* to g if there exists a Schwarz function ω, analytic in Δ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order subordination

$$\phi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.1}$$

then p is a solution of the differential subordination (1.1). Similarly, if p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.2}$$

then p is a solution of the differential superordination (1.2). (If f is subordinate to F, then F is called to be superordinate to f.) Also, an analytic function q_1 is a dominant if $p \prec q_1$ for all p satisfying (1.1)and an analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.2) and . An univalent dominant $\widetilde{q_1}$ that satisfies $\widetilde{q_1} \prec q$ for all dominant q of (1.1) is said to be the best dominant and an univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant. Recently Miller and Mocanu [6] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [6], Bulboacă [2] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [1]. Shanmugam et al. [14] obtained sufficient conditions for a normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \quad \text{and} \quad q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. On the other hand, Obradović and Owa [7] obtained subordination results for the quantity $\left(\frac{f(z)}{z}\right)^\mu$. A detailed investigation of starlike functions of complex order and convex functions of complex order using Briot–Bouquet differential subordination technique has been studied very recently by Srivastava and Lashin [20].

Let $\mathcal{A}_p$ denote the class of all *analytic* and p-valent functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \Delta), \tag{1.3}$$

and $\mathcal{A} := \mathcal{A}_1$, where $p \in \mathbb{N} := \{1, 2, 3, \cdots\}$. For any two analytic functions f given by (1.3) and g given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

their Hadamard product (or convolution) is the function $f * g$ defined by

$$(f * g)(z) := z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \tag{1.4}$$

we choose g as a fixed function in $\mathcal{A}_p$ such that $(f * g)(z)$ exist for any $f(z) \in \mathcal{A}_p$. For various choices of b_n we get different linear operators which has been studied in recent past.

For example, if the coefficient of b_n in (1.4) are chosen as

$$\left(\frac{n+\lambda}{p+\lambda}\right)^k \ (\ \lambda \geq 0;\ k \in \mathbb{Z}),$$

then the convolution (1.4) yields the operator $J_p(\lambda, k)f := \mathcal{A}_p \longrightarrow \mathcal{A}_p$ called the multiplier transformation(see also [3]), and when $\lambda = 0$ it is interesting to note that it lead to the the p-valent Sălăgean operator $D_p^k f(z)$ introduced by Shenan *et al.* [18]. Further, if

$g(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \dots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \dots (\beta_m)_{n-p}} \frac{z^n}{(n-p)!}$, then the convolution (1.4) gives the Dziok and Srivastava operator [4]

$$\Lambda(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m; z)f(z) \equiv H_{l,m}^p f(z) := (f * g)(z);$$

where $\alpha_1, \alpha_2, \cdots \alpha_l, \beta_1, \beta_2, \cdots \dots, \beta_m$ are complex parameters, $\beta_j \notin \{0, -1, -2, \cdots\}$ for $j = 1, 2, \cdots, m$, $l \leq m+1, l, m \in \mathbb{N} \cup \{0\}$. Here $(a)_\nu$ denotes the well-known Pochhammer symbol (or shifted factorial). Special cases of Dziok and Srivastava operator [4] includes the Hohlov linear operator, Carlson-Shaffer operator $L_p(a, c)$, p-valent Ruscheweyh operator $D^{\lambda+p-1}$ [9] as well as its generalized version, the Bernardi-Libera-Livingston operator and Srivastava-Owa fractional derivative operator.

In an earlier investigation, a sequence of results using differential subordination with convolution for the univalent case has been studied by Shanmugam [13]. A systematic study of the subordination and superordination using certain operators under the univalent case has also been studied by Shanmugam *et al.* [15, 16].

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results for the $p-$valent functions involving the Hadamard product. Furthermore, as special cases, we also obtain corresponding results of Obradović and Tuneski [8], Ponnusamy and Rajasekaran [10], Ravichandran [11], Ravichandran and Darus [12], Shanmugam *et al.* [14, 17], Singh [19] and Tuneski [21].

§2. Main Results

In order to investigate our subordination and superordination results, we recall the following known results.

Definition 2.1. [6, Definition 2, p. 817] Denote by Q, the set of all functions f that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem A [5, Theorem 3.4h, p. 132] Let q be an univalent function in Δ and let θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that

1. Q is starlike univalent in Δ, and
2. $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) = \Re\left(\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right) > 0$ for all $z \in \Delta$.

If ψ is analytic in Δ, with $\psi(0) = q(0)$, $\psi(\Delta) \subset D$ and $\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $\psi(z) \prec q(z)$ and q is the best dominant.

Theorem B [2] Let the function q be univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that

1. $\Re\left[\dfrac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ for all $z \in \Delta$,
2. Q(z)= $zq'(z)\varphi(q(z))$ is starlike univalent in Δ.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$ and q is the best subordinant.

We now prove the following result involving differential subordination between analytic functions.

Theorem 2.2. *Let the function q be analytic and univalent in Δ such that $q(z) \neq 0$. Let $z \in \Delta$, α, δ, ξ, γ_1, δ_1, δ_2, $\delta_3 \in \mathbb{C}$ and suppose at least one of δ_1, δ_2, $\delta_3 \in \mathbb{C}$ is non-zero. Suppose q satisfies*

(2.1)
$$\Re\left(1+\left(\frac{\xi q^2(z)+2\delta q^3(z)-\gamma_1}{\delta_1 q^2(z)+\delta_2 q(z)+\delta_3}\right)-\frac{zq'(z)}{q(z)}\left(\frac{\delta_2 q(z)+2\delta_3}{\delta_1 q^2(z)+\delta_2 q(z)+\delta_3}\right)+\frac{zq''(z)}{q'(z)}\right)>0$$

and

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\left(\frac{\delta_2 q(z)+2\delta_3}{\delta_1 q^2(z)+\delta_2 q(z)+\delta_3}\right)\right)>0. \tag{2.2}$$

Let

(2.3)
$$\Psi(f,g,\mu,\xi,\beta,\delta,\gamma_1,\delta_1,\delta_3):=\begin{cases} \alpha+\xi\left(\frac{z(f*g)'(z)}{p(f*g)(z)}\right)^{\mu} \\ +\delta\left(\frac{z(f*g)'(z)}{p(f*g)(z)}\right)^{2\mu}+\gamma_1\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \\ +\mu\left[1+\frac{z(f*g)''(z)}{(f*g)'(z)}-\frac{z(f*g)'(z)}{(f*g)(z)}\right]\left\{\delta_2+\delta_1\left\{\frac{z(f*g)'(z)}{p(f*g)(z)}\right\}^{\mu}\right\} \\ +\delta_3\mu\left[1+\frac{z(f*g)''(z)}{(f*g)'(z)}-\frac{z(f*g)'(z)}{(f*g)(z)}\right]\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \end{cases}$$

for some $\mu \in \mathbb{C}\setminus\{0\}$. If $f \in \mathcal{A}_p$ satisfies the following subordination

(2.4)
$$\Psi(f,g,\mu,\xi,\delta,\gamma_1,\delta_1,\delta_3) \prec \alpha+\xi q(z)+\delta(q(z))^2+\frac{\gamma_1}{q(z)}+\delta_1 zq'(z)+\delta_2\frac{zq'(z)}{q(z)}+\delta_3\frac{zq'(z)}{(q(z))^2},$$

then

$$\left(\frac{1}{p}\frac{z(f*g)'(z)}{(f*g)(z)}\right)^{\mu} \prec q(z) \tag{2.5}$$

and q is the best dominant.

Proof. Define the function ψ by

$$\psi(z):=\left(\frac{1}{p}\frac{z(f*g)'(z)}{(f*g)(z)}\right)^{\mu} \tag{2.6}$$

so that, by a straightforward computation, we have

$$\frac{z\psi'(z)}{\psi(z)}=\mu\left[1+\frac{z(f*g)''(z)}{(f*g)'(z)}-\frac{z(f*g)'(z)}{(f*g)(z)}\right]$$

which, in light of hypothesis (2.4) yields

$$\alpha+\xi\psi(z)+\delta(\psi(z))^2+\frac{\gamma_1}{\psi(z)}+\delta_1 z\psi'(z)+\delta_2\frac{z\psi'(z)}{\psi(z)}+\delta_3\frac{z\psi'(z)}{(\psi(z))^2}$$
$$\prec \alpha+\xi q(z)+\delta(q(z))^2+\frac{\gamma_1}{q(z)}+\delta_1 zq'(z)+\delta_2\frac{zq'(z)}{q(z)}+\delta_3\frac{zq'(z)}{(q(z))^2}.$$

By setting

$$\theta(\omega) := \alpha + \xi\omega + \delta\omega^2 + \frac{\gamma_1}{\omega} \qquad \text{and} \qquad \phi(\omega) := \delta_1 + \frac{\delta_2}{\omega} + \frac{\delta_3}{\omega^2},$$

we obtain

$$\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)).$$

It can be easily observed that θ and ϕ are analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C} \setminus \{0\}).$$

Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \delta_1 zq'(z) + \delta_2 \frac{zq'(z)}{q(z)} + \delta_3 \frac{zq'(z)}{(q(z))^2}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 zq'(z) + \delta_2 \frac{zq'(z)}{q(z)} + \delta_3 \frac{zq'(z)}{(q(z))^2},$$

we find from (2.2) that Q is starlike univalent in Δ and that

$$\begin{aligned} \Re\left(\frac{zh'(z)}{Q(z)}\right) &= \Re\left\{1 + \left(\frac{\xi q^2(z) + 2\delta q^3(z) - \gamma_1}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3}\right)\right. \\ &\left. -\frac{zq'(z)}{q(z)}\left\{\frac{\delta_2 q(z) + 2\delta_3}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3}\right\} + \frac{zq''(z)}{q'(z)}\right\} > 0, \end{aligned}$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \gamma_1,\ \delta_1, \delta_2,\ \delta_3 \in \mathbb{C})$$

by the hypothesis (2.1) and (2.2). The assertion (2.5) now follows by an application of Theorem A. $\square$

For the choices $p = 1$, $\mu = 1$, $g(z) = \dfrac{z}{1-z}$, $\alpha = \gamma_1 = \delta_2 = \delta_3 = 0$, $\xi = 1 - \delta$, $\delta_1 = \delta$, and assuming $0 < \delta \leq 1$, in Theorem 2.2, we have

Corollary 2.3. *[11, Theorem 3, p. 44] If q is convex univalent and $0 < \delta \leq 1$,*

$$\Re\left(\frac{1-\delta}{\delta} + 2q(z) + (1 + \frac{zq''(z)}{q'(z)})\right) > 0$$

and

$$\frac{zf'(z)}{f(z)} + \delta\frac{z^2 f''(z)}{f(z)} \prec (1-\delta)q(z) + \delta q^2(z) + \delta zq'(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant

For the choices $p = 1$, $g(z) = \dfrac{z}{1-z}$, $\alpha = \delta = \delta_2 = \delta_3 = \gamma_1 = 0$, in Theorem 2.2, we get the following corollary.

Corollary 2.4. *Let $\xi, \delta_1 \in \mathbb{C}$ and $\mu \neq 0 \in \mathbb{C}$. Let q be univalent in Δ and satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{-\Re\left(\frac{\xi}{\delta_1}\right), 0\right\}.$$

If $f \in \mathcal{A}$, and

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(\mu\delta_1\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] + \xi\right) \prec \delta_1 zq'(z) + \xi q(z),$$

then

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu} \prec q(z)$$

and the dominant q is the best dominant.

We remark here that Corollary 2.4 is an improvement of the corresponding result obtained by Singh [19].

Remark 2.5. For $q(z) = 1 + \dfrac{\lambda}{1+\xi}z$ and $\delta_1 = 1$, in Corollary 2.4, we get the result obtained by Singh [19, Theorem 1 (iii), p.571] and by setting $q(z) = \displaystyle\int_0^1 \frac{1 - \lambda z t^{\xi}}{1 + \lambda z t^{\xi}} dt$ and $\xi = 1$ in Corollary 2.4, we obtain another recent result of Singh [19, Theorem 3, p.573].

For the choices $p = 1$, $g(z) = \dfrac{z}{1-z}$, $\alpha = \delta_1 = \delta = \delta_3 = \gamma_1 = 0$, $\mu = 1$, and $\xi = 1$ in Theorem 2.2, we get the following result obtained by Ravichandran and Darus [12].

Corollary 2.6. *Let $\delta_2 \neq 0$ be a complex number. Let $q(z) \neq 0$ be univalent in Δ and let*

$$Q(z) := \xi\frac{zq'(z)}{q(z)} \quad \text{and} \quad h(z) := q(z) + Q(z).$$

Suppose that either (i) $h(z)$ is convex, or (ii) $Q(z)$ is starlike univalent in Δ. Further assume that

$$\Re\left\{\frac{q(z)}{\delta_2} + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \quad (z \in \Delta).$$

If

$$(1 - \delta_2)\frac{zf'(z)}{f(z)} + \delta_2\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z) + \delta_2\frac{zq'(z)}{q(z)},$$

then $\frac{zf'(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

By taking $q(z) := 1 + z$, we observe that the function q is non vanishing and the function $zq'(z)/q(z) = \dfrac{z}{1+z}$ is starlike. Also, letting the function $h(z) := 1 + z + \dfrac{\delta_2 z}{1+z}$, we have

$$\begin{aligned}\Re\frac{zh'(z)}{Q(z)} &= \Re\left[\frac{1+z}{\delta_2} + \frac{1}{1+z}\right]\\ &\geq \frac{1}{2} + \Re\left[\frac{1}{\delta_2} - \frac{1}{|\delta_2|}\right]\\ &\geq 0\end{aligned}$$

provided

$$\Re\left[\frac{1}{|\delta_2|} - \frac{1}{\delta_2}\right] < \frac{1}{2}.$$

For $p = 1$, $g(z) = \dfrac{z}{1-z}$, $\alpha = \delta_1 = \delta = \delta_3 = \gamma_1 = 0$, $\xi = 1$, $\mu = 1$ and $q(z) = 1 + z$ in Theorem 2.2, we have the following corollary.

Corollary 2.7. *Let $\delta_2 \in \mathbb{C}$ satisfies*

$$\Re\left[\frac{1}{|\delta_2|} - \frac{1}{\delta_2}\right] < \frac{1}{2}.$$

If

$$(1-\delta_2)\frac{zf'(z)}{f(z)} + \delta_2\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + z + \frac{\delta_2 z}{1+z},$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

The following result of Ponnusamy and Rajasekaran [10] follows from our corollary 2.7.

Corollary 2.8. *(Ponnusamy and Rajasekaran [10]) If $f \in \mathcal{A}$ satisfies*

$$(1-\delta_2)\frac{zf'(z)}{f(z)} + \delta_2\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + z + \frac{\delta_2 z}{1+z} \quad (\delta_2 \geq 0),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1.$$

The function

$$q(z) := \frac{2(1-z)}{2-z} \tag{2.7}$$

maps Δ onto the convex region $|q(z) - 2/3| < 2/3$ and satisfies the conditions of Theorem 2.2. Hence our Theorem 2.2, for the function $q(z)$ given by (2.7), reduces to the following:

Corollary 2.9. *Let* $\delta_2 > 0$. *If*

$$(1-\delta_2)\frac{zf'(z)}{f(z)} + \delta_2\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{2-(4+\delta_2)z+2z^2}{(1-z)(2-z)}$$

then

$$\left|\frac{zf'(z)}{f(z)} - \frac{2}{3}\right| < \frac{2}{3}.$$

Let $h(z) := \dfrac{2-(4+\delta_2)z+2z^2}{(1-z)(2-z)}$. For $2/3 < \delta_2 \leq 1$, with $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, we have $\Re h(z) = \dfrac{12+2\delta_2-12\cos\theta}{10-8\cos\theta} \geq \dfrac{3\delta_2}{2}$. Thus $h(\Delta)$ contains the half-plane $\Re h(z) < 3\delta_2/2$. In this case, our Corollary 2.9 gives the following result of Ponnusamy and Rajasekaran [10]:

Corollary 2.10. *Let* $2/3 < \delta_2 \leq 1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\Re\left[(1-\delta_2)\frac{zf'(z)}{f(z)} + \delta_2\left(1+\frac{zf''(z)}{f'(z)}\right)\right] < \frac{3\delta_2}{2},$$

then

$$\left|\frac{zf'(z)}{f(z)} - \frac{2}{3}\right| < \frac{2}{3}.$$

Remark 2.11. For the choices $p = 1$, $g(z) = \dfrac{z}{1-z}$, $\alpha = \xi = \delta_1 = \delta = \delta_2 = \gamma_1 = 0$, $\mu = 1$ $q(z) = \dfrac{1+Az}{1+Bz}$, $(-1 \leq B < A \leq 1)$ in Theorem 2.2, we get the result obtained by Tuneski [21].

For the choices $p = 1$, $g(z) = \dfrac{z}{1-z}$, $\alpha = \xi = \delta_1 = \delta = \delta_2 = \gamma_1 = 0$, $\mu = 1$ $q(z) = \dfrac{1+z}{1-z}$, in Theorem 2.2, we get the result obtained by Obradovič and Tuneski [8].

Corollary 2.12. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{2z}{(1+z)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$$

.

Theorem 2.13. *Let q be analytic and univalent in Δ such that $q(z) \neq 0$. Let $z \in \Delta$, α, δ, ξ, γ_1, δ_1, δ_2, $\delta_3 \in \mathbb{C}$ and suppose at least one of δ_1, δ_2, δ_3 is non–zero. Let q satisfies (2.1)and (2.2). Let*

(2.8)

$$\Psi_1(f,g,\mu,\xi,\beta,\delta,\gamma_1,\delta_1,\delta_3) := \begin{cases} \alpha + \xi\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \\ +\delta\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{2\mu} + \gamma_1\left(\frac{z(f*g)'(z)}{p(f*g)(z)}\right)^{\mu} \\ +\mu\left[\frac{z(f*g)'(z)}{(f*g)(z)} - 1 - \frac{z(f*g)''(z)}{(f*g)'(z)}\right]\left\{\delta_2 + \delta_1\left\{\frac{p(f*g)(z)}{z(f*g)'(z)}\right\}^{\mu}\right\} \\ +\delta_3\mu\left[\frac{z(f*g)'(z)}{(f*g)(z)} - 1 - \frac{z(f*g)''(z)}{(f*g)'(z)}\right]\left(\frac{z(f*g)'(z)}{p(f*g)(z)}\right)^{\mu} \end{cases}.$$

If $f \in \mathcal{A}_p$ satisfies the following subordination

(2.9)

$$\Psi_1(f,g,\mu,\xi,\delta,\gamma_1,\delta_1,\delta_3) \prec \alpha+\xi q(z)+\delta(q(z))^2+\frac{\gamma_1}{q(z)}+\delta_1 zq'(z)+\delta_2\frac{zq'(z)}{q(z)}+\delta_3\frac{zq'(z)}{(q(z))^2}$$

for some $\mu \in \mathbb{C} \setminus \{0\}$, then

$$\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \prec q(z) \tag{2.10}$$

and q is the best dominant.

Proof. Let the function ψ be defined by

$$\psi(z) := \left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \tag{2.11}$$

Evidently,

$$\frac{z\psi'(z)}{\psi(z)} = \mu\left[\frac{z(f*g)'(z)}{(f*g)(z)} - 1 - \frac{z(f*g)''(z)}{(f*g)'(z)}\right]$$

which, in light of hypothesis (2.9) yields

$$\alpha + \xi\psi(z) + \delta(\psi(z))^2 + \frac{\gamma_1}{\psi(z)} + \delta_1 z\psi'(z) + \delta_2\frac{z\psi'(z)}{\psi(z)} + \delta_3\frac{z\psi'(z)}{(\psi(z))^2}$$

$$\prec \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 zq'(z) + \delta_2\frac{zq'(z)}{q(z)} + \delta_3\frac{zq'(z)}{(q(z))^2},$$

Letting

$$\theta(\omega) := \alpha + \xi\omega + \delta\omega^2 + \frac{\gamma_1}{\omega} \qquad \text{and} \qquad \phi(\omega) := \delta_1 + \frac{\delta_2}{\omega} + \frac{\delta_3}{\omega^2}$$

and following the steps of Theorem 2.2, the assertions (2.1)and (2.2), the result follows by an application of Theorem A. □

Remark 2.14. For the choices $g(z) = \frac{z}{1-z}$, $\alpha = \delta = \gamma_1 = \delta_2 = \delta_3 = 0$, Theorem 2.2 coincides with the result obtained by Shanmugam et al. [14].

Remark 2.15. For the choices $g(z) = \frac{z}{1-z}$, $\alpha = \delta = \delta_2 = \delta_3 = \gamma_1 = 0$, $q(z) = 1 + \frac{\lambda}{1+\xi} z$ and $\delta_1 = 1$ in Theorem 2.2, we get the result obtained by Singh [19, Theorem 1 (iii), p.571].

Next, by appealing to Theorem B we prove Theorem 2.16 and Theorem 2.17 below.

Theorem 2.16. *Let q be analytic and univalent in Δ such that $q(z) \neq 0$. Let $z \in \Delta$, δ, ξ, γ_1, δ_1, δ_2, $\delta_3 \in \mathbb{C}$ and $\mu \in \mathbb{C} \setminus \{0\}$. Suppose that q satisfies (2.2) and*

$$\Re \left[\frac{2\delta(q(z))^3 + \xi(q(z)) - \gamma_1}{\delta_1(q(z))^2 + \delta_2 q(z) + \delta_3} \right] > 0, \tag{2.12}$$

If $f \in \mathcal{A}_p$, $\left(\frac{1}{p} \frac{z(f{*}g)'(z)}{(f{*}g)(z)} \right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$, and $\Psi(f,\, g,\, \mu,\, \xi,\, \delta,\, \gamma_1,\, \delta_1,\, \delta_2,\, \delta_3)$ is univalent in Δ, where $\Psi(f,\, g,\, \mu,\, \xi,\, \delta,\, \gamma_1,\, \delta_1,\, \delta_2,\, \delta_3)$ is as defined in (2.3), then

$$\alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 z q'(z) + \delta_2 \frac{zq'(z)}{q(z)} + \delta_3 \frac{zq'(z)}{(q(z))^2}$$
$$\prec \Psi(f,\, g,\, \mu,\, \xi,\, \delta,\, \gamma_1,\, \delta_1,\, \delta_2,\, \delta_3)$$

implies

$$q(z) \prec \left(\frac{1}{p} \frac{z(f{*}g)'(z)}{(f{*}g)(z)} \right)^{\mu} \tag{2.13}$$

and q is the best subordinant .

Proof. Defining ψ by (2.6), following steps of Theorem 2.2, and by setting

$$\vartheta(w) := \alpha + \xi\omega + \delta\omega^2 + \frac{\gamma_1}{\omega} \quad \text{and} \quad \varphi(w) := \delta_1 + \frac{\delta_2}{\omega} + \frac{\delta_3}{\omega^2},$$

it is easily observed that ϑ and φ are analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\varphi(w) \neq 0.$$

In view of the condition (2.12) and since q is univalent, it is routine to show that (1) and (2) of Theorem B are satisfied. The assertion (2.13) follows by an application of Theorem B. $\square$

Theorem 2.17. *Let q be analytic and univalent in Δ such that $q(z) \neq 0$. Let $z \in \Delta$, δ, ξ, γ_1, δ_1, δ_2, $\delta_3 \in \mathbb{C}$ and $\mu \in \mathbb{C} \setminus \{0\}$. Suppose that q satisfies (2.12). If $f \in \mathcal{A}_p$, $\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$, and $\Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$ is univalent in Δ where $\Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$ is as defined in (2.8), then*

$$\alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 z q'(z) + \delta_2 \frac{zq'(z)}{q(z)} + \delta_3 \frac{zq'(z)}{(q(z))^2}$$
$$\prec \Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$$

implies

$$q(z) \prec \left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \tag{2.14}$$

and q is the best subordinant.

Proof. Let the function ψ be defined by ψ by (2.11). By setting

$$\vartheta(w) := \alpha + \xi\omega + \delta\omega^2 + \frac{\gamma_1}{\omega} \quad \text{and} \quad \varphi(w) := \delta_1 + \frac{\delta_2}{\omega} + \frac{\delta_3}{\omega^2},$$

it is easily observed that the functions ϑ and φ are analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\varphi(w) \neq 0, \quad (w \in \mathbb{C} \setminus \{0\}).$$

The assertion (2.14) follows by an application of Theorem B. $\square$

Combining the corresponding subordination and superordination results, we get the following sandwich theorems.

Theorem 2.18. *Let q_1 and q_2 be univalent in Δ such that q_1 and q_2 satisfy (2.2), $q_1(z) \neq 0$ and $q_2(z) \neq 0$. Further, suppose q_1 and q_2 satisfy (2.12) and (2.1). Let $z \in \Delta$, δ, ξ, γ_1, δ_1, δ_2, $\delta_3 \in \mathbb{C}$ and $\mu \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{A}_p$, $\left(\frac{1}{p}\frac{z(f*g)'(z)}{(f*g)(z)}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$ and $\Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$ defined by (2.3) is univalent in Δ, then*

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \frac{\gamma_1}{q_1(z)} + \delta_1 z q_1'(z) + \delta_2 \frac{zq_1'(z)}{q_1(z)} + \delta_3 \frac{zq_1'(z)}{(q_1(z))^2}$$
$$\prec \Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$$
$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \frac{\gamma_1}{q_2(z)} + \delta_1 z q_2'(z) + \delta_2 \frac{zq_2'(z)}{q_2(z)} + \delta_3 \frac{zq_2'(z)}{(q_2(z))^2}$$

implies

$$q_1(z) \prec \left(\frac{1}{p}\frac{z(f*g)'(z)}{(f*g)(z)}\right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

Theorem 2.19. *Let q_1 and q_2 be univalent in Δ such that q_1 and q_2 satisfy (2.2), $q_1(z) \neq 0$ and $q_2(z) \neq 0$. Further, suppose q_1 and q_2 satisfy (2.12) and (2.1). Let $z \in \Delta$, δ, ξ, γ_1, $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$ and $\mu \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{A}_p$, $\left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap Q$ and $\Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$ defined by (2.8) is univalent in Δ, then*

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \frac{\gamma_1}{q_1(z)} + \delta_1 z q_1'(z) + \delta_2 \frac{zq_1'(z)}{q_1(z)} + \delta_3 \frac{zq_1'(z)}{(q_1(z))^2}$$

$$\prec \Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$$

$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \frac{\gamma_1}{q_2(z)} + \delta_1 z q_2'(z) + \delta_2 \frac{zq_2'(z)}{q_2(z)} + \delta_3 \frac{zq_2'(z)}{(q_2(z))^2}$$

implies

$$q_1(z) \prec \left(\frac{p(f*g)(z)}{z(f*g)'(z)}\right)^{\mu} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

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