SUT Journal of Mathematics
Vol. 44, No. 1 (2008), 153–168

Reduction of the codimension for degenerate submanifolds

Jean-Pierre Ezin, Mouhamadou Hassirou*and Joël Tossa

(Received April 18, 2007; Revised February 7, 2008)

Abstract. We give in this paper sufficient conditions for r-lightlike submanifolds M of dimension m, which is not totally geodesic in an $(m+n)$-dimensional semi-Riemannian manifold of constant curvature c to admit a reduction of codimension. We consider proper r-lightlike, coisotrope and totally lightlike submanifolds, generalizing thus previous results on isotropic submanifolds [1] as well as in the Riemannian case developed in [2, 5, 10].

AMS 2000 *Mathematics Subject Classification.* 53C50.

Key words and phrases. Lightlike submanifolds, Screen distribution on degenerate submanifolds, Totally umbilical of lightlike submanifolds, Reduction of codimension.

§1. Introduction and basic facts

This paper deals with the reduction of the codimension of lightlike submanifolds in semi-Riemannian manifolds. Assume (M, g) is an m-dimensional r-lightlike submanifold which is not totally geodesic in an $(m+n)$-dimensional $(n \neq m)$ semi-Riemannian manifold of constant curvature c. The reduction of the codimension consists of finding a sufficient condition for M to be immersed into an $(m+p)$-dimensional totally geodesic submanifold of constant curvature, where $p < n$. The substantial codimension is then the smallest codimension that an immersion can be reduced to. We generalize results obtained on the subject when the ambient space is Riemannian [5, 10] and the ones obtained in the semi-Riemannian case [1] where lightlike isotropic submanifolds have

*Current affiliation : Département de Mathématiques et d'Informatique, Université Abdou Moumouni, Niamey (Niger). Partially supported by ANTSI / UNESCO during this work.

been considered. We also give a sufficient condition for a totally umbilical coisotropic submanifold [4] of pseudo-Euclidean space to admit a reduction of codimension.

Reduction of codimension is often used in geometry. The following classical property of curves in Euclidean n-space $\mathbb{R}^n$ is a motivating example for our study. Consider a curve $c : (a, b) \to \mathbb{R}^n$. Suppose for $j < n$, its curvatures $k_1, \ldots, k_{j-1}$, do not vanish and k_j is identically null. It is well known that c is then contained in a j-dimensional affine subspace. From a physics point of view the universe we live in is usually represented as a 4-dimensional subspace embedded into a $(4 + d)$-dimensional spacetime. This idea has attracted and still attracts the attention of many physicists and cosmologists. Also the imbedding of the exact solutions of Einstein equations into higher dimensional semi-Euclidean space is expected to provide a better understanding of their intrinsic geometry. In both cases the problem to be solved is to find out the lowest codimension of the imbedding under consideration in order to obtain a theoretical framework in which fundamental laws of physics might present some unification. The Kaluza-Klein scheme that takes into account the mutual interaction between matter and metric is a stimulating example.

The present paper aims to furnish a contribution to studies in those directions. It is organized as follows. We give in the preliminaries in section 2, basic formulas concerning geometric objects on lightlike submanifolds, which is now the classic reference in this subject. Proofs of the main results are given in section 3 and finally we construct examples to illustrate our motivations in section 4.

§2. Preliminaries

2.1. Preliminary Recollections

For the convenience of the reader, we start with an overview of geometry of lightlike submanifolds, using notations and results of [3]. The fundamental difference between the theory of lightlike (or degenerate) submanifolds (M, g), and the classical theory of submanifolds of a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ comes from the fact that

$$Rad(TM) = TM \cap TM^{\perp} \neq \{0\}. \tag{2.1}$$

Given an integer $r > 0$, the submanifold M is said to be r-lightlike (or r-degenerate) if the rank of $Rad(TM)$ is equal to r everywhere. We have four cases of lightlike submanifolds:

- The proper r-lightlike submanifolds, where $0 < r < min(m, n)$. In this case, we have $Rad(TM) \subseteq TM$ and $Rad(TM) \subseteq TM^{\perp}$ then there exist non-degenerate screen distributions $S(TM)$ and $S(TM^{\perp})$, complementary vector subbundle to $Rad(TM)$ in TM and in $TM^{\perp}$ respectively such that,

$$\begin{aligned} TM &= Rad(TM) \perp S(TM). \\ TM^{\perp} &= Rad(TM) \perp S(TM^{\perp}). \end{aligned}$$

The subbundle $S(TM^{\perp})$ is called transversal screen distribution. Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\overline{M}$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$\begin{aligned} T\overline{M}_{|M} &= TM \oplus tr(TM) \\ &= S(TM) \perp S(TM^{\perp}) \perp (Rad(TM) \oplus ltr(TM)) \end{aligned} \tag{2.2}$$

where

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{2.3}$$

- The coisotropic submanifolds, with $1 \leq r = n < m$. In this case, relation (2.2) becomes

$$\begin{aligned} T\overline{M}_{|M} &= TM \oplus ltr(TM) \\ &= S(TM) \perp (Rad(TM) \oplus ltr(TM)) \end{aligned} \tag{2.4}$$

- The isotropic submanifold case, with $1 \leq r = m < n$. In this case, $Rad(TM) = TM \subsetneq TM^{\perp}$ and $S(TM) = \{0\}$. The relation (2.2) is expressed as

$$\begin{aligned} T\overline{M}_{|M} &= TM \oplus tr(TM) \\ &= (TM \oplus ltr(TM)) \perp S(TM^{\perp}). \end{aligned} \tag{2.5}$$

Null curves are examples of isotropic submanifolds.

- The totally lightlike submanifolds, where $1 < r = n = m$. We have in this case $Rad(TM) = TM = TM^{\perp}$, $S(TM) = S(TM^{\perp}) = \{0\}$ and

$$T\overline{M}_{|M} = TM \oplus ltr(TM). \tag{2.6}$$

Null curves of two dimensional manifolds are examples of totally lightlike submanifolds.

2.2. The Induced Connection

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(TM) \tag{2.7}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V \quad \forall X \in \Gamma(TM),\, V \in \Gamma(tr(TM)) \tag{2.8}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ are in $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. We suppose $S(TM^\perp) \neq \{0\}$ and we denote by L and S the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively. Using the relation (2.3), relations (2.7) and (2.8) become respectively

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \forall X, Y \in \Gamma(TM) \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$ and

$$\overline{\nabla}_X V = -A_V X + D^l_X V + D^s_X V \tag{2.10}$$

$\forall X \in \Gamma(TM),\ \forall V \in \Gamma(tr(TM))$, where

$$D^l_X V = L(\nabla^t_X V) \;\; D^s_X V = S(\nabla^t_X V).$$

Then we have for all $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$

$$\begin{aligned} \nabla^l_X(LV) = D^l_X(LV) \quad &\text{and} \quad \nabla^s_X(SV) = D^s_X(SV). \\ D^l(X, SV) = D^l_X(SV) \quad &\text{and} \quad D^s(X, LV) = D^s_X(LV). \end{aligned}$$

The applications ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$, respectively. We call them respectively lightlike connection and the screen transversal connection on M. Relation (2.10) can also be written as

$$\overline{\nabla}_X V = -A_V X + D^l(X, SV) + D^s(X, LV) + \nabla^l_X(LV) + \nabla^s_X(SV).$$

These geometric objects verify the following relations [3, p.156]:

$$\overline{g}(h^s(X,Y), W) + \overline{g}(Y, D^l(X,W)) = \overline{g}(A_W X, Y) \tag{2.11}$$
$$\overline{g}(h^l(X,Y), \xi) + \overline{g}(Y, h^l(X,\xi)) + \overline{g}(Y, \nabla_X \xi) = 0 \tag{2.12}$$
$$\overline{g}(W, D^s(X,N)) = \overline{g}(A_W X, N) \tag{2.13}$$
$$\overline{g}(A_N X, N') = \overline{g}(A_{N'} X, Y) \tag{2.14}$$
$$\overline{g}(A_N X, PY) = \overline{g}(N, \nabla_X PY) \tag{2.15}$$
$$h^l_i(X, \xi_j) = h^l_j(X, \xi_i) \tag{2.16}$$

where

$$X, Y \in \Gamma(TM), \;\; N \in \Gamma(ltr(TM)), \;\; \xi_i \in \Gamma(Rad(TM)), \;\; W \in \Gamma(S(TM^{\perp}))$$

and h_i^l are such that $h_i^l(X,Y) = g(\overline{\nabla}_X Y, \xi_i)$. Then h_i^l does not depend on the choice of $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$ and are zero on $Rad(TM)$. Consequently the second fundamental form h^l is identically equal to zero on an isotropic and on a totally lightlike submanifolds.

So, D^l is a shape application form of r-lightlike and isotropic submanifolds. We have

$$(2.17) \qquad (\nabla_X g)(X,Y) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y)$$

$$(2.18) \qquad (\nabla_X^t \overline{g})(V,V') = -(\overline{g}(A_V X, V') + \overline{g}(A_{V'} X, V).$$

Hence, the induced connections ∇ and ∇^t are not metric in general. As a consequence, we have

1. The induced connection ∇ of the Levi-Civita connection $\overline{\nabla}$ of $(\overline{M}, \overline{g})$ is metric on isotropic and totally lightlike submanifolds (M, g).
2. A proper r-lightlike or a coisotropic submanifold (M, g) admits a metric connection if and only if h^l vanishes identically on M.

Let $f : M^m \longrightarrow M^{m+n}$ be an isometric immersion of an m-dimensional r-lightlike $(1 \leq r \leq min(m,n))$ submanifold into an $(m+n)$-dimensional semi-Riemannian manifold. *The first transversal space of* f at $x \in M$ is the subspace

$$T_1(x) = span\{h^l(X,Y) + h^s(X,Y), \quad X, Y \in T_x M\}, \quad x \in M.$$

Proposition 2.1. *Suppose that the induced connection ∇ on M is a metric connection, then the first transversal space of f, $T_1(x)$ is characterized by*

$$T_1(x) = \{W \in S(T_x M^{\perp}) \subset tr(T_x M), D^l(.,W) = 0 \quad and \quad A_W = 0\}^{\perp}$$

for all $x \in M$.

Proof. Recall that ∇ is metric $\Longleftrightarrow h^l = 0$ and we have

$$T_1(x) = span\{h^s(X,Y), \quad X, Y \in T_x M\} \quad x \in M.$$

Suppose

$N(x) = \{W \in S(T_x M^{\perp}) \subset tr(T_x M), D^l(.,W) = 0 \quad \text{and} \quad A_W = 0\}^{\perp}$.

Let $V \in T_1(x)$ and $W \in N^{\perp}(x)$ such that $V = h^s(X,Y)$.

$$\begin{aligned} \overline{g}(V,W) = \overline{g}(h^s(X,Y),W) &= \overline{g}(A_W X, Y) - \overline{g}(D^l(X,W),Y) \\ &= 0. \end{aligned}$$

We have for every $V \in T_1(x)$, $\overline{g}(V,W)=0$, $\forall W \in N^{\perp}(x)$ and $\forall x \in M$. Then V lies in $(N^{\perp}(x))^{\perp} = N(x)$, $T_1(x) \subset N(x)$.

Conversely, taken $V \in T_1^{\perp}(x)$ such that $\forall X, Y \in T_xM$

$$\overline{g}(h^s(X,Y),V) = \overline{g}(A_V X, Y) - \overline{g}(D^l(X,V),Y) = 0.$$

If $Y \in Rad(T_xM)$, then $\overline{g}(D^l(X,V),Y) = 0$ and $D^l(X,.) = 0$ for all X. If $Y \in S(T_xM)$, then $\overline{g}(A_V X, Y) = 0$ and $A_V = 0$.

Hence $V \in N^{\perp}(x)$ and $N(x) = (N^{\perp})^{\perp}(x) \subset T_1(x)$. □

Let $f : M^m \longrightarrow \mathbb{R}_q^{m+n}$ be an isometric immersion of an m-dimensional coisotropic submanifold into an $(m+n)$-dimensional semi-Riemannian manifold. Define *the first radical space of f* at $x \in M$ to be the subspace

$$R_1(x) = span\{\xi \in Rad(T_xM),\ \exists X \in T_xM,\ \dot{A}_\xi X \neq 0\}, \quad x \in M.$$

The first transversal space then becomes

$$T_1'(x) = \{h^l(X,Y),\ X,Y \in T_xM\}, \quad x \in M.$$

Proposition 2.2. *If $(M^m, g, S(TM))$ is a non totally geodesic coisotropic submanifold, then for $x \in M$, $T_1'(x)$ is characterized by $R_1(x)$.*

Proof. Let $x \in M$ and π_x a projection of T_xM on $S(T_xM)$. If $U \in T_1'(x)$ and $U \neq 0$, then there exists $X, Y \in T_xM$ such that $U = h^l(\pi_x(Y), X)$. Moreover there exists $\xi \in rad(T_xM)$ and $(\xi \neq 0)$, such that $g(h^l(\pi_x(Y),X),\xi) \neq 0$. Hence from relation (2.12), one has $g(h^l(X,\pi_x(Y)),\xi) = g(\dot{A}_\xi X, \pi_x(Y)) \neq 0$ and $\xi \in R_1(x)$. Conversely, if $\xi \in R_1(x)$ then there exists $X \in T_xM$ such that $\dot{A}_\xi X \neq 0$. Hence $g(\dot{A}_\xi X, \dot{A}_\xi X) = g(h(X, \dot{A}_\xi X),\xi) \neq 0$ and $U = h(X, \dot{A}_\xi X) \in R_1(x)$. □

Let $x \in M$ and P and $\widetilde{P}$ be subbundle in $Rad(TM)$ and in $ltr(TM)$ respectively. We say that P and $\widetilde{P}$ are corresponding subbundles, if for all $\xi_x \in P(x)$ there exists $N_x \in \widetilde{P}(x)$ such that $g(\xi_x, N_x) = 1$ and $g(\xi_x, N_x') = 0$ for all $N_x' \in ltr(T_xM) \backslash \widetilde{P}(x)$ and vice versa.

Proposition 2.3. *Let P be vector subbundle of constant rank in $Rad(TM)$ which contains $R_1(x)$ for all $x \in M$ and $\overline{P}(x)$ the complementary of $P(x)$ in $Rad(T_xM)$. If $P(x)$ and $\overline{P}(x)$ are parallel w.r.t the $\dot{\nabla}^t$ then their corresponding subbundles $\widetilde{P}(x) \supset T_1(x)$ and $\widetilde{\overline{P}}(x)$ in $ltr(TM)$ respectively are parallel w.r.t. ∇^t.*

Proof. Let $x \in M$ and $\xi \in \overline{P}(x)$. Then $\dot{A}_\xi = 0$ and for all, $U \in \widetilde{P}(x)$, $\overline{g}(\xi, U) = 0$. Let $X \in T_xM$, we have

$$\begin{aligned} \overline{\nabla}_X \overline{g}(\xi, U) = 0 &\iff \overline{g}(\dot{\nabla}^t_X \xi, U) + \overline{g}(\xi, \nabla^t_X U) = 0 \\ &\iff \overline{g}(\dot{\nabla}^t_X \xi, U) = -\overline{g}(\xi, \nabla^t_X U) = 0 \end{aligned}$$

$\overline{g}(\xi, \nabla^t_X U) = 0 \Longleftrightarrow \nabla^t_X U \in \widetilde{P}(x)$. Thus $\widetilde{P}(x)$ is parallel.

It's the same with $\overline{P}(x)$ and $\widetilde{\overline{P}}(x)$. □

2.3. The main results

Suppose that $(\overline{M}_c^{m+n}, \overline{g})$ is an $(m+n)$-dimensional complete and simply connected semi-Riemannian manifold with constant sectional curvature c and $f : M^m \longrightarrow \overline{M}^{m+n}$ an isometric immersion of the lightlike submanifold M^m in $\overline{M}^{m+n}$.

Theorem 2.1. *Let $f : M^m \longrightarrow \overline{M}^{m+n}$ be an isometric immersion of a r-lightlike submanifold $(1 \leq r \leq m,\ r \neq n)$ $(M, g, S(TM), S(TM^\perp))$ into $(\overline{M}_c^{m+n}, \overline{g})$. Suppose that*

1. *the induced linear connection ∇ on M and the transversal linear connection ∇^t on the transversal subbundle $tr(TM)$ are metric ones.*
2. *there exists a screen transversal subbundle P of $S(TM^\perp)$ of constant rank p $(p < n)$, parallel w.r.t the connection ∇^s on $S(TM^\perp)$, such that*

$$T_1(x) \subset P(x), \quad \forall x \in M$$

where $T_1(x)$ is the first transversal space of f at $x \in M$.

Then the codimension of f can be reduced to p.

The difference which exists between Theorem 1 of [1] and Theorem 2.1 (above), is that the latter is more general. Because in this case the subbundle $S(TM^\perp) \neq \{0\}$ contrary to the isotropic case where $S(TM^\perp) = \{0\}$ (and A_W is not defined).

Instead of a screen transversal subbundle as in Theorem 2.1, in the coisotropic submanifold we use a radical subbundle. We have

Theorem 2.2. *Let $f : M^m \longrightarrow \mathbb{R}^{m+n}$ be an isometric immersion of a lightlike coisotropic submanifold $(M, g, S(TM))$ into a pseudo-Euclidean space $(\mathbb{R}_q^{m+n}, \overline{g})$. Suppose there exists a radical subbundle P of $Rad(TM)$ of constant rank p $(p < n)$, parallel w.r.t the connection $\dot{\nabla}^t$ on $Rad(TM)$, such that its complementary in $Rad(T_xM)$ is also parallel and*

$$R_1(x) \subset P(x), \quad \forall x \in M$$

where $R_1(x)$ is the first radical space of f at $x \in M$. Then the codimension of f can be reduced to p.

Now suppose that $(M, S(TM), g)$ is a coisotropic submanifold of semi-Riemannian $(\mathbb{R}_q^{m+n}, \overline{g})$. The submanifold M is said to be totally umbilical in $\overline{M}$, if and only if

$$h^l(X,Y) = \overline{g}(X,Y)N, \ \forall X, Y \in \Gamma(TM), \ N \in tr(TM) \tag{2.19}$$

and h the second fundamental form [7]. Then N is called an umbilical vector field.

If ξ is a nonzero vector fields in $\Gamma(Rad(TM))$ such that $\overline{g}(\xi, N) = 1$ then

$$\overline{g}(h^l(X,Y), \xi) = \overline{g}(X,Y), \text{ and } \overline{g}(\xi, N) = 1.$$

This definition does not depend on the choice of screen distribution [3, Theo 2.1, pg 157].

Then we have the following.

Theorem 2.3. *Let $f : M^m \longrightarrow \mathbb{R}^{m+n}$ be a totally umbilical isometric immersion of a lightlike coisotropic submanifold $(M, g, S(TM))$ into a pseudo-Euclidean space $(\mathbb{R}_q^{m+n}, \overline{g})$. Suppose that the umbilical vector field is parallel w.r.t the connection ∇^t on $ltr(TM)$. Then the codimension of f can be reduced to* 1.

§3. Proof of Theorems

3.1. Proof of Theorem 2.1

Recall that P is parallel w.r.t. ∇^s if for all

$$X \in \Gamma(TM) \text{ and } W \in \Gamma(P), \nabla^s_X W \in \Gamma(P).$$

As c is constant, we have three possible cases.

Case $c = 0$.

Let $x_0 \in M$, we have to prove that $f(M) \subset T_{x_0}M \oplus P(x_0)$. Let η be a vector of $P^\perp(x_0)$, the complementary orthogonal bundle of $P(x_0)$ in $S(TM^\perp)$ and η_t the parallel transport of vector η in $\Gamma(P^\perp)$ along the regular curve $\gamma : I \longrightarrow M$ $(I \subset \mathbb{R})$ through x_0.

Since $\overline{g}$ is non degenerate on $S(TM^\perp)$, if $P^\perp$ is parallel then P is parallel. Hence

$$\eta_t = \nabla^s_{\dot{\gamma}}\eta \in \Gamma(P^\perp(\gamma(t))), \forall t \in I$$

and

$$\overline{\nabla}_{\dot{\gamma}}\eta_t = -A_{\eta_t}\dot{\gamma} + D^l(\dot{\gamma}, \eta_t) + \nabla^s_{\dot{\gamma}}\eta_t$$

but

$$\eta_t = \nabla^s_{\dot{\gamma}}\eta \in \Gamma(P^\perp(\gamma(t))) \Longrightarrow A_{\eta_t}\dot{\gamma} = 0 \quad \text{and} \quad D^l(\dot{\gamma}, \eta_t) = 0,$$

as η_t is parallel transport of η along γ in $\Gamma(P^\perp)$, $\nabla^s_{\dot\gamma}\eta_t = 0$, $\forall t \in I$.

Thus, we have $\overline{\nabla}_{\dot\gamma}\eta_t = 0 \Longrightarrow \eta_t = \eta = cste$ in $\mathbb{R}^{m+n}_q$

$$\frac{d}{dt}(\overline{g}(f(\gamma(t)) - f(x_0), \eta_t)) = \overline{g}(f_\star\dot\gamma, \eta) = 0$$

$$\Longrightarrow \quad f(\gamma(t)) - f(x_0) \in (P^\perp(\gamma(t)))^\perp = P(\gamma(t)).$$

As γ and η are arbitrary,

$$f(M) \subset T_{x_0}(M) \oplus P(x_0) \cong \mathbb{R}^{n+p}$$

and $\mathbb{R}^{n+p}$ is totally geodesic in $\mathbb{R}^{n+m}_q$.

Case $c > 0$.

Then M^m is isometrically immersed in the pseudosphere $\overline{M}^{m+n} = S^{m+n}_q$ by an immersion $f : M^m \longrightarrow S^{m+n}_q$. Denote by $i : S^{m+n}_q \longrightarrow \mathbb{R}^{m+n+1}_q$ the canonical injection of S^{m+n}_q in $\mathbb{R}^{m+n+1}_q$ and consider the isometric immersion $\widehat{f} = i \circ f : M^m \longrightarrow \mathbb{R}^{m+n+1}_q$.

We have the corresponding vector spaces

$$tr(\widehat{T}_x M) = tr(T_x M) \oplus < f(x) >$$

where

$$< f(x) > := span\{f(x)\} \subset S(\widehat{T}_x M^\perp).$$

We deduce

$$\widehat{T}_1(x) = T_1(x) \oplus < f(x) > \subset P(x) \oplus < f(x) > = \widehat{P}(x).$$

The complementaries $\widehat{P}(x)$ in $S(\widehat{T}_x M^\perp)$ and $P(x)$ in $S(T_x M^\perp)$ coincide; $\widehat{P}^\perp(x) = P^\perp(x)$, and parallel w.r.t. the connection $\nabla^s = \widehat{\nabla}^s_{|S(T_x M^\perp)}$.

As $< f(x) > \subset \widehat{P}(x)$ and $P(x)$ is parallel w.r.t. the connection $\widehat{\nabla}^s_{|S(T_x M^\perp)}$ in $S(\widehat{T}_x M^\perp)$, then $\forall X \in \Gamma(TM)$ and $W \in \widehat{P}(x)^\perp$,

$$\begin{aligned} \overline{g}(\widehat{\nabla}^s_X f(x), W) &= \overline{\nabla}_X \overline{g}(f(x), W) + \overline{g}(f(x), \widehat{\nabla}^s_X W) \\ &= 0 \ \ (\text{car} \widehat{\nabla}^s_X W \in \widehat{P}^\perp(x)) \end{aligned}$$

so that $\widehat{\nabla}^s_X f(x) \in \widehat{P}(x)$. Hence $\widehat{P}(x)$ is parallel w.r.t. the connection $\widehat{\nabla}^s$.

As $ltr(T_x M) = ltr(\widehat{T}_x M)$ is parallel, then $\forall N \in ltr(T_x M)$

$$\begin{aligned} \overline{\nabla}_X < f(x), N > &= 0 \\ &= < \overline{\nabla}_X f(x), N > + < f(x), \overline{\nabla}_X N > \\ &= - < \widehat{A}_{f(x)} X, N > \end{aligned}$$

and hence $\widehat{\nabla}^t$ is a metric connection. As in the case $c = 0$, we obtain

$$\widehat{f}(M) \subset \widehat{T}_x M \oplus \widehat{P}(x) = T_x M \oplus P(x) \oplus f(x) \cong \mathbb{R}^{m+p+1}$$
$$f(M) \subset S_q^{m+n} \cap \mathbb{R}^{m+p+1}.$$

This ends the proof of the case $c > 0$

Case $c < 0$.

The proof of this case is similar to the second case $c > 0$. We consider an immersion $\hat{f} = M^m \longrightarrow \mathbb{R}_{q+1}^{m+n+1}$ such that $\hat{f} = i \circ f$ where $i : \mathbb{H}^{m+p} \longrightarrow \mathbb{R}_{q+1}^{m+n+1}$ is the canonical injection of pseudo-hyperbolic $\mathbb{H}^{m+n}$ into $\mathbb{R}_{q+1}^{m+n+1}$, we have

$$\hat{f}(M) \subset \hat{T}_x M \oplus \hat{P}(x) = T_x M \oplus P(x) \oplus f(x) \cong \mathbb{R}^{m+p+1}$$
$$f(M) \subset \mathbb{H}_c^{m+n} \cap \mathbb{R}^{m+p+1}.$$

This ends the proof of Theorem 2.1. □

Corollary 3.1. *Let $f : M^m \longrightarrow \overline{M}^{m+n}$ be an isometric immersion of a r-lightlike submanifold $(1 \leq r \leq m,\ r \neq n)$ $(M, g, S(TM), S(TM^{\perp}))$ into $(\overline{M}_c^{m+n}, \overline{g})$. If the induced connection ∇ and the transversal connection ∇^t are metric ones, then the substantial codimension of f is less than or equal to $n - r$.*

Proof. As ∇ and ∇^t are metric, $ltr(TM)$ is parallel w.r.t. the connection ∇^t. Hence $S(TM^{\perp})$ is also parallel w.r.t. ∇^t. In particular $S(TM^{\perp})$ is parallel w.r.t. ∇^s. Since $T_1(x) \subset S(T_x M^{\perp})$ and $S(TM^{\perp})$ has a constant rank $n - r$, there exists a parallel distribution P of constant rank in $S(TM^{\perp})$ such that $T_1(x) \subset P(x) \subseteq S(T_x M^{\perp})$. Hence f admits a reduction of its codimension to the rank of P $(0 < rank(P) \leq rank(S(T_x M^{\perp})) = n - r)$. □

An isometric immersion f is said to be an 1-regular if the first transversal (radical) space has a constant rank.

Corollary 3.2. *If f is an 1-regular immersion and $T_1 = S(TM^{\perp})$, then the codimension of f can be reduced to $n - r$.*

As a consequence of Theorem 2.1, we have

Corollary 3.3. *The totally geodesic submanifold Q^{m+p} of $\mathbb{R}_q^{m+n}$ obtained after reduction of codimension and which contains $f(M^m)$ is a degenerate submanifold of $\mathbb{R}_q^{m+n}$. Moreover*

- *If $p < n - r$, then Q^{m+p} is r-lightlike.*
- *If $p = n - r$, then Q^{m+p} is coisotropic.*

Proof. Since $h^l = 0$ and $T_1(x) \subset P(x) \subset S(TM^\perp)$, then $\forall x \in M,\ \ T_xM \oplus P(x)$ has r lightlike vectors fields. Hence Q is r-degenerate because

$$T_xQ = T_xM \oplus P(x) = S(T_xM) \oplus Rad(T_xM) \oplus P(x), \qquad \forall x \in M.$$

Moreover

$$T_x\overline{M} = T_xM \oplus P(x) \oplus P^\perp(x) \oplus ltr(T_xM) = T_xQ + \overline{tr(T_xM)}, \qquad \forall x \in M$$

where $\overline{tr(T_xM)} = P^\perp(x) \oplus ltr(T_xM)$.

If $p < n - r$, then the rank of $P^\perp(x)$ is zero. Hence Q is an r degenerate manifold.

If $p = n - r$, then the rank of $P^\perp(x)$ is zero and $\overline{tr(T_xM)} = ltr(T_xM)$. Hence Q is lightlike coisotrope submanifold. □

Proof of Theorem 2.2. The idea of proof is identical to that of Theorem 2.1 apart from some technical use for radical subbundle. Let $x \in M$, we will prove that $f(M) \subset T_xM \oplus \widetilde{P}(x)$ and that $T_xM \oplus \widetilde{P}(x)$ is totally geodesic in $\mathbb{R}^{m+n}_q$.

Let η be a vector of $\overline{P}(x)$ and η_t the parallel transport of η in $\overline{P}$ along an arbitrary smooth curve $\gamma : I \longrightarrow M$ $(I \subset \mathbb{R})$ through x. The relation (2.7) gives

$$\overline{\nabla}_{\dot\gamma}\eta_t = \nabla_{\dot\gamma}\eta_t + h^l(\dot\gamma, \eta_t) \quad \forall I \in \mathbb{R}.$$

With the Weingarten relation we have

$$\nabla_{\dot\gamma}\eta_t = -\dot{A}_{\eta_t}\dot\gamma + \dot\nabla_{\dot\gamma}\eta_t$$

$\eta_t \in \Gamma(\overline{P}) \Longrightarrow \dot{A}_{\eta_t}\dot\gamma = 0$. As η_t is obtained by parallel transport of η along γ in $\Gamma(\overline{P})$, $\dot\nabla^t_{\dot\gamma}\eta_t = 0$, $\forall t \in I$. Hence we have $\nabla_{\dot\gamma}\eta_t = 0$.

With relation (2.12), $h(\eta_t, \dot\gamma) = 0$, and $\overline{\nabla}_{\dot\gamma}\eta_t = 0$ yields $\eta_t = \eta = cste$

$$\frac{d}{dt}(\overline{g}(f(\gamma(t)) - f(x), \eta_t)) = \overline{g}(f_\star\dot\gamma, \eta) = 0.$$

As γ and η are arbitrary and $f(\gamma(t)) - f(x) \in ltr(M)$, we have

$$f(\gamma(t)) - f(x) \in \widetilde{P}(\gamma(t)).$$

Hence

$$f(M) \subset T_x(M) \oplus \widetilde{P}(x) \equiv \mathbb{R}^{m+p}.$$

$\mathbb{R}^{m+p}$ is totally geodesic in $\mathbb{R}^{m+n}_q$. □

Corollary 3.4. *Let $f : M^m \longrightarrow \mathbb{R}^{m+n}_q$ be a 1-regular immersion. If R_1 is a parallel subbundle of rank $p < n$, then f has a substantial codimension p.*

3.2. Proof of Theorem 2.3

For totally umbilical coisotropic submanifold M, $T_1(x) = span\{N_x\}$, for each $x \in M$ and as N is a parallel vector field, T_1 is then a distribution of constant rank 1. Then the first radical space R_1 is also parallel and of constant rank 1. Use Theorem 2.2 to complete the proof. □

Remark 3.1. In the Theorem 2.3, one can replace the condition on the vector field N by $\nabla^t_{\xi_i} N = \alpha(\xi_i)N$ (parallel along the $Rad(TM)$ subbundle), where $\alpha(\xi_i)$ is a smooth function of M, because we have

$$\nabla^t_X N = 0, \quad \forall X \in \Gamma(S(TM)). \tag{3.1}$$

§4. Examples

4.1. *r*-lightlike submanifold

We consider the surface M of Euclidean space $\mathbb{R}^4_2$ with semi-Riemannian metric of signature $sig(g) = (-, -, +, +)$ by equations:

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{R}^4_2 \\ (v^1, v^2) & \longmapsto & (x^1, x^2, x^3, x^4) \end{array}$$

where

$$\begin{cases} x^1 &= v^1 \\ x^2 &= v^2 \\ x^3 &= \frac{1}{\sqrt{2}}(v^1 + v^2) \\ x^4 &= \frac{1}{2}\log(1 + (v^1 - v^2)^2) \end{cases}$$

$TM = span\{V_1, V_2\}$
with

$$\begin{aligned} V_1 = \frac{\partial}{\partial v^1} &= \frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3} + \frac{(x^1 - x^2)}{(1 + (x^1 - x^2)^2)}\frac{\partial}{\partial x^4} \\ V_2 = \frac{\partial}{\partial v^2} &= \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3} - \frac{(x^1 - x^2)}{(1 + (x^1 - x^2)^2)}\frac{\partial}{\partial x^4} \end{aligned}$$

and $TM^\perp = span\{H_1, H_2\}$,
where

$$\begin{aligned} H_1 &= \frac{\partial}{\partial x^1} + \sqrt{2}\frac{\partial}{\partial x^3} \\ H_2 &= 2(x^2 - x^1)\frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1)\frac{\partial}{\partial x^3} + (1 + (x^2 - x^1)^2)\frac{\partial}{\partial x^4}, \end{aligned}$$

moreover $H_1 = V_1 + V_2$, and

$$Rad(TM) = TM \cap TM^{\perp} = span\{\xi = H_1\}$$

is a distribution of constant rank 1. Hence the surface M is a 1-lightlike surface of $\mathbb{R}^4_2$. The vector subbundle $S(TM^{\perp})$, complementary to $rad(TM)$ in $TM^{\perp}$ is spanned by H_2.

$$S(TM^{\perp}) = span\{H_2\}.$$

The construction of lightlike transversal vector bundle $ltr(TM)$ gives:

$$ltrTM = span\left\{N = -\frac{1}{2}\frac{\partial}{\partial x^1} + \frac{1}{2}\frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3}\right\}$$

and $\overline{g}(N, N) = 0,\ \overline{g}(N, \xi) = 1$.

Put $H_1 = \xi$, $H_2 = W_2$ and $U = \sqrt{2}(1 + (x^1 - x^2))V_2$.

Therefore

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) = span\{N, W\}.$$

An easy computation gives

$$\overline{\nabla}_U U = 2(1 + (x^2 - x^1)^2)\left\{2(x^2 - x^1)\frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1)\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}\right\}$$

$\overline{\nabla}_\xi U = \overline{\nabla}_X \xi = \overline{\nabla}_X N = 0 \qquad \forall X \in \Gamma(TM)$.

Using the Gauss and Weingarten relations, we obtain

$$h^l = 0, \quad h^s(X, \xi) = 0, \quad h^s(U, U) = W$$

$\nabla_X U = \dfrac{2\sqrt{2}(x^1 - x^2)^3}{(1 + (x^1 - x^2)^2)} X^2 U$ with $X = X^1\xi + X^2 U \in \Gamma(TM)$, $A_\xi = 0,\ D^l(X, W) = 0,\ A_W \xi = 0$ and $A_W U = -2U$.

So, the surface M is non totally geodesic and the induced and transversal connections ∇ and ∇^t respectively are metric connections. The first transversal space is given by

$$T_1(x) = \{h^s(X, Y), X, Y \in \Gamma(TM)\} = S(T_x M^{\perp}).$$

The distribution T_1 is of constant rank 1. Hence M admits a reduction of its codimension to 1.

4.2. Coisotropic submanifold

Let M be a submanifold of $\mathbb{R}^5_2$, Euclidean space of $\mathbb{R}^5$ with semi-Riemannian metric of signature $sig(g) = (-,-,+,+,+)$. Suppose M is defined by equations:

$$
\text{(4.1)} \qquad \begin{cases} x^1 &= u \\ x^2 &= ((v^1)^2 + (v^2)^2)^{\frac{1}{2}} \\ x^3 &= v^1 \\ x^4 &= u \\ x^5 &= v^2 \end{cases}
$$

The tangent bundle is given by $TM = span\{U_1, U_2, U_3\}$ where

$$
\begin{aligned}
U_1 &= \frac{\partial}{\partial u} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4} \\
U_2 &= \frac{\partial}{\partial v^1} = \frac{x^3}{x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \\
U_3 &= \frac{\partial}{\partial v^2} = \frac{x^5}{x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^5}
\end{aligned}
$$

and the cotangent bundle is $TM^\perp = span\{\xi_1 = U_1, \xi_2 = x^3U_2 + x^5U_3\}$.

Then the radical subbundle is given by $Rad(TM) = TM \cap TM^\perp = TM^\perp$. Thus M is coisotropic.

The construction of lightlike transversal subbundle, $ltr(TM)$ gives:

$$ltrTM = span\{N_1, N_2\}$$

where

$$N_1 = \frac{1}{2}\left(-\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}\right) \text{ and } N_2 = \frac{1}{2(x^3)^2}\left(-x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} - x^5\frac{\partial}{\partial x^5}\right)$$

with $\overline{g}(N_i, N_j) = 0$, $\overline{g}(N_i, \xi_j) = \delta_{ij}$.

Put $TM = span\{\xi_1, \xi_2, V\}$ where $V = x^2U_3$. With a direct computation $\mathbb{R}^5_2$, we obtain

$$
\begin{gathered}
\overline{\nabla}_V\xi_1 = \overline{\nabla}_{\xi_2}\xi_1 = \overline{\nabla}_{\xi_1}\xi_2 = \overline{\nabla}_{\xi_1}V = 0, \quad \overline{\nabla}_V\xi_2 = V. \\
\overline{\nabla}_{\xi_2}\xi_2 = \xi_2, \quad \overline{\nabla}_{\xi_2}V = V, \quad \overline{\nabla}_V V = x^2\frac{\partial}{\partial x^2} + x^5\frac{\partial}{\partial x^5}.
\end{gathered}
$$

Thus the Gauss and the Weigentern formulas give

$$
\begin{gathered}
\nabla_V\xi_2 = V, \quad \nabla_{\xi_1}\xi_2 = \nabla_{\xi_1}V = 0, \quad \nabla_{\xi_2}\xi_2 = \xi_2, \quad \nabla_{\xi_2}V = V, \\
\nabla_V V = \frac{1}{2}\xi_2, \quad \nabla_X\xi_1 = 0 \qquad \forall X \in \Gamma(TM)
\end{gathered}
$$

and

$$h_1^l(X,Y)=0, \quad h_2^l(X,\xi)=0, \quad h_2^l(V,V)=-{(x^3)}^2\neq 0 \ \ \forall x\in M.$$

Then the induced connection ∇ on M, is not a metric connection, thus M is no totally geodesic. Moreover we have $\dot{A}_{\xi_1}X=\dot{A}_{\xi_2}\xi=0$, $\dot{A}_{\xi_2}V=-V$, $\forall X\in \Gamma(TM)$ and $\xi\in\Gamma(TM^{\perp})$. Therefore the first transversal space and the first radical space are

$$T_1(x)=span\{h^l(V,V)\}=span\{N_2\} \ \text{ and } \ R_1(x)=span\{\xi_2\}.$$

We have $\overline{g}(\overline{\nabla}_V\xi_2,N_1)=0$. So $R_1(x)$ is parallel $\forall x\in M$ and the rank of R_1 is constant equal to 1. The map f is 1-regular and admits a substantial codimension 1. Moreover, we have

$$h^l(X,Y)=\overline{g}(X,Y)N_2.$$

References

[1] C. Atindogbe, J-P. Ezin, J. Tossa, *Reduction of the codimension for lightlike isotropic submanifolds.* J. Geom. Phys. 42 (1-2), (2002), 1–11.

[2] M. Dajczer and all, *Submanifolds and isometric immersion,* Math. Lect. Series 13 (1990).

[3] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications.* Mathematics and Its Applications, Kluwer Acad. Publishers Dordrecht, (1996).

[4] K. L. Duggal and D. H. Jin, *Half Lightlike submanifolds of codimension 2,* Math. J. Toyama Univ. vol. 22 (1999), 121–161.

[5] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geom. 5 (1971), 333–340.

[6] A. Fridman, *Isotropic Embedding of Riemannian Manifolds into Euclidean Spaces* Rev. of Moder. Phys. 37 (1) 1965, 201–203.

[7] K. Honda, *Some Lightlike Submanifolds,* SUT Jour. of Math. vol. 37, (1), (2001) 69–78.

[8] A. Nersessian, E. Ramos, *Massive spinning particles and Geometry of null curves* Phys Lett B. 445 (1998), 123–128.

[9] J. Rosen, *Embedding of Various Relativistic Riemannian Spaces in Pseudo-Riemannian Spaces* Rev. of Moder. Phys. 37 (1) 1965, 204–214.

[10] K. Tenenblat, R. Tribuzy, *Reduction of the codimension of isometric immersions in space forms*, Topology 25 (4) (1986) 541–548.

Jean Pierre Ezin
Université d'Abomey-Calavi (UAC)
Institut de Mathématiques et de Sciences Physiques(IMSP)ICAC-3 (Affiliate Centre of the ICTP - Trieste, Italy
BP 613 Porto-Novo (Rép. du Bénin)
E-mail: jp.ezin@imsp-uac.org

Mouhamadou Hassirou
Université d'Abomey-Calavi (UAC)
Institut de Mathématiques et de Sciences Physiques(IMSP)ICAC-3 (Affiliate Centre of the ICTP - Trieste, Italy
BP 613 Porto-Novo (Rép. du Bénin)
E-mail: hassirou@imsp-uac.org

Joël Tossa
Université d'Abomey-Calavi (UAC)
Institut de Mathématiques et de Sciences Physiques(IMSP)ICAC-3 (Affiliate Centre of the ICTP - Trieste, Italy
BP 613 Porto-Novo (Rép. du Bénin)
E-mail: joel.tossa@imsp-uac.org