SUT Journal of Mathematics
Vol. 44, No. 1 (2008), 107–123

On an orbit algebra induced by the Auslander-Reiten translation for the enveloping algebra of a self-injective Nakayama algebra

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(Received February 6, 2008; Revised May 27, 2008)

Abstract. Let A be a basic self-injective Nakayama algebra over an algebraically closed field. In this paper, we investigate the ring structure of the orbit algebra $\mathbb{A}(\tau_{A^e}; A) = \bigoplus_{i\geq 0} \underline{\mathrm{Hom}}_{A^e}\left(\tau^i_{A^e}(A), A\right)$, where A^e is the enveloping algebra of A and τ_{A^e} is the Auslander-Reiten translation for A^e.

AMS 2000 *Mathematics Subject Classification.* 16D50, 16G30.

Key words and phrases. Self-injective Nakayama algebra, Auslander-Reiten translation, orbit algebra, Hochschild cohomology.

§1. Introduction

Let K be an algebraically closed field, and let R be a finite dimensional self-injective algebra over K. We denote by R^{op} the opposite algebra of R, and by $\mathrm{mod}(R)$ the category of finitely generated left R-modules. Recall from [ASS] that the *projectively stable category* $\underline{\mathrm{mod}}(R)$ of $\mathrm{mod}(R)$ is defined to be the category whose objects are the same as those of $\mathrm{mod}(R)$ and the morphism set $\underline{\mathrm{Hom}}_R(M, N)$ for M, N in $\underline{\mathrm{mod}}(R)$ is the factor space $\mathrm{Hom}_R(M, N)/\mathscr{P}(M, N)$, where $\mathscr{P}(M, N)$ is a subspace of $\mathrm{Hom}_R(M, N)$ consisting of all morphisms which factor through a projective module in $\mathrm{mod}(R)$. Dually, the *injectively stable category* $\overline{\mathrm{mod}}(R)$ of $\mathrm{mod}(R)$ is also defined. However, since R is self-injective, we obtain $\underline{\mathrm{mod}}(R) = \overline{\mathrm{mod}}(R)$.

Let M be a module in $\mathrm{mod}(R)$, and let $P_1 \xrightarrow{\rho_1} P_0 \xrightarrow{\rho_0} M \longrightarrow 0$ be a minimal projective presentation of M. Applying the functor $(-)^t := \mathrm{Hom}_R(-, R)$, we have the exact sequence of right R-modules:

$$0 \longrightarrow M^t \xrightarrow{\rho_0^t} P_0^t \xrightarrow{\rho_1^t} P_1^t \longrightarrow \mathrm{Coker}\,\rho_1^t \longrightarrow 0.$$

Then, by setting $\mathrm{Tr}_R(M) := \mathrm{Coker}\,\rho_1^t$, we obtain the duality $\mathrm{Tr}_R : \underline{\mathrm{mod}}(R) \longrightarrow \underline{\mathrm{mod}}(R^{\mathrm{op}})$ called the *transpose duality*. Moreover, we have the self-duality $\tau_R := D\mathrm{Tr}_R : \underline{\mathrm{mod}}(R) \longrightarrow \underline{\mathrm{mod}}(R)$ called the *Auslander-Reiten translation* (see [ARS], [ASS]), where D denotes the usual duality $\mathrm{Hom}_K(-, K)$. In this paper, we study a graded algebra over K induced by τ_R in the case where R is the enveloping algebra of a self-injective Nakayama algebra.

Let s be a positive integer and K an algebraically closed field, and let Γ be the cyclic quiver with s vertices $e_0, e_1, \dots, e_{s-1}$ and s arrows $a_0, a_1, \dots, a_{s-1}$, where each a_t $(0 \le t \le s-1)$ starts at e_t and ends at e_{t+1}. Here, we regard the index t of e_t modulo s. We denote by $K\Gamma$ the path algebra of Γ over K, and by X the sum of all arrows in $K\Gamma$: $X = a_0 + \cdots + a_{s-1}$. Moreover, we denote the K-algebra $K\Gamma/(X^k)$ $(k \ge 2)$ by A. It is known that A is a basic self-injective Nakayama algebra (see [ASS]). Note that the enveloping algebra $A^e := A \otimes_K A^{\mathrm{op}}$ is also a self-injective algebra. Recall that the *τ_{A^e}-orbit algebra* of A, denoted by $\mathbb{A}(\tau_{A^e}; A)$ as in [P], is a graded K-algebra defined as follows: $\mathbb{A}(\tau_{A^e}; A)$ is the direct sum of the K-vector spaces

$$\mathbb{A}(\tau_{A^e}; A) = \bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A).$$

The multiplication $\underline{f} \cdot \underline{g}$ of homogeneous elements $\underline{f} \in \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^m(A), A)$ and $\underline{g} \in \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^n(A), A)$ is the composition $\underline{f} \circ \tau_{A^e}^m(\underline{g}) \in \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^{m+n}(A), A)$.

In [P], Pogorzały describes the ring structure of $\mathbb{A}(\tau_{A^e}; A)$ by using a Galois covering of A^e in the case where the τ_{A^e}-period of A equals one, that is, $k \equiv 2 \pmod{s}$. See Remark 2.6 for $k \equiv 1 \pmod{s}$. In this paper, under the condition that $s \ge 2$ and $k \equiv 0 \pmod{s}$, we find a basis of the K-space $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ $(i \ge 0)$ by using an injective hull of $\tau_{A^e}^i(A)$ and determine the ring structure of $\mathbb{A}(\tau_{A^e}; A)$.

This paper is organized as follows: In Section 2, we will define an automorphism of categories $(-)_{\alpha^n} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ for any integer n and an automorphism α of A, and prove that $\mathbb{A}(\tau_{A^e}; A)$ is isomorphic to the orbit algebra $\bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ induced by $(-)_{\alpha^{k-2}}$ (Lemma 2.1). Next, we explicitly give a K-basis of $\mathrm{Hom}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ (Proposition 2.3). Moreover, in the case $s \ge 2$ and $k \equiv 0 \pmod{s}$, we find a K-basis of $\mathscr{P}(A_{\alpha^{-2i}}, A)$ $(i \ge 0)$ by means of the injective hull of A^e-module $A_{\alpha^{-2i}}$ given in [F], and we give a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ $(i \ge 0)$ (Theorem 2.5). In Section 3, we give a presentation of $\mathbb{A}(\tau_{A^e}; A)$ by the generators and the relations in the case $s \ge 2$ and $k \equiv 0 \pmod{s}$ (Theorems 3.2, 3.3).

§2. The stable homomorphisms

Let s be a positive integer, and let Γ be the cyclic quiver with s vertices $e_0, e_1, \ldots, e_{s-1}$ and s arrows $a_0, a_1, \ldots, a_{s-1}$, where each a_i starts at e_i and ends at e_{i+1}. Here, we regard the index i of e_i modulo s. Denote by X the sum of all arrows in the path algebra $K\Gamma$, and by A the algebra $K\Gamma/(X^k)$ ($k \geq 2$) as in Section 1. Furthermore, for simplicity, we denote a coset in A by one of its representative elements in $K\Gamma$. Then clearly the set $\{X^j e_\ell \mid 0 \leq \ell \leq s-1,\ 0 \leq j \leq k-1\}$ is a K-basis of A, and so $\dim_K A = ks$.

Our purpose in this section is to give a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ for $i \geq 0$ in the case $k \equiv 0 \pmod{s}$ (Theorem 2.5). However, the discussion in the subsections 2.1 and 2.2 are valid for arbitrary $k \geq 2$.

2.1. The algebra $\mathbb{A}(\tau_{A^e}; A)$ and an automorphism α of A

Let $\alpha : A \longrightarrow A$ be an algebra automorphism defined by $\alpha(e_t) = e_{t-1},\ \alpha(a_t) = a_{t-1}$ for $0 \leq t \leq s-1$. Then clearly $\alpha^s = \mathrm{id}_A$ holds. For any integer n and M in $\mathrm{mod}(A^e)$, we denote by M_{α^n} the left A^e-module, equivalently, the A-bimodule defined as follows: M_{α^n} has the underlying K-space M, and the operation $\cdot$ of A from the right is given by $m \cdot a = m\alpha^n(a)$ for $a \in A$, $m \in M_{\alpha^n}$, and the operation of A from the left is the usual one. Moreover, for any left A^e-homomorphism $f : M \longrightarrow N$, we define the A^e-homomorphism $f_{\alpha^n} : M_{\alpha^n} \longrightarrow N_{\alpha^n}$ by $f_{\alpha^n}(m) = f(m)$ for $m \in M_{\alpha^n}$. Then we have the automorphism of categories $(-)_{\alpha^n} : \mathrm{mod}(A^e) \longrightarrow \mathrm{mod}(A^e)$ with the inverse $(-)_{\alpha^{-n}} : \mathrm{mod}(A^e) \longrightarrow \mathrm{mod}(A^e)$ (see [H]). It is easy to see that φ is in $\mathscr{P}(M, N)$ if and only if φ_{α^n} is in $\mathscr{P}(M_{\alpha^n}, N_{\alpha^n})$. Hence the functor $(-)_{\alpha^n}$ induces the automorphism of $\underline{\mathrm{mod}}(A^e)$. We also denote this functor by $(-)_{\alpha^n}$.

It is shown in [F, Theorem] that $\tau_{A^e}^i(A) \simeq A_{\alpha^{i(k-2)}}$ as left A^e-modules for each $i \geq 0$. So, we immediately have an isomorphism $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ of K-spaces. In the following, we show that, in fact, there is an isomorphism $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ for each $i \geq 0$ which provides an isomorphism of algebras between $\mathbb{A}(\tau_{A^e}; A)$ and the orbit algebra $\bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ induced by $(-)_{\alpha^{k-2}}$.

Lemma 2.1. *There exists an isomorphism of K-spaces*

$$\theta_i : \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$$

for each $i \geq 0$ such that

$$\bigoplus_{i \geq 0} \theta_i : \mathbb{A}(\tau_{A^e}; A) \xrightarrow{\sim} \bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$$

is an isomorphism of graded K-algebras.

Proof. First note that $\tau_{A^e} \simeq \mathscr{N}\Omega^2_{A^e}$ as functors, where $\Omega_{A^e} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the *syzygy functor* and $\mathscr{N} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the *Nakayama functor* $D\mathrm{Hom}_{A^e}(-, A^e)$ (see [ARS]). Moreover Ω_{A^e} and $\mathscr{N}$ are commutative as functors, and so $\tau^i_{A^e} \simeq \mathscr{N}^i\Omega^{2i}_{A^e}$ for all $i \geq 0$ as functors.

We show the following statement from which the lemma easily follows: For each integers $i, j \geq 0$, there exists an isomorphism $\eta_{i,j} : \mathscr{N}^i\Omega^{2i}_{A^e}(A_{\alpha^j}) \longrightarrow A_{\alpha^{i(k-2)+j}}$ in $\underline{\mathrm{mod}}(A^e)$ such that, for any integers $\ell, p, q \geq 0$ and a morphism $\underline{f} : A_{\alpha^p} \longrightarrow A_{\alpha^q}$ in $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^p}, A_{\alpha^q})$, the square

$$\begin{array}{ccc} \mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^p}) & \xrightarrow{\mathscr{N}^\ell\Omega^{2\ell}_{A^e}(\underline{f})} & \mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^q}) \\ \wr\downarrow \eta_{\ell,p} & & \wr\downarrow \eta_{\ell,q} \\ A_{\alpha^{\ell(k-2)+p}} & \xrightarrow{(\underline{f})_{\alpha^{\ell(k-2)}}} & A_{\alpha^{\ell(k-2)+q}} \end{array}$$

in $\underline{\mathrm{mod}}(A^e)$ commutes.

It is shown in [EH, Section 4] that $\Omega^{2\ell}_{A^e}(A) \simeq A_{\alpha^{-\ell k}}$ for $\ell \geq 0$ as left A^e-modules, and then we easily have an isomorphism of A^e-modules $\mu_{t,r} : \Omega^{2t}_{A^e}(A_{\alpha^r}) \longrightarrow A_{\alpha^{-tk+r}}$ for $t, r \geq 0$ such that the following square in $\underline{\mathrm{mod}}(A^e)$ commutes for any $\ell, p, q \geq 0$ and $\underline{f} \in \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^p}, A_{\alpha^q})$:

$$\begin{array}{ccc} \Omega^{2\ell}_{A^e}(A_{\alpha^p}) & \xrightarrow{\Omega^{2\ell}_{A^e}(\underline{f})} & \Omega^{2\ell}_{A^e}(A_{\alpha^q}) \\ \wr\downarrow \underline{\mu_{\ell,p}} & & \wr\downarrow \underline{\mu_{\ell,q}} \\ A_{\alpha^{-\ell k+p}} & \xrightarrow{(\underline{f})_{\alpha^{-\ell k}}} & A_{\alpha^{-\ell k+q}}. \end{array}$$

Since $\nu := \alpha^{1-k} \otimes \alpha^{k-1} : A^e \longrightarrow A^e$ is a Nakayama automorphism of A^e (see [F, Appendix]), we have $\mathscr{N} \simeq F_\nu$ as functors, where $F_\nu : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the functor defined as follows: For M in $\underline{\mathrm{mod}}(A^e)$, $F_\nu(M)$ has the underlying K-space M, and the operation $\cdot$ of A^e is given by $(a \otimes b^{\mathrm{op}}) \cdot m = \nu(a \otimes b^{\mathrm{op}})m = \alpha^{1-k}(a)m\alpha^{k-1}(b)$ for $a \otimes b^{\mathrm{op}} \in A^e$ and $m \in F_\nu(M)$. Also, for $\underline{f} \in \underline{\mathrm{Hom}}_{A^e}(M, N)$, $F_\nu(\underline{f})$ is the coset $\underline{{}_\nu f} \in \underline{\mathrm{Hom}}_{A^e}(F_\nu(M), F_\nu(N))$, where ${}_\nu f \in \mathrm{Hom}_{A^e}(F_\nu(M), F_\nu(N))$ is given by ${}_\nu f(m) := f(m)$ for $m \in F_\nu(M)$.

Applying $\mathscr{N}^\ell$ to the square above yields the following commutative square in $\underline{\mathrm{mod}}(A^e)$:

$$\begin{array}{ccc} \mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^p}) & \xrightarrow{\mathscr{N}^\ell\Omega^{2\ell}_{A^e}(\underline{f})} & \mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^q}) \\ \wr\downarrow \xi_{\ell,p} & & \wr\downarrow \xi_{\ell,q} \\ F^\ell_\nu(A_{\alpha^{-\ell k+p}}) & \xrightarrow{\underline{{}_\nu(f_{\alpha^{-\ell k}})}} & F^\ell_\nu(A_{\alpha^{-\ell k+q}}). \end{array}$$

Moreover there exists the following commutative square in $\underline{\mathrm{mod}}(A^e)$:

$$\begin{array}{ccc} F_\nu^\ell(A_{\alpha^{-\ell k+p}}) & \xrightarrow{\underline{\nu(f_{\alpha^{-\ell k}})}} & F_\nu^\ell(A_{\alpha^{-\ell k+q}}) \\ \wr\downarrow \underline{\alpha^{\ell(k-1)}} & & \wr\downarrow \underline{\alpha^{\ell(k-1)}} \\ A_{\alpha^{\ell(k-2)+p}} & \xrightarrow{\underline{f_{\alpha^{\ell(k-2)}}}} & A_{\alpha^{\ell(k-2)+q}}. \end{array}$$

In the above square, the left vertical map $\underline{\alpha^{\ell(k-1)}}$ is defined by

$$\underline{\alpha^{\ell(k-1)}}(x) = \alpha^{\ell(k-1)}(x) \quad \text{for } x \in F_\nu^\ell(A_{\alpha^{-\ell k+p}}),$$

and it is verified that $\underline{\alpha^{\ell(k-1)}}$ is an A^e-homomorphism between the A^e-modules $F_\nu^\ell(A_{\alpha^{-\ell k+p}})$ and $A_{\alpha^{\ell(k-2)+p}}$. Similarly the right vertical map $\underline{\alpha^{\ell(k-1)}}$ is defined and it is also an A^e-homomorphism.

We will show the commutativity of this square. Let $q-p \equiv z \pmod s$ $(0 \le z \le s-1)$. Let $\underline{f}(e_t) = \sum_{u=0}^{k-1}\sum_{v=1}^{s} k_{u,v}^{(t)} X^u e_v$ for each t $(1 \le t \le s)$, where $k_{u,v}^{(t)} \in K$. Then we have

$$\underline{f}(e_t) = \sum_{j_t=0}^{n_t} k_{z+j_t s, w_t}^{(t)} X^{z+j_t s} e_{w_t},$$

where $t+p-q \equiv w_t \pmod s$ $(1 \le w_t \le s)$, because $\underline{f}(e_t) = \underline{f}((e_t \otimes e_{t+p}^{\mathrm{op}})\cdot e_t) = (e_t \otimes e_{t+p}^{\mathrm{op}}) \cdot \underline{f}(e_t) = e_t \underline{f}(e_t) e_{t+p-q}$. Furthermore, $\alpha(X) = X$ implies $X\underline{f}(e_t) = \underline{f}(e_{t+1})X$. Hence it follows that $k_{z+rs,w_1}^{(1)} = k_{z+rs,w_2}^{(2)} = \cdots = k_{z+rs,w_s}^{(s)}$ for each r $(0 \le r \le n)$ and $k_{z+r's,w_t}^{(t)} = 0$ for $r' > n$, where $n = \min\{n_1,\ldots,n_s\}$. Therefore we have

$$\underline{f}(1) = \sum_{t=1}^{s} \underline{f}(e_t) = \sum_{i=0}^{n} k_{z+is,w_1}^{(1)} X^{z+is}.$$

Hence we have $\underline{\alpha^j}(\underline{f}(1)) = \underline{f}(1)$ for any j. Finally, for $x \in F_\nu^\ell(A_{\alpha^{-\ell k+p}})$, we get

$$\begin{aligned}(\underline{f_{\alpha^{\ell(k-2)}}} \circ \underline{\alpha^{\ell(k-1)}})(x) &= \underline{f_{\alpha^{\ell(k-2)}}}(\underline{\alpha^{\ell(k-1)}}(x)) \\ &= \underline{f_{\alpha^{\ell(k-2)}}}(\underline{\alpha^{\ell(k-1)}}((\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot 1)) \\ &= \underline{f_{\alpha^{\ell(k-2)}}}((\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot 1) \\ &= (\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot \underline{f_{\alpha^{\ell(k-2)}}}(1) \\ &= \alpha^{\ell(k-1)}(x)\underline{f}(1)\end{aligned}$$

and

$$(\underline{\alpha^{\ell(k-1)}} \circ \underline{\nu(f_{\alpha^{-\ell k}})})(x) = \underline{\alpha^{\ell(k-1)}}(\underline{\nu(f_{\alpha^{-\ell k}})}(x))$$

$$\begin{aligned}
&= \underline{\alpha^{\ell(k-1)}}(\underline{{}_{\nu}(f_{\alpha^{-\ell k}})}((\alpha^{\ell(k-1)}(x)\otimes 1^{\mathrm{op}})\cdot 1)) \\
&= \underline{\alpha^{\ell(k-1)}}((\alpha^{\ell(k-1)}(x)\otimes 1^{\mathrm{op}})\cdot \underline{{}_{\nu}(f_{\alpha^{-\ell k}})}(1)) \\
&= (\alpha^{\ell(k-1)}(x)\otimes 1^{\mathrm{op}})\cdot \underline{\alpha^{\ell(k-1)}}(\underline{{}_{\nu}(f_{\alpha^{-\ell k}})}(1)) \\
&= \alpha^{\ell(k-1)}(x)\underline{f}(1).
\end{aligned}$$

So the square is commutative.

Combining the last two squares, we have the desired isomorphism $\eta_{i,j}$. □

2.2. The spaces of homomorphisms

Next we will give a K-basis of $\mathrm{Hom}_{A^e}(\tau^i_{A^e}(A), A)$ for $i \geq 0$. We will use the following lemma, which is an analogue of [EH, Lemma 2.1]. The proof is straightforward.

Lemma 2.2. *Let n be any integer. Then the map*

$$\mathrm{Hom}_{A^e}(A_{\alpha^n}, A) \longrightarrow {}_{\alpha^n}Z := \{x \in A \mid xy = \alpha^n(y)x \text{ for any } y \in A\}$$

given by $f \longmapsto f(1)$ is an isomorphism of K-spaces.

If $s = 1$, then we easily see that the τ_{A^e}-period of A equals one by [F, Corollary 3.7], and so the ring structure of $\mathbb{A}(\tau_{A^e}; A)$ is described in [P]. Therefore, in the rest of this paper, we assume $s \geq 2$. Also, for any integer z, denote by $\overline{z}$ the unique integer r $(0 \leq r \leq s-1)$ such that $z \equiv r \pmod s$, and let m be the unique integer such that $k = ms + \overline{k}$.

First we consider the K-space $\mathrm{Hom}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ for each $i \geq 0$. We identify $\mathrm{Hom}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ with ${}_{\alpha^{i(k-2)}}Z$ via the isomorphism in Lemma 2.2. Then we have the following proposition.

Proposition 2.3. *Let i be any non-negative integer, and set $d = -i(k-2)$. Then we have the isomorphism of K-spaces*

$$\operatorname{Hom}_{A^e}(A_{\alpha^{-d}}, A) = {}_{\alpha^{-d}}Z$$

$$= \begin{cases} \bigoplus_{j=0}^{m} KX^{js+\overline{d}} & \textit{if } \ \overline{k}-1 \neq \overline{d} < \overline{k}, \\ \Big(\bigoplus_{j=0}^{m-1} KX^{js+\overline{k}-1}\Big) \oplus \Big(\bigoplus_{\ell=0}^{s-1} KX^{ms+\overline{k}-1}e_\ell\Big) & \textit{if } \ \overline{k}-1 = \overline{d}, \\ \bigoplus_{j=0}^{m-1} KX^{js+\overline{d}} & \textit{if } \ \overline{k} \leq \overline{d} \neq s-1, \\ \bigoplus_{j=0}^{m-1} KX^{js+s-1} & \textit{if } \ \overline{d} = s-1 \textit{ and } \overline{k} \neq 0, \\ \Big(\bigoplus_{j=0}^{m-2} KX^{js+s-1}\Big) \oplus \Big(\bigoplus_{\ell=0}^{s-1} KX^{ms-1}e_\ell\Big) & \textit{if } \ \overline{d} = s-1 \textit{ and } \overline{k} = 0. \end{cases}$$

Proof. Take any $x \in {}_{\alpha^{-d}}Z$ and let $x = \sum_{j=0}^{k-1}\sum_{\ell=0}^{s-1} k_{j,\ell}X^j e_\ell$, where $k_{j,\ell} \in K$. Then we have $xe_t = xe_te_t = \alpha^{-d}(e_t)xe_t = e_{t+d}xe_t$ for each t $(0 \leq t \leq s-1)$. Furthermore, if j $(0 \leq j \leq k-1)$ satisfies $j \not\equiv d \pmod{s}$, then since $e_{t+d-j}e_t = 0$ we get $e_{t+d}X^je_t = X^je_{t+d-j}e_t = 0$. Thus we have

$$\sum_{j=0}^{k-1} k_{j,t}X^je_t = \sum_{\substack{0\leq j\leq k-1,\\ j\equiv d \,(\mathrm{mod}\, s)}} k_{j,t}X^je_t \quad \text{for each } t \ (0 \leq t \leq s-1),$$

and hence $k_{j,t} = 0$ for every t $(0 \leq t \leq s-1)$ and j $(0 \leq j \leq k-1)$ such that $j \not\equiv d \pmod{s}$. Then we have

$$x = \begin{cases} \sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}}e_\ell & \text{if } \ \overline{d} < \overline{k}, \\ \sum_{j=0}^{m-1}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}}e_\ell & \text{if } \ \overline{k} \leq \overline{d}. \end{cases}$$

Next, note that $xX = \alpha^{-d}(X)x = Xx$ holds. We consider the case $\overline{d} < \overline{k}$. If $\overline{d} \neq \overline{k}-1$, then since $xX = Xx$ we have

$$\sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}+1}e_{\ell-1} = \sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}+1}e_\ell.$$

So, for every $0 \leq j \leq m$ and $0 \leq \ell \leq s-1$, we obtain $k_{js+\overline{d},\ell+1} = k_{js+\overline{d},\ell}$, where we put $k_{js+\overline{d},s} := k_{js+\overline{d},0}$. Hence $k_{js+\overline{d},0} = k_{js+\overline{d},\ell}$ for $0 \leq j \leq m$ and

$0 \le \ell \le s-1$. This yields

$$x = \sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},0} X^{js+\overline{d}} e_\ell = \sum_{j=0}^{m} k_{js+\overline{d},0} X^{js+\overline{d}} \in \bigoplus_{j=0}^{m} K X^{js+\overline{d}}.$$

Therefore ${}_{\alpha^{-d}}Z \subseteq \bigoplus_{j=0}^{m} K X^{js+\overline{d}}$. Conversely, $X^{js+\overline{d}}$ belongs to ${}_{\alpha^{-d}}Z$, because $X^{js+\overline{d}} e_u = e_{u+d} X^{js+\overline{d}} = \alpha^{-d}(e_u) X^{js+\overline{d}}$ and $X^{js+\overline{d}} X = X^{js+\overline{d}+1} = X X^{js+\overline{d}} = \alpha^{-d}(X) X^{js+\overline{d}}$ for any $0 \le j \le m$ and $0 \le u \le s-1$. This shows $\bigoplus_{j=0}^{m} K X^{js+\overline{d}} \subseteq {}_{\alpha^{-d}}Z$. Therefore ${}_{\alpha^{-d}}Z = \bigoplus_{j=0}^{m} K X^{js+\overline{d}}$. On the other hand, if $\overline{d} = \overline{k}-1$, then since $xX = Xx$ we have

$$\sum_{j=0}^{m-1}\sum_{\ell=0}^{s-1} k_{js+\overline{k}-1,\ell} X^{js+\overline{k}} e_{\ell-1} = \sum_{j=0}^{m-1}\sum_{\ell=0}^{s-1} k_{js+\overline{k}-1,\ell} X^{js+\overline{k}} e_\ell.$$

So, for every $0 \le j \le m-1$ and $0 \le \ell \le s-1$, we obtain $k_{js+\overline{k}-1,\ell+1} = k_{js+\overline{k}-1,\ell}$, where we put $k_{js+\overline{k}-1,s} := k_{js+\overline{k}-1,0}$. Hence $k_{js+\overline{k}-1,0} = k_{js+\overline{k}-1,\ell}$ for $0 \le j \le m-1$ and $0 \le \ell \le s-1$. Then it follows that

$$\begin{aligned} x &= \sum_{j=0}^{m-1}\sum_{\ell=0}^{s-1} k_{js+\overline{k}-1,0} X^{js+\overline{k}-1} e_\ell + \sum_{\ell=0}^{s-1} k_{ms+\overline{k}-1,\ell} X^{ms+\overline{k}-1} e_\ell \\ &= \sum_{j=0}^{m-1} k_{js+\overline{k}-1,0} X^{js+\overline{k}-1} + \sum_{\ell=0}^{s-1} k_{ms+\overline{k}-1,\ell} X^{ms+\overline{k}-1} e_\ell \\ &\in \Big(\bigoplus_{j=0}^{m-1} K X^{js+\overline{k}-1}\Big) \oplus \Big(\bigoplus_{\ell=0}^{s-1} K X^{ms+\overline{k}-1} e_\ell\Big). \end{aligned}$$

Thus ${}_{\alpha^{-d}}Z = {}_{\alpha^{-\overline{k}+1}}Z \subseteq (\bigoplus_{j=0}^{m-1} K X^{js+\overline{k}-1}) \oplus (\bigoplus_{\ell=0}^{s-1} K X^{ms+\overline{k}-1} e_\ell)$. Conversely, it is easy to check that the equations $X^{js+\overline{k}-1} e_u = \alpha^{-\overline{k}+1}(e_u) X^{js+\overline{k}-1}$ and $X^{js+\overline{k}-1} X = \alpha^{-\overline{k}+1}(X) X^{js+\overline{k}-1}$ hold for every $0 \le j \le m-1$ and $0 \le u \le s-1$. Hence $X^{js+\overline{k}-1}$ is in ${}_{\alpha^{-\overline{k}+1}}Z$ for $0 \le j \le m-1$. Moreover, it follows that $X^{ms+\overline{k}-1} e_\ell$ is in ${}_{\alpha^{-\overline{k}+1}}Z$ for $0 \le \ell \le s-1$. Actually, for $0 \le \ell \le s-1$ and $0 \le u \le s-1$, we easily obtain the equations

$$(X^{ms+\overline{k}-1} e_\ell) e_u = \left\{ \begin{array}{ll} 0 & \text{if } u \ne \ell \\ X^{ms+\overline{k}-1} e_\ell & \text{if } u = \ell \end{array} \right\} = \alpha^{-\overline{k}+1}(e_u)(X^{ms+\overline{k}-1} e_\ell)$$

and

$$(X^{ms+\overline{k}-1} e_\ell) X = 0 = \alpha^{-\overline{k}+1}(X)(X^{ms+\overline{k}-1} e_\ell),$$

which mean that $X^{ms+\overline{k}-1} e_\ell$ is in ${}_{\alpha^{-\overline{k}+1}}Z$ for each $0 \le \ell \le s-1$. Accordingly, it follows that $(\bigoplus_{j=0}^{m-1} K X^{js+\overline{k}-1}) \oplus (\bigoplus_{\ell=0}^{s-1} K X^{ms+\overline{k}-1} e_\ell) \subseteq {}_{\alpha^{-\overline{k}+1}}Z$. Therefore, we get the desired equation in this case.

The desired equations in the case $\overline{k} \leq \overline{d}$ are shown in the similar way above. □

2.3. Factor through projectives

Next we will give a basis of the K-space $\mathscr{P}(A_{\alpha^{-2i}}, A)$ for $i \geq 0$ in the case $\overline{k} = 0$. Until the end of this paper, we assume $\overline{k} = 0$, i.e., $k = ms$.

Let i be an integer. Then, from [F, Lemma 4.5], we can describe an injective hull of the left A^e-module $A_{\alpha^{i(k-2)}} = A_{\alpha^{-2i}}$ as follows:

$$0 \longrightarrow A_{\alpha^{-2i}} \xrightarrow{\ \iota\ } \bigoplus_{\ell=0}^{s-1} Ae_{\ell+1} \otimes e_{\ell-2i}A,$$

where ι is given by

$$\iota(e_u) = e_u\Big(\sum_{j=0}^{ms-1} X^j \otimes X^{ms-j-1}\Big)e_{u-2i} \quad \text{for} \quad 0 \leq u \leq s-1.$$

In the following lemma, we regard $\mathscr{P}(A_{\alpha^{-2i}}, A)$ as a subspace of ${}_{\alpha^{-2i}}Z$ by means of the isomorphism in Lemma 2.2.

Lemma 2.4. *Let i be any non-negative integer.*

(1) *If $-2i \not\equiv 1 \pmod s$, then we have $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$.*

(2) *If $-2i \equiv 1 \pmod s$, then we have*
  (a) *if* $\operatorname{char} K \mid m$, *then* $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$; *and*
  (b) *if* $\operatorname{char} K \nmid m$, *then the set* $\{X^{ms-1}\}$ *is a basis of* $\mathscr{P}(A_{\alpha^{-2i}}, A)$.

Proof. Let φ be in $\mathscr{P}(A_{\alpha^{-2i}}, A)$. Then, we easily obtain an A^e-homomorphism $h : \bigoplus_{\ell=0}^{s-1} Ae_{\ell+1} \otimes e_{\ell-2i}A \longrightarrow A$ such that $\varphi = h\iota$. Hence, for each u $(0 \leq u \leq s-1)$, we have

$$\varphi(e_u) = h\iota(e_u) = \sum_{j=0}^{ms-1} X^j h(e_{u-j} \otimes e_{u-2i-j-1})X^{ms-j-1}.$$

Case $-2i \not\equiv 1 \pmod s$: Since $u - j \not\equiv u - 2i - j - 1 \pmod s$ for j $(0 \leq j \leq ms-1)$, we obtain $e_{u-j} \neq e_{u-2i-j-1}$ for j $(0 \leq j \leq ms-1)$. Then it is easy to see that $h(e_{u-j} \otimes e_{u-2i-j-1})$ is in the radical $(X)/(X^{ms})$ of A, and so $\varphi(e_u) = 0$ for each $0 \leq u \leq s-1$. This means $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$.

Case $-2i \equiv 1 \pmod s$: Since $u - j \equiv u - 2i - j - 1 \pmod s$ for j $(0 \leq j \leq ms-1)$, we have $e_{u-j} = e_{u-2i-j-1}$ for j $(0 \leq j \leq ms-1)$. Thus $h(e_{u-j} \otimes e_{u-2i-j-1}) = h(e_{u-j} \otimes e_{u-j})$ holds. We write $h(e_w \otimes e_w) = b_w e_w +$

$\sum_{i=1}^{m-1} b_{w,i}X^{is}e_w$ with $b_w, b_{w,i} \in K$ $(1 \le i \le m-1)$ for each $0 \le w \le s-1$. Then it follows that

$$\begin{aligned}\varphi(e_u) &= \sum_{j=0}^{ms-1} X^j\Big(b_{\overline{u-j}}e_{u-j} + \sum_{i=1}^{m-1} b_{\overline{u-j},i}X^{is}e_{u-j}\Big)X^{ms-j-1}\\ &= \sum_{j=0}^{ms-1} X^j b_{\overline{u-j}}e_{u-j}X^{ms-j-1}\\ &= \Big(\sum_{j=0}^{ms-1} b_{\overline{u-j}}\Big)e_uX^{ms-1} = m\Big(\sum_{j=0}^{s-1} b_j\Big)e_uX^{ms-1}.\end{aligned}$$

So we get

$$\varphi(1) = \sum_{u=0}^{s-1}\varphi(e_u) = m\Big(\sum_{j=0}^{s-1} b_j\Big)\Big(\sum_{u=0}^{s-1} e_u\Big)X^{ms-1} = m\Big(\sum_{j=0}^{s-1} b_j\Big)X^{ms-1}.$$

Conversely, take any $c \in K$, and let $\varphi : A_{\alpha^{-2i}} \longrightarrow A$ be the A^e-homomorphism given by $\varphi(1) = mcX^{ms-1}$. Then φ factors through ι. In fact, let $\eta : \bigoplus_{\ell=0}^{s-1} Ae_\ell \otimes e_\ell A \longrightarrow A$ be the A^e-homomorphism given by

$$\eta(e_\ell \otimes e_\ell) = \begin{cases} ce_0 & \text{if } \ell = 0,\\ 0 & \text{if } 1 \le \ell \le s-1.\end{cases}$$

Then, for every u $(0 \le u \le s-1)$, we obtain

$$\begin{aligned}\eta\iota(e_u) &= \eta\left(e_u\Big(\sum_{j=0}^{ms-1} X^j \otimes X^{ms-j-1}\Big)e_{u+1}\right)\\ &= \sum_{j=0}^{ms-1} X^j\eta\left(e_{u-j}\otimes e_{u-j}\right)X^{ms-j-1}\\ &= \sum_{\ell=0}^{m-1} X^{u+\ell s}ce_0X^{ms-u-\ell s-1}\\ &= mce_uX^{ms-1}.\end{aligned}$$

So one have $\eta\iota(1) = \sum_{u=0}^{s-1}\eta\iota(e_u) = \sum_{u=0}^{s-1} mce_uX^{ms-1} = mcX^{ms-1} = \varphi(1)$, which shows $\varphi = \eta\iota$. Consequently, we obtain

$$\mathscr{P}(A_{\alpha^{-2i}}, A) = \{\varphi \in \mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A) \mid \varphi(1) = mcX^{ms-1} \text{ for } c \in K\}.$$

Thus, if $\operatorname{char} K \mid m$, then $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$; and if $\operatorname{char} K \nmid m$, then by identifying $\mathscr{P}(A_{\alpha^{-2i}}, A)$ with a subspace of ${}_{\alpha^{-2i}}Z$ via the isomorphism in Lemma 2.2 we obtain a K-basis $\{X^{ms-1}\}$ of $\mathscr{P}(A_{\alpha^{-2i}}, A)$. This completes the proof. □

2.4. The spaces of stable homomorphisms

Finally, we will find a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)$ $(i \geq 0)$. If, for each $i \geq 0$, we denote by ${}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ the image of $\mathscr{P}(A_{\alpha^{-2i}}, A)$ under the isomorphism in Lemma 2.2, then we have the isomorphism of K-spaces

$$\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A) \xrightarrow{\sim} {}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}};\ \underline{f} \longmapsto f(1) + {}_{\alpha^{-2i}}Z_{\mathrm{pr}}. \tag{2.1}$$

In the following theorem, we regard $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ $\big(\simeq \underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)\big)$ as ${}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ for $i \geq 0$ by using the isomorphism above.

Theorem 2.5. *Let $k = ms$ for $m \geq 1$ and $s \geq 2$. Then, for any non-negative integer i, we have the following*:

(1) *If $-2i \not\equiv 1 \pmod{s}$, then the set*

$$\{X^{\overline{2i}+js} \mid 0 \leq j \leq m-1\}$$

is a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)$.

(2) *If $-2i \equiv 1 \pmod{s}$, then we have*

  (a) *if* $\mathrm{char}\, K \mid m$, *then the set*

$$\{X^{js+s-1}, X^{ms-1}e_\ell \mid 0 \leq j \leq m-2,\ 0 \leq \ell \leq s-1\}$$

  is a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)$; and

  (b) *if* $\mathrm{char}\, K \nmid m$, *then the set*

$$\{X^{js+s-1}, Y_\ell \mid 0 \leq j \leq m-2,\ 0 \leq \ell \leq s-2\}$$

  is a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)$ where $Y_\ell := \sum_{j=0}^{\ell} X^{ms-1}e_j$ for $0 \leq \ell \leq s-2$.

Proof. Since $\overline{k} = 0$ and $\overline{-i(k-2)} = \overline{2i}$, it follows from Proposition 2.3 that, if $\overline{2i} \neq s-1$, that is, $-2i \not\equiv 1 \pmod{s}$, then the set

$$\left\{X^{\overline{2i}+js} \;\middle|\; 0 \leq j \leq m-1\right\} \tag{2.2}$$

is a K-basis of $\mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A)$; and if $\overline{2i} = s-1$, that is, $-2i \equiv 1 \pmod{s}$, then the set

$$\left\{X^{js+s-1}, X^{ms-1}e_\ell \;\middle|\; 0 \leq j \leq m-2,\ 0 \leq \ell \leq s-1\right\} \tag{2.3}$$

is a K-basis of $\mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A)$. So, if $-2i \not\equiv 1 \pmod{s}$, then by Lemma 2.4 (1) we have a K-basis

$$\left\{X^{\overline{2i}+js} \;\middle|\; 0 \leq j \leq m-1\right\}$$

of $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$; and if $-2i \equiv 1 \pmod{s}$ and $\operatorname{char} K \mid m$, then by Lemma 2.4 (2)(a) we obtain a K-basis

$$\left\{X^{js+s-1}, X^{ms-1}e_\ell \;\middle|\; 0 \leq j \leq m-2,\ 0 \leq \ell \leq s-1\right\}$$

of $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$. On the other hand, if $-2i \equiv 1 \pmod{s}$ and $\operatorname{char} K \nmid m$, then by Lemma 2.4 (2)(b) we have a K-basis

$$\left\{X^{js+s-1}, Y_\ell \;\middle|\; 0 \leq j \leq m-2,\ 0 \leq \ell \leq s-2\right\}$$

of $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$, where we put $Y_\ell := \sum_{j=0}^{\ell} X^{ms-1}e_j \in \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ for $0 \leq \ell \leq s-2$. □

Remark 2.6. We consider the case $k \equiv 1 \pmod{s}$. Then, A is exactly a symmetric algebra (see [T, Lemma 3.1]), and hence A^e is also a symmetric algebra (see [EN, Proposition 2]). So $\tau^i_{A^e}(A) \simeq \Omega^{2i}_{A^e}(A)$ for $i \geq 0$ as A^e-modules, which yields $\underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A) \simeq \underline{\mathrm{Hom}}_{A^e}(\Omega^{2i}_{A^e}(A), A)$ as K-spaces for each $i \geq 0$. Moreover, since A is self-injective, we have $\underline{\mathrm{Hom}}_{A^e}(\Omega^{2i}_{A^e}(A), A) \simeq \mathrm{Ext}^{2i}_{A^e}(A, A)$ for each $i \geq 1$. Therefore $\underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)$ is isomorphic to the $2i$th Hochschild cohomology group $\mathrm{HH}^{2i}(A) := \mathrm{Ext}^{2i}_{A^e}(A, A)$ for each $i \geq 1$. In [H], Holm computes the dimension of $\mathrm{HH}^{2i}(A)$ $(i \geq 0)$ and describes the even Hochschild cohomology ring $\mathrm{HH}^{ev}(A) = \bigoplus_{i \geq 0} \mathrm{HH}^{2i}(A)$ (see also [EH]).

§3. The ring structure of $\mathbb{A}(\tau_{A^e}; A)$

Throughout this section, we keep the notation from Section 2, and assume that $\overline{k} = 0$, i.e., $k = ms$ $(m \geq 1,\ s \geq 2)$. The purpose in this section is to give the generators and the relations of $\mathbb{A}(\tau_{A^e}; A) = \bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(\tau^i_{A^e}(A), A)$ as K-algebra, explicitly, in the similar way in [EH] and [H].

Since, by Lemma 2.1, the algebra $\mathbb{A}(\tau_{A^e}; A)$ is isomorphic to the orbit algebra $\bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ induced by the functor $(-)_{\alpha^{-2}}$, it suffices to consider the algebra $\bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$. As in Theorem 2.5, for each $i \geq 0$, we identify $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ with ${}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ via the isomorphism (2.1).

The following lemma says that the multiplication $\cdot$ in $\bigoplus_{i \geq 0} {}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}} = \bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ is induced by that of A. Here, for simplicity, we set $\mathbb{A}_i := {}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ $(i \geq 0)$ and denote a coset $x + {}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ in $\mathbb{A}_i$ $(i \geq 0)$ by $[x]$.

Lemma 3.1. *Let i and j be any non-negative integers. Then $xy = yx$ in A for $x \in {}_{\alpha^{-2i}}Z$ and $y \in {}_{\alpha^{-2j}}Z$. Furthermore, for $[x] = x + {}_{\alpha^{-2i}}Z_{\mathrm{pr}} \in \mathbb{A}_i$ and $[y] = y + {}_{\alpha^{-2j}}Z_{\mathrm{pr}} \in \mathbb{A}_j$, the multiplication $[x] \cdot [y]$ in $\bigoplus_{i \geq 0} \mathbb{A}_i$ is given by $[x] \cdot [y] = [xy] \in \mathbb{A}_{i+j}$. Consequently, $[x]$ and $[y]$ are commutative.*

Proof. For each $\ell \geq 0$, ${}_{\alpha^{-2\ell}}Z$ has K-basis (2.2) if $-2\ell \not\equiv 1 \pmod{s}$, and has K-basis (2.3) if $-2\ell \equiv 1 \pmod{s}$. Therefore, we easily see that $x \in {}_{\alpha^{-2i}}Z$ and $y \in {}_{\alpha^{-2j}}Z$ are commutative.

Now, by Lemma 2.2, there exist A^e-homomorphisms $f \in \mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A)$ and $g \in \mathrm{Hom}_{A^e}(A_{\alpha^{-2j}}, A)$ satisfying $f(1) = x$ and $g(1) = y$. Moreover the multiplication $\underline{f} \cdot \underline{g}$ in the orbit algebra $\bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ is given by

$$\underline{f} \cdot \underline{g} = \underline{f} \circ (\underline{g})_{\alpha^{-2i}} = \underline{f} \circ \underline{g_{\alpha^{-2i}}} = \underline{f \circ g_{\alpha^{-2i}}} \in \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2(i+j)}}, A).$$

Then, since

$$[f \circ g_{\alpha^{-2i}}(1)] = [f \circ g(1)] = [f(g(1))] = [f(g(1)1)] = [g(1)f(1)] = [yx] = [xy]\,,$$

it follows that $[x] \cdot [y] = [xy]$. □

Now we give the generators and the relations of the algebra $\bigoplus_{i \geq 0} \mathbb{A}_i$ ($\simeq \mathbb{A}(\tau_{A^e}; A)$). Note that $\bigoplus_{i \geq 0} \mathbb{A}_i$ is a commutative graded K-algebra by Lemma 3.1.

First we consider the case when s is even. We put $s = 2t$ for an integer $t \geq 1$. Then, for each $i \geq 0$, since $-2i \not\equiv 1 \pmod{2t}$, by Theorem 2.5 (1) we obtain the K-basis

$$\{X^{\overline{2i}+2tj} \mid 0 \leq j \leq m-1\}$$

of $\mathbb{A}_i$. It is easy to see that, if we set $i = qt + r$ $(0 \leq r \leq t-1)$, then this basis can be written as

$$\{X^{2r+2tj} \mid 0 \leq j \leq m-1\}.$$

Here note that $\mathbb{A}_{i+t} = \mathbb{A}_i$ holds for each $i \geq 0$. We set $y_0 := X^{2t} \in \mathbb{A}_0$. Then, by Lemma 3.1, we have

$$y_0^j = X^{2tj} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_0$, and we have the following relation:

(1) $y_0^m = 0$.

Next, we put $y_1 := X^2 \in \mathbb{A}_1$. Then, for $1 \leq i \leq t-1$, we obtain

$$y_0^j \cdot y_1^i = X^{2jt+2i} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_i$. Furthermore, we set $y_t := 1 \in \mathbb{A}_t (= \mathbb{A}_0)$. Then, for any $\ell \geq t$, by letting $\ell = qt + r$ $(0 \leq r \leq t-1)$, we have

$$y_0^j \cdot y_1^r \cdot y_t^q = X^{2jt+2r} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_\ell$, and we have the following relation:

(2) $y_0 \cdot y_t = y_1^t$.

Summarizing these results, we have the following theorem.

Theorem 3.2. *Let $k = ms$ and $s = 2t$ $(t \geq 1)$. Then $\mathbb{A}(\tau_{A^e}; A)$ is isomorphic to the commutative graded K-algebra $K[y_0, y_1, y_t]/(y_0^m, y_0 \cdot y_t - y_1^t)$, where* $\deg y_i = i$ $(i = 0, 1, t)$.

Next we consider the case when s is odd. We put $s = 2t + 1$ for an integer $t \geq 1$. For each $i \geq 0$ with $i \not\equiv t \pmod{2t+1}$, since $-2i \not\equiv 1 \pmod{2t+1}$, by Theorem 2.5 (1) we obtain the K-basis

$$\{X^{\overline{2i}+(2t+1)j} \mid 0 \leq j \leq m-1\}$$

of $\mathbb{A}_i$. It is easy to see that, if we set $i = q(2t+1) + r$ $(0 \leq r \leq 2t,\ r \neq t)$, then this basis can be written as follows:

$$\{X^{2r+(2t+1)j} \mid 0 \leq j \leq m-1\} \quad \text{if } 0 \leq r \leq t-1,$$

and

$$\{X^{2r-(2t+1)+(2t+1)j} \mid 0 \leq j \leq m-1\} \quad \text{if } t+1 \leq r \leq 2t.$$

On the other hand, for each $i \geq 0$ with $i \equiv t \pmod{2t+1}$, by Theorem 2.5 (2) we have the following K-basis of $\mathbb{A}_i$:

$$\{X^{2t+(2t+1)j}, X^{(2t+1)m-1}e_\ell \mid 0 \leq j \leq m-2,\ 0 \leq \ell \leq 2t\} \quad \text{if char}\, K \mid m,$$

and

$$\{X^{2t+j(2t+1)}, Y_\ell \mid 0 \leq j \leq m-2,\ 0 \leq \ell \leq 2t-1\} \quad \text{if char}\, K \nmid m,$$

where $Y_\ell := \sum_{j=0}^{\ell} X^{(2t+1)m-1}e_j \in \mathbb{A}_i$ for $0 \leq \ell \leq 2t-1$.

First assume $\text{char}\, K \mid m$. We put

$$z_0 := X^{2t+1} \in \mathbb{A}_0.$$

Then by Lemma 3.1 we have

$$X^{(2t+1)j} = z_0^j \quad \text{for } 1 \leq j \leq m-1$$

in $\mathbb{A}_0$, and we obtain the following relation:

(1) $z_0^m = 0$.

We set

$$z_1 := X^2 \in \mathbb{A}_1 \quad \text{and} \quad z_{t,\ell} := X^{m(2t+1)-1}e_\ell \in \mathbb{A}_t \quad \text{for } 0 \leq \ell \leq 2t.$$

Then for each $1 \leq i \leq t$ we have

$$X^{2i+(2t+1)j} = z_0^j \cdot z_1^i \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_i$, and we obtain the following relations:

(2) $z_0^{m-1} \cdot z_1^t = \sum_{\ell=0}^{2t} z_{t,\ell}$,

(3) $z_0 \cdot z_{t,\ell} = 0$ for $0 \leq \ell \leq 2t$,

(4) $z_1 \cdot z_{t,\ell} = 0$ for $0 \leq \ell \leq 2t$,

(5) $z_{t,u} \cdot z_{t,v} = 0$ for $0 \leq u, v \leq 2t$.

Next we set $z_{t+1} := X \in \mathbb{A}_{t+1}$. Then for each $t+1 \leq i \leq 2t$ we have

$$X^{2i-(2t+1)+(2t+1)j} = z_0^j \cdot z_1^{i-(t+1)} \cdot z_{t+1} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_i$, and we obtain the following relations:

(6) $z_1^{t+1} = z_0 \cdot z_{t+1}$,

(7) $z_{t+1} \cdot z_{t,\ell} = 0$ for $0 \leq \ell \leq 2t$.

Furthermore, we set $z_{2t+1} := 1 \in \mathbb{A}_{2t+1}(= \mathbb{A}_0)$. Then, for any $\ell \geq 2t+1$, let $\ell = q(2t+1) + r$ $(0 \leq r \leq 2t)$. If $0 \leq r \leq t-1$, then

$$X^{2r+(2t+1)j} = z_0^j \cdot z_1^r \cdot z_{2t+1}^q \quad \text{for } 0 \leq j \leq m-1.$$

If $r = t$, then

$$X^{2t+(2t+1)j} = z_0^j \cdot z_1^t \cdot z_{2t+1}^q, \quad X^{m(2t+1)-1}e_\ell = z_{t,\ell} \cdot z_{2t+1}^q \quad \text{for } 0 \leq j \leq m-2.$$

If $t+1 \leq r \leq 2t$, then

$$X^{2r-(2t+1)+(2t+1)j} = z_0^j \cdot z_1^{r-(t+1)} \cdot z_{t+1} \cdot z_{2t+1}^q \quad \text{for } 0 \leq j \leq m-1.$$

So we obtain the following relations:

(8) $z_0 \cdot z_{2t+1} = z_1^t \cdot z_{t+1}$,

(9) $z_{t+1}^2 = z_1 \cdot z_{2t+1}$.

Next we assume $\operatorname{char} K \nmid m$. As in the above we put $z_0 := X^{2t+1} \in \mathbb{A}_0$, $z_1 := X^2 \in \mathbb{A}_1$, $z_{t+1} := X \in \mathbb{A}_{t+1}$, and $z_{2t+1} := 1 \in \mathbb{A}_{2t+1}$. Moreover, we set $z'_{t,\ell} := Y_\ell$ for $0 \leq \ell \leq 2t-1$. Then these elements are generators of $\bigoplus_{i \geq 0} \mathbb{A}_i$. Thus we obtain the relations (1), (6), (8) and (9) above and the following relations:

(2′) $z_0^{m-1} \cdot z_1^t = 0$,

(3′) $z_0 \cdot z'_{t,\ell} = 0$ for $0 \leq \ell \leq 2t-1$,

(4′) $z_1 \cdot z'_{t,\ell} = 0$ for $0 \leq \ell \leq 2t-1$, (5′) $z_{t,u} \cdot z_{t,v}$ for $0 \leq u, v \leq 2t-1$,

(7′) $z_{t+1} \cdot z'_{t,\ell} = 0$ for $0 \leq \ell \leq 2t-1$.

Summarizing these results, we have the following theorem.

Theorem 3.3. *Let* $k = ms$ *and* $s = 2t+1$ $(t \geq 1)$. *If* $\operatorname{char} K \mid m$, *then* $\mathbb{A}(\tau_{A^e}; A)$ *is a commutative graded algebra with generators* $z_0, z_1, z_{t,\ell}$ $(0 \leq \ell \leq 2t), z_{t+1}, z_{2t+1}$ *where* $\deg z_i = i$ $(i = 0, 1, t+1, 2t+1)$ *and* $\deg z_{t,\ell} = t$, *and relations*

$$z_0^m = 0, \quad z_{t+1}^2 = z_1 \cdot z_{2t+1}, \quad z_1^{t+1} = z_0 \cdot z_{t+1}, \quad z_0 \cdot z_{2t+1} = z_1^t \cdot z_{t+1},$$

$$z_0^{m-1} \cdot z_1^t = \textstyle\sum_{\ell=0}^{2t} z_{t,\ell}, \quad z_{t,u} \cdot z_{t,v} = 0 \text{ for } 0 \leq u, v \leq 2t,$$

$$z_j \cdot z_{t,\ell} = 0 \text{ for } j = 0, 1, t+1 \text{ and } 0 \leq \ell \leq 2t.$$

And if $\operatorname{char} K \nmid m$, *then* $\mathbb{A}(\tau_{A^e}; A)$ *is a commutative graded algebra with generators* $z_0, z_1, z'_{t,\ell}$ $(0 \leq \ell \leq 2t-1), z_{t+1}, z_{2t+1}$ *where* $\deg z_i = i$ $(i = 0, 1, t+1, 2t+1)$ *and* $\deg z'_{t,\ell} = t$, *and relations*

$$z_0^m = 0, \quad z_{t+1}^2 = z_1 \cdot z_{2t+1}, \quad z_1^{t+1} = z_0 \cdot z_{t+1}, \quad z_0 \cdot z_{2t+1} = z_1^t \cdot z_{t+1},$$

$$z_0^{m-1} \cdot z_1^t = 0, \quad z'_{t,u} \cdot z'_{t,v} = 0 \text{ for } 0 \leq u, v \leq 2t-1,$$

$$z_j \cdot z'_{t,\ell} = 0 \text{ for } j = 0, 1, t+1 \text{ and } 0 \leq \ell \leq 2t-1.$$

Acknowledgement

The authors would like to thank the referee for valuable suggestions. And we would like to express our thanks to Professor Katsunori Sanada and Doctor Manabu Suda for many suggestions.

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