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Vertex-disjoint t-claws in graphs

Shuya Chiba

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Abstract. Let $\delta(G)$ denote the minimum degree of a graph G. We prove that for $t \geq 4$ and $k \geq 2$, a graph G of order at least $(t+1)k + \frac{11}{6}t^2$ with $\delta(G) \geq k + t - 1$ contains k pairwise vertex-disjoint copies of $K_{1,t}$.

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§1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G), E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. A graph F is called a t-claw if F is isomorphic to $K_{1,t}$.

Let H be a fixed connected graph, and let $k \geq 2$ be a fixed integer. In this paper, we are concerned with the existence of k pairwise vertex-disjoint copies of H in a graph G. The main theorem of this paper deals with the case where $|V(G)| > k|V(H)|$, but we start with results which deal with the case where $|V(G)| = k|V(H)|$. For $H = K_t$ with $t \geq 2$, Hajnal and Szemerédi [6] proved that if $|V(G)| = kt$ and $\delta(G) \geq \frac{t-1}{t}|V(G)|$, then G contains k pairwise vertex-disjoint copies of K_t (see also Corrádi and Hajnal [1]). For $H = P_t$ with $t(\geq 3)$ odd, it is easy to see that if $|V(G)| = kt$ and $\delta(G) \geq \frac{|V(G)|-2}{2}$, then G contains k pairwise vertex-disjoint copies of P_t (for results concerning the case where it is assumed that G is connected, the reader is referred to Johansson [7] and Enomoto, Kaneko and Tuza [4]).

Note that $P_3 \cong K_{1,2}$ is a 2-claw. Thus letting $t = 2$ in the above result, we obtain the following proposition.

Proposition 1. *Let $k \geq 2$ be an integer, and let G be a graph of order $3k$ such that $\delta(G) \geq (3k-2)/2$. Then G contains k pairwise vertex-disjoint 2-claws.*

In the case where H is a 3-claw, Egawa, Fujita and Ota [2] proved the following theorem.

Theorem 2 (Egawa, Fujita and Ota [2]). *Let $k \geq 2$ be an integer, and let G be a graph of order $4k$ such that $\delta(G) \geq 2k$. Then G contains k pairwise vertex-disjoint 3-claws, unless k is odd and G is isomorphic to $K_{2k,2k}$.*

In Proposition 1 and Theorem 2, the condition on the minimum degree is sharp. However, if we assume that the order of G is slightly greater than $3k$ or $4k$, then a much weaker condition on the minimum degree guarantees the existence of k pairwise vertex-disjoint 2-claws or 3-claws.

Theorem 3 (Ota [8]). *Let $k \geq 2$ be an integer, and let G be a graph of order at least $3k+2$ such that $\delta(G) \geq k+1$. Then G contains k pairwise vertex-disjoint 2-claws.*

Theorem 4 (Egawa and Ota [3]). *Let $k \geq 2$ be an integer, and let G be a graph of order at least $4k+6$ such that $\delta(G) \geq k+2$. Then G contains k pairwise vertex-disjoint 3-claws.*

Based on these results, Ota [8] made the following conjecture.

Conjecture 5 (Ota [8]). *Let $t \geq 2$, $k \geq 2$ be integers, and let G be a graph of order at least $(t+1)k + t^2 - t$ such that $\delta(G) \geq k+t-1$. Then G contains k pairwise vertex-disjoint t-claws.*

As is shown in [8], in this conjecture, the condition on the minimum degree of G is sharp in the sense that for any fixed t and k, there exists a graph of arbitrarily large order which has minimum degree $k+t-2$ but does not contain k vertex-disjoint t-claws and, if k is suffciently large compared with t, then the condition on the order of G is also sharp in the sense that there exists a graph G with $|V(G)| = (t+1)k + t^2 - t - 1$ and $\delta(G) \geq k+t-1$ such that G does not contain k vertex-disjoint t-claws. Theorems 3 and 4 above show that the conjecture is true for $t = 2, 3$. For $t \geq 4$, Ota [8; Theorem 1] proved the following theorem.

Theorem 6 (Ota [8]). *Let $t \geq 4$, $k \geq 2$ be intgers, and let G be a graph of order at least $(t+1)k + 2t^2 - 3t - 1$ such that $\delta(G) \geq k+t-1$. Then G contains k pairwise vertex-disjoint t-claws.*

The coefficient -3 of t in the lower bound on $|V(G)|$ was improved to -4 by Fujita in [5].

Theorem 7 (Fujita [5]). *Let $t \geq 4$, $k \geq 2$ be intgers, and let G be a graph of order at least $(t+1)k + 2t^2 - 4t + 2$ such that $\delta(G) \geq k+t-1$. Then G contains k pairwise vertex-disjoint t-claws.*

The purpose of this paper is to improve the coefficient of t^2 as follows.

Main Theorem *Let $t \geq 4$, $k \geq 2$ be intgers, and let G be a graph of order at least $(t+1)k + \frac{11}{6}t^2$ such that $\delta(G) \geq k + t - 1$. Then G contains k pairwise vertex-disjoint t-claws.*

We need the following notation and terminology. Let G be a graph. For a vertex $v \in V(G)$, we denote by $N(v) = N_G(v)$ and $d_G(v)$ the set of vertices adjacent to v and the degree of v, respectively; thus $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, we let $\langle S \rangle = \langle S \rangle_G$ denote the subgraph of G induced by S. For disjoint subsets S and T of $V(G)$, we let $E(S,T) = E_G(S,T)$ denote the set of edges of G joining a vertex in S and vertex in T. When S or T contains of a single vertex, say $S = \{x\}$ or $T = \{y\}$, we write $E(x,T)$ or $E(S,y)$ for $E(S,T)$.

§2. Preparation for the proof of the main theorem

By way of contradiction, suppose that there exists a graph G with $|V(G)| \geq (t+1)k + \frac{11}{6}t^2$ and $\delta(G) \geq k+t-1$ such that G does not contain k pairwise vertex-disjoint t-claws. By Theorem 7, we have $|V(G)| \leq (t+1)k + 2t^2 - 4t + 1$. Since $\lceil \frac{11}{6}t^2 \rceil \geq 2t^2 - 4t + 2$ for $4 \leq t \leq 23$, this implies $t \geq 24$. We may assume that G is an edge-maximal counterexample. Then G contains $k-1$ vertex-disjoint t-claws, say $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$. Set $H = G - (\bigcup_{i=1}^{k-1} V(C^{(i)}))$. Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be the K_t components of H, i.e.,the components of H isomorphic to K_t. Define $U = \bigcup_{\alpha=1}^{s} V(P^{(\alpha)})$ and $W = V(H) - U$. We may assume that $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$ are chosen so that $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$ is as large as possible.

By assumption, H contains no t-claw, or equivalently, every vertex of H has degree at most $t-1$. We define $n = |V(H)|$. Since $n = |V(G)| - (t+1)(k-1)$, we have $\frac{11}{6}t^2 + t + 1 \leq n \leq 2t^2 - 3t + 2$. For each i, let $a^{(i)}$ be the center of $C^{(i)}$ and $B^{(i)} = \{b_1^{(i)}, b_2^{(i)}, \ldots, b_t^{(i)}\}$ be the set of leaves of $C^{(i)}$. In the following argument, we sometimes fix i and set $C = C^{(i)}$. In such cases, we write $a, B, b_1, b_2, \ldots, b_t$ instead of $a^{(i)}, B^{(i)}, b_1^{(i)}, b_2^{(i)}, \ldots, b_t^{(i)}$, respectively.

We first state seven lemmas concerning the number of edges between $V(C^{(i)})$ and $V(H)$, which are proved in [5; Lemmas 2.1 through 2.7]. Fix i with $1 \leq i \leq k-1$. Thus as mentioned in the preceding paragraph, a denotes the center of $C = C^{(i)}$, and $B = \{b_1, b_2, \ldots, b_t\}$ denotes the set of leaves of C.

Lemma 2.1. *Let $v \in V(H)$, and suppose that $d_H(v) + |E(B,v)| \geq t$. Then $|E(a, V(H) - \{v\} - N_H(v))| \leq t - 1 - d_H(v)$.*

Lemma 2.2. *If $E(a, V(H)) \neq \emptyset$, then $|E(b_p, V(H))| \leq t$ for every $b_p \in B$.*

Lemma 2.3. *If* $|((N(b_p) \cup N(b_q)) \cap V(H)| \geq 2t-1$ *for* $b_p, b_q \in B$ *with* $p \neq q$, *then* $|E(b_p, V(H))| \leq t-2$ *or* $|E(b_q, V(H))| \leq t-2$.

Lemma 2.4. *Let* $v \in V(H)$, *and suppose that* $d_H(v) + |E(B, v)| \geq t+1$. *Then* $|E(b_p, V(H) - \{v\} - N_H(v))| \leq t-2$ *for every* $b_p \in B$.

Lemma 2.5. *Let* P *be a* K_t *component of* H, *and suppose that there exists* $v \in V(H) - V(P)$ *such that* $d_H(v) + |E(V(C), v)| \geq t+1$. *Then* $E(V(C), V(P)) = \emptyset$.

Lemma 2.6. *Let* P *be a* K_t *component of* H, *and suppose that there exists* $v \in V(H) - V(P)$ *such that* $|E(V(C), v)| \geq 2$. *Then* $E(B, V(P)) = \emptyset$, *and hence it follows that* $|E(V(C), V(P))| \leq t$.

Lemma 2.7. *Let* P *be a* K_t *component of* H, *and suppose that* $E(V(C), V(H) - V(P)) \neq \emptyset$. *Then* $|E(V(C), V(P))| \leq t$.

In the rest of this section, we consider the case where $s \geq t+1$. For each α with $1 \leq \alpha \leq t+1$, we take $u_\alpha \in V(P^{(\alpha)})$. Since

$$\sum_{i=1}^{k-1} \sum_{\alpha=1}^{t+1} |E(V(C^{(i)}), u_\alpha)| = \sum_{\alpha=1}^{t+1} (d_G(u_\alpha) - (t-1)) \geq (t+1)k,$$

there exists an index i with $1 \leq i \leq k-1$ such that $\sum_{\alpha=1}^{t+1} |E(V(C^{(i)}), u_\alpha)| > t+1$. Then there exist two edges xu_α and yu_β joining $V(C^{(i)})$ and $\{u_1, u_2, \ldots, u_{t+1}\}$ with $x, y \in V(C^{(i)})$, $x \neq y$ and $\alpha \neq \beta$. Replacing $C^{(i)}$ by t-claws contained in $\langle \{x\} \cup V(P^{(\alpha)}) \rangle$ and $\langle \{y\} \cup V(P^{(\beta)}) \rangle$, we obtain k vertex-disjoint t-claws in G. This is a contradiction.

§3. The case where $s = t$

We continue with the notation of the preceding section. In order to prove the main theorem, we shall choose some $C^{(i)}$'s and show that they together with some vertices in H contain more t-claws, which contradicts the assumption that G is a counterexample. In this section, we consider the case where $s = t$. For each α with $1 \leq \alpha \leq t$, we take a vertex $u_\alpha \in V(P^{(\alpha)})$, and let $v \in W$. Define

$$J = \{i \mid 1 \leq i \leq k-1, \ \ |E(V(C^{(i)}), \{u_1, u_2, \ldots, u_t, v\})| \geq t+2\}.$$

The following two lemmas are proved in [5; Lemmas 3.1 and 3.2].

Lemma 3.1. *Suppose that* $C = C^{(i)}$ *satisfies* $|E(V(C), \{u_1, u_2, \ldots, u_t, v\})| \geq t+2$. *Then the following hold.*

(i) $2 \leq |E(V(C), v)| \leq t$.

(ii) $E(B, \{u_1, u_2, \ldots, u_t, v\}) = \emptyset$.

Lemma 3.2. $\sum_{i \in J} |E(V(C^{(i)}), v)| \geq |J| + t + 1$.

We may assume that $J = \{1, 2, \ldots, m\}$ where $m = |J|$, and $|E(V(C^{(1)}), v)| \geq |E(V(C^{(2)}), v)| \geq \cdots \geq |E(V(C^{(m)}), v)| \geq 2$. By Lemmas 3.1(i) and 3.2, there exists $l \in J$ with $2 \leq l \leq m$ such that

$$\sum_{i=1}^{l} (|E(V(C^{(i)}), v)| - 1) \geq t \tag{3.1}$$

and such that $\sum_{i=1}^{l-1}(|E(V(C^{(i)}), v)| - 1) \leq t - 1$. By Lemma 3.1(i), we also have $\sum_{j=1}^{i}(|E(V(C^{(j)}), v)| - 1) \leq t - 1$ for each $1 \leq i \leq l - 1$, and $l - 1 \leq \sum_{i=1}^{l-1}(|E(V(C^{(i)}), v)| - 1) \leq t - 1$. Thus $2 \leq l \leq t$. The following lemma is proved in [5; Lemma 3.3].

Lemma 3.3. *We have* $|E(a^{(i)}, \{u_1, u_2, \ldots, u_t\})| \geq i$ *for each* $1 \leq i \leq l$.

Now by Lemma 3.3, we may assume that we can take l independent edges $a^{(i)}u_i$, $1 \leq i \leq l$. On the other hand, (3.1) implies that $\sum_{i=1}^{l} |E(B^{(i)}, v)| \geq t$. Hence we can take $X \subset N(v) \cap \left(\bigcup_{i=1}^{l} B^{(i)}\right)$ with $|X| = t$. Then each of $\langle X \cup \{v\} \rangle$ and $\langle \{a^{(i)}\} \cup V(P^{(i)}) \rangle$ for $1 \leq i \leq l$ contains a t-claw. These are $l+1$ vertex-disjoint t-claws in $\langle (\bigcup_{i=1}^{l} V(C^{(i)})) \cup V(H) \rangle$, which contradicts the assumption that G is a counterexample.

§4. Counting argument

Throughout the rest of this paper, we assume that $s \leq t - 1$. In this section, we find a good vertex in H that can be used later to find an extra t-claw. The lemmas proved in this section are actually proved in [5], but we include their proofs for the convenience of the reader. Recall that U is the set of vertices contained in the K_t components of H, and $W = V(H) - U$. We define

$$I = \{i \mid 1 \leq i \leq k-1, \;\; E(V(C^{(i)}), W) = \emptyset\},$$
$$J = \{i \mid 1 \leq i \leq k-1, \;\; i \notin I, \;\; |E(V(C^{(i)}), V(H))| \geq n - s + 1\}.$$

Note that since $n \geq \frac{11}{6}t^2 + t + 1$ and $s \leq t - 1$, we have $|E(V(C^{(i)}), V(H))| \geq \frac{11}{6}t^2 + 3$ for each $i \in J$.

Lemma 4.1. *There exists* $v \in W$ *such that*

$$d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \geq |J| + t.$$

Proof. Suppose that

$$d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \leq |J| + t - 1 \text{ for all } v \in W. \tag{4.1}$$

We first claim that $|E(V(C^{(i)}), U)| \leq t(t+1)$ for each $i \in I$. If $V(C^{(i)})$ is joined by edges to at most one component of $\langle U \rangle$, then the claim is obvious. If $V(C^{(i)})$ is joined to at least two components of $\langle U \rangle$, then by Lemma 2.7, $|E(V(C^{(i)}), U)| \leq ts < t(t+1)$. Thus the claim follows. Note that this claim implies that

$$\sum_{i \in I} |E(V(C^{(i)}), U)| \leq t(t+1)|I|. \tag{4.2}$$

For $i \in J$, since $E(V(C^{(i)}), W) \neq \emptyset$, it follows from Lemma 2.7 that $|E(V(C^{(i)}), U)| \leq ts$. Hence

$$\sum_{i \in J} |E(V(C^{(i)}), U)| \leq ts|J|. \tag{4.3}$$

By the definition of I,

$$\sum_{i \in I} |E(V(C^{(i)}), W)| = 0. \tag{4.4}$$

By (4.1),

$$\sum_{v \in W} \Big(d_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \Big) \leq (|J| + t - 1)(n - ts). \tag{4.5}$$

For $i \notin I \cup J$, we have $|E(V(C^{(i)}), V(H))| \leq n - s$ by the definition of J. Hence

$$\sum_{i \notin I \cup J} |E(V(C^{(i)}), V(H))| \leq (n - s)(k - 1 - |I| - |J|). \tag{4.6}$$

Now we estimate the following weighted sum of the degrees of vertices in H in two ways: $\frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v \in W} d_G(v)$. First, since $\delta(G) \geq k + t - 1$,

$$\begin{aligned} \frac{t-1}{t} \sum_{u \in U} d_G(u) + \sum_{v \in W} d_G(v) &\geq (k + t - 1)\Big(\frac{t-1}{t}|U| + |W|\Big) \\ &= (k + t - 1)(n - s). \end{aligned} \tag{4.7}$$

On the other hand, by (4.2) through (4.6),

$$
\begin{aligned}
&\frac{t-1}{t}\sum_{u\in U} d_G(u) + \sum_{v\in W} d_G(v)\\
&= \frac{t-1}{t}\sum_{u\in U}\Big(d_H(u) + \sum_{i=1}^{k-1}|E(V(C^{(i)}), u)|\Big)\\
&\qquad + \sum_{v\in W}\Big(d_H(v) + \sum_{i=1}^{k-1}|E(V(C^{(i)}), v)|\Big)\\
&= \frac{t-1}{t}\Big(\sum_{u\in U} d_H(u) + \Big(\sum_{i\in I} + \sum_{i\in J} + \sum_{i\notin I\cup J}\Big)|E(V(C^{(i)}), U)|\Big)\\
&\qquad + \sum_{v\in W}\Big(d_H(v) + \sum_{i\in J}|E(V(C^{(i)}), v)|\Big) + \sum_{i\notin I\cup J}|E(V(C^{(i)}), W)|\\
&\le \frac{t-1}{t}\Big(\sum_{u\in U} d_H(u) + \sum_{i\in I}|E(V(C^{(i)}), U)| + \sum_{i\in J}|E(V(C^{(i)}), U)|\Big)\\
&\qquad + \sum_{v\in W}\Big(d_H(v) + \sum_{i\in J}|E(V(C^{(i)}), v)|\Big) + \sum_{i\notin I\cup J}|E(V(C^{(i)}), V(H))|\\
&\le \frac{t-1}{t}(t(t-1)s + t(t+1)|I| + ts|J|) + (|J| + t - 1)(n - ts)\\
&\qquad + (n-s)(k-1-|I|-|J|)\\
&= (k+t-1)(n-s) + (t^2-1)|I| - (n-s)(|I|+1)\\
&\le (k+t-1)(n-s) + (t^2-1)|I| - \Big(\frac{11}{6}t^2 + 2\Big)(|I|+1)\\
&= (k+t-1)(n-s) - \Big(\frac{5}{6}t^2 + 3\Big)|I| - \Big(\frac{11}{6}t^2 + 2\Big).
\end{aligned}
$$

This contradicts (4.7), which completes the proof of Lemma 4.1. □

In the following argument, we consider the vertices in W satisfying the condition in Lemma 4.1. We define

$$W_0 = \{v \in W \mid d_H(v) + \sum_{i\in J}|E(V(C^{(i)}), v)| \ge |J| + t\},$$

which is not empty by Lemma 4.1. We also define

$$
\begin{aligned}
W_1 &= \{v \in W \mid \text{there exists } i \in J \text{ such that } d_H(v) + |E(V(C^{(i)}), v)| \ge t + 1\},\\
W_2 &= \{v \in W - W_1 \mid \text{there exists } J_0 \subset J \text{ with } 2 \le |J_0| \le t - d_H(v)\\
&\qquad\qquad \text{such that } d_H(v) + \sum_{i\in J_0}|E(V(C^{(i)}), v)| \ge |J_0| + t\}.
\end{aligned}
$$

Lemma 4.2. *The following statements hold:*

(i) $W_0 \subset W_1 \cup W_2$.

(ii) *If* v *is a vertex in* W_0 *with* $d_H(v) = t-1$*, then* $v \in W_1$.

Proof. Suppose that $v \in W_0$. By the definition of W_0,

$$\sum_{i \in J} (|E(V(C^{(i)}), v)| - 1) \geq t - d_H(v).$$

Thus there exists $J_0 \subset J$ with $1 \leq |J_0| \leq t - d_H(v)$ such that $|E(V(C^{(i)}), v)| - 1 \geq 1$ for each $i \in J_0$ and

$$\sum_{i \in J_0} (|E(V(C^{(i)}), v)| - 1) \geq t - d_H(v).$$

This proves (i). Further if $d_H(v) = t-1$, then $|J_0| = 1$. Thus (ii) holds. $\square$

Lemma 4.3. *Suppose that* $W_1 = \emptyset$*. Fix* $C = C^{(i)}$ *with* $i \in J$*, and let* $b_p \in B$*. Suppose that* $E(B, U) = \emptyset$ *and* $|E(b_p, V(H))| \geq t+1$*, and let* $x_1, x_2, \ldots, x_{t-1}$ *be* $t-1$ *vertices in* $N(b_p) \cap V(H)$*. Then the following inequality holds:*

$$|E(a, V(H))| + |E(b_p, V(H))| + \sum_{i=1}^{t-1} (d_H(x_i) + |E(V(C), x_i)|) \geq |E(V(C), V(H))| + t - 1 + |E(\langle a, x_1, x_2, \ldots, x_{t-1} \rangle)|.$$

Proof. First we claim that $|E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \leq \sum_{i=1}^{t-1} d_H(x_i) - e$, where $e = |E(\langle x_1, x_2, \ldots, x_{t-1} \rangle)|$. We replace C by the t-claw with center b_p contained in $\langle a, b_p, x_1, x_2, \ldots, x_{t-1} \rangle$. Let $H' = \langle (V(H) - \{x_1, x_2, \ldots, x_{t-1}\}) \cup (V(C) - \{a, b_p\}) \rangle$, and let U' be the union of the vertex sets of the K_t components of H'. Also we set $S = (B - \{b_p\}) \cap U'$.

If $S = \emptyset$, then the claim immediately follows from the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$. Thus we may assume that $S \neq \emptyset$. Let $y \in N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})$. If there exists $b_q \in S$ such that $b_q y \notin E(G)$, then each of $\langle \{b_q, a\} \cup N_{H'}(b_q) \rangle$ and $\langle \{b_p, x_1, x_2, \ldots, x_{t-1}, y\} \rangle$ contains a t-claw, a contradiction. Thus $by \in E(G)$ for every $b \in S$. Hence there exists a K_t component P' of H' such that $\{y\} \cup S \subset V(P')$. Note that $|N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})| \geq 2$ by the assumption that $|E(b_p, V(H))| \geq t+1$. Since the above observation holds for any choice of $y \in N(b_p) \cap (V(H) - \{x_1, \ldots, x_{t-1}\})$, it follows that $1 \leq |S| \leq t-2$. This implies that $\langle V(C) - \{b_p\} \rangle \not\cong K_t$. On the other hand, since $W_1 = \emptyset$ and $d_H(y) + |E(V(C), y)| \geq d_H(y) + |E(B, y)| = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + |E(y, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| + |S| + 1 = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + d_{P'}(y) + 1 = |E(y, \{x_1, x_2, \ldots, x_{t-1}\})| + t$, we obtain $E(y, \{x_1, x_2, \ldots, x_{t-1}\}) = \emptyset$ and $d_H(y) = t - |E(B, y)|$.

Now replace C by the t-claw contained in $\langle b_p, x_1, x_2, \ldots, x_{t-1}, y \rangle$, and set $H'' = \langle (V(C) - \{b_p\}) \cup (V(H) - \{x_1, x_2, \ldots, x_{t-1}, y\}) \rangle$. Then since $\langle V(C) -$

$\{b_p\}\rangle$ is connected and not isomorphic to K_t, the union of the vertex sets of the K_t components of H'' coincides with U. Therefore it follows from the maximality of $|E(\langle W\rangle)| + \frac{2}{t}|E(\langle U\rangle)|$ that

$$\begin{aligned}
0 &\le \left(\sum_{i=1}^{t-1} d_H(x_i) + d_H(y) - e\right) - \big\{(|E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
&\quad - |E(B - \{b_p\}, y)|) + |E(a, B - \{b_p\})|\big\} \\
&= \left(\sum_{i=1}^{t-1} d_H(x_i) + d_H(y) - e\right) - \big\{(|E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
&\quad - (|E(B, y)| - 1)) + (t-1)\big\} \\
&= \left(\sum_{i=1}^{t-1} d_H(x_i) - e\right) - |E(B - \{b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})|,
\end{aligned}$$

as claimed. Consequently

$$\begin{aligned}
&|E(a, V(H))| + |E(b_p, V(H))| + \sum_{i=1}^{t-1} |E(V(C), x_i)| \\
&= |E(V(C), V(H)| + |E(\{a, b_p\}, \{x_1, x_2, \ldots, x_{t-1}\})| \\
&\qquad - |E(V(C) - \{a, b_p\}, V(H) - \{x_1, x_2, \ldots, x_{t-1}\})| \\
&\ge |E(V(C), V(H))| + (t - 1 + |E(a, \{x_1, x_2, \ldots, x_{t-1}\})|) - \left(\sum_{i=1}^{t-1} d_H(x_i) - e\right) \\
&= |E(V(C), V(H)| + t - 1 + |E(\langle a, x_1, x_2, \ldots, x_{t-1}\rangle)| - \sum_{i=1}^{t-1} d_H(x_i).
\end{aligned}$$

This completes the proof of Lemma 4.3. □

§5. Property of J_0

We continue with the notation of the preceding sections. In this section and the next section, we consider the case where $W_1 = \emptyset$.

Case 1: $W_1 = \emptyset$.

We take a vertex $v \in W_0$, and fix it. By Lemma 4.2(i), $v \in W_2$. Also by Lemma 4.2(ii), $d_H(v) \le t - 2$. By the definition of W_2, there exists $J_0 \subseteq J$ with $2 \le |J_0| \le t - d_H(v)$ such that $d_H(v) + \sum_{i \in J_0}(|E(V(C^{(i)}), v)| - 1) \ge t$. We choose such a subset J_0 of J so that $|J_0|$ is as small as possible. Then

$$d_H(v) + \sum_{j \in J_0 - \{i\}} (|E(V(C^{(j)}), v)| - 1) \le t - 1 \text{ for each } i \in J_0, \tag{5.1}$$

and hence

$$|E(V(C^{(i)}), v)| \geq 2 \text{ for each } i \in J_0. \tag{5.2}$$

By (5.1) and (5.2), we have

$$d_H(v) + \sum_{i \in J_0 - M} (|E(V(C^{(i)}), v)| - 1) \leq t - |M| \tag{5.3}$$

for any nonempty subset M of J_0. By Lemma 2.6, (5.2) implies

$$E(B^{(i)}, U) = \emptyset \text{ for each } i \in J_0. \tag{5.4}$$

Lemma 5.1. *For each $C = C^{(i)}$ with $i \in J_0$, one of the following statements hold:*

(i) $|E(a, V(H) - \{v\} - N_H(v))| \geq t - d_H(v)$; *or*

(ii) $E(a, V(H)) = \emptyset$ *and* $|E(b_p, W)| \geq \frac{5}{6}t^2 + 3t + 1$ *for some* $b_p \in B$.

Proof. Since $i \in J_0 \subset J$,

$$|E(V(C), V(H))| \geq n - s + 1 \geq \frac{11}{6}t^2 + 3. \tag{5.5}$$

Suppose that (i) does not hold. Then $|E(a, V(H))| \leq (t - 1 - d_H(v)) + (1 + |N_H(v)|) = t$. If $E(a, V(H)) \neq \emptyset$, then by Lemma 2.2, $|E(b, V(H))| \leq t$ for every $b \in B$, and hence $|E(V(C), V(H))| = |E(a, V(H))| + |E(B, V(H))| \leq t + t^2$, which contradicts (5.5). Thus $E(a, V(H)) = \emptyset$. This together with (5.4) implies $E(V(C), V(H)) = E(B, W)$. Hence by (5.5), there exists $b_p \in B$ such that $|E(b_p, W)| \geq t + 1$. Take $x_1, x_2, \ldots, x_{t-1} \in N(b_p) \cap W$. Since $W_1 = \emptyset$, $d_H(x_i) + |E(x_i, V(C))| \leq t$ for each $1 \leq i \leq t - 1$. Consequently by Lemma 4.3, $|E(b_p, W)| = |E(b_p, V(H))| \geq |E(V(C), V(H))| + t - 1 - (\sum_{i=1}^{t-1}\{d_H(x_i) + |E(x_i, V(C))|\}) \geq \frac{11}{6}t^2 + 3 + t - 1 - t(t-1) = \frac{5}{6}t^2 + 2t + 2 > 2t - 1$. By Lemma 2.3, this implies that $|E(b, W)| \leq t - 2$ for each $b \in B - \{b_p\}$. Therefore it follows from (5.5) that $|E(b_p, W)| \geq \frac{11}{6}t^2 + 3 - (t-1)(t-2) \geq \frac{5}{6}t^2 + 3t + 1$. This completes the proof of Lemma 5.1. □

We may assume that $J_0 = \{i \mid 1 \leq i \leq |J_0|\}$. We may also assume that there exists an integer h with $0 \leq h \leq |J_0|$ such that $C = C^{(i)}$ satisfies (i) in Lemma 5.1 for all $1 \leq i \leq h$, and $C = C^{(i)}$ satisfies (ii) in Lemma 5.1 for all $h + 1 \leq i \leq |J_0|$. Let $J_{0,1} = \{i \mid 1 \leq i \leq h\}$ and $J_{0,2} = \{i \mid h + 1 \leq i \leq |J_0|\}$. For $C = C^{(i)}$ with $i \in J_{0,2}$, we may assume that $b_1^{(i)}$ is the vertex $b_p^{(i)}$ satisfying the condition of Lemma 5.1(ii). Our first aim is to obtain an upper bound for

$\sum_{i\in J_{0,2}} |E(V(C^{(i)}), V(H))|$ (see Lemma 5.4). Recall that $d_H(v) \le t-2$. Thus it follows from (5.4) and Lemma 5.1(ii) that for each $i \in J_{0,2}$,

$$\begin{aligned}(5.6)\quad |E(b_1^{(i)}, V(H)-\{v\}-N_H(v))| &= |E(b_1^{(i)}, W)| - |E(b_1^{(i)}, \{v\}\cup N_H(v))| \\ &\ge |E(b_1^{(i)}, W)| - 1 - d_H(v) \\ &\ge |E(b_1^{(i)}, W)| - t + 1 \\ &\ge \frac{5}{6}t^2 + 2t + 2.\end{aligned}$$

Lemma 5.2. $|J_{0,2}| \ge \frac{5}{6}t - 1$.

Proof. Suppose that $|J_{0,2}| \le \lceil \frac{5}{6}t \rceil - 2$. Recall that

$$\sum_{i\in J_0} (|E(V(C^{(i)}), v)| - 1) \ge t - d_H(v).$$

This inequality implies that for each i $(1 \le i \le |J_0|)$, we can choose a subset $X^{(i)} \subset N(v) \cap V(C^{(i)})$ so that

$$|X^{(i)}| \le |E(V(C^{(i)}), v)| - 1, \quad \sum_{i=1}^{|J_0|} |X^{(i)}| = t - d_H(v)$$

and

$$a^{(i)} \notin X^{(i)} \text{ for } 1 \le i \le h, \;\; b_1^{(i)} \notin X^{(i)} \text{ for } h+1 \le i \le |J_0|.$$

Then we can find a t-claw with center v in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$. Note that by the condition in Lemma 5.1(ii), we have $a^{(i)} \notin X^{(i)}$ also for $h+1 \le i \le |J_0|$.

We define $Y^{(i)} = V(C^{(i)}) - X^{(i)}$ for $1 \le i \le h$ and $Y^{(i)} = \{a^{(i)}, b_1^{(i)}\}$ for $h+1 \le i \le |J_0|$. We take disjoint subsets $Z^{(i)}$ of $V(H) - \{v\} - N_H(v)$ for $1 \le i \le |J_0|$ so that

$$\begin{aligned} &Z^{(i)} \subset N(a^{(i)}),\;\; |Z^{(i)}| = |X^{(i)}| \text{ for } 1 \le i \le h, \\ &Z^{(i)} \subset N(b_1^{(i)}),\;\; |Z^{(i)}| = t-1 \text{ for } h+1 \le i \le |J_0|. \end{aligned}$$

This can be done by determining $Z^{(i)}$ from $i = 1$ up to $|J_0|$, because for $1 \le i \le h$,

$$\sum_{j=1}^{i} |X^{(j)}| \le \sum_{j=1}^{|J_0|} |X^{(j)}| = t - d_H(v) \le |E(a^{(i)}, V(H) - \{v\} - N_H(v))|$$

by Lemma 5.1(i) and, if $J_{0,2} \neq \emptyset$, then for $h+1 \leq i \leq |J_0|$,

$$\begin{aligned}\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) &\leq \sum_{j=1}^{h}(|E(V(C^{(j)}), v)| - 1) + (t-1)|J_{0,2}| \\ &\leq (t - |J_{0,2}|) + (t-1)|J_{0,2}| = t + (t-2)|J_{0,2}| \\ &\leq t + (t-2)\Big(\Big\lceil \frac{5}{6}t \Big\rceil - 2\Big) \leq t + (t-2)\Big(\frac{5}{6}t - 1\Big) \\ &\leq \frac{5}{6}t^2 - \frac{10}{6}t + 2 \\ &\leq |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|\end{aligned}$$

by (5.3) and (5.6). Then for each i with $1 \leq i \leq |J_0|$, $\langle Y^{(i)} \cup Z^{(i)} \rangle$ contains a t-claw with center $a^{(i)}$ or $b_1^{(i)}$ depending on whether $i \leq h$ or $i \geq h+1$. Obviously these t-claws and the t-claw in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$ are pairwise vertex-disjoint. This contradicts the assumption that G is a counterexample, and completes the proof of Lemma 5.2. □

Lemma 5.3.

$$\sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| \leq \frac{7}{6}t^3.$$

Proof. Suppose that $\sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| > \lfloor \frac{7}{6}t^3 \rfloor$. Set

$$A = \Big\{ i \in J_{0,2} \;\Big|\; |E(b_1^{(i)}, W)| \geq t^2 - \frac{5}{6}t + 2 \Big\}.$$

We show that

$$|A| \geq \Big\lfloor \frac{1}{6}t \Big\rfloor + 1. \tag{5.7}$$

Suppose that $|A| \leq \frac{1}{6}t$. Recall that $n \leq 2t^2 - 3t + 2$ (see the second paragraph of Section 2). Since $|J_{0,2}| \leq |J_0| \leq t$ and $A \subset J_{0,2}$, we get

$$\begin{aligned}\sum_{i \in J_{0,2}} |E(b_1^{(i)}, W)| &= \sum_{i \in A} |E(b_1^{(i)}, W)| + \sum_{i \in J_{0,2} - A} |E(b_1^{(i)}, W)| \\ &\leq |A|(2t^2 - 3t + 2) + (t - |A|)\Big(t^2 - \frac{5}{6}t + 2\Big) \\ &= |A|\Big(t^2 - \frac{13}{6}t\Big) + t\Big(t^2 - \frac{5}{6}t + 2\Big) \\ &\leq \frac{1}{6}t\Big(t^2 - \frac{13}{6}t\Big) + t\Big(t^2 - \frac{5}{6}t + 2\Big) \\ &= \frac{7}{6}t^3 + 2t - \frac{43}{36}t^2 \leq \frac{7}{6}t^3,\end{aligned}$$

a contradiction. Thus (5.7) is proved.

By (5.7), we may assume that there exists an integer h' with $h \le h' \le |J_0| - \lfloor \frac{1}{6}t \rfloor - 1$ such that $|E(b_1^{(i)}, W)| \le t^2 - \lfloor \frac{5}{6}t \rfloor + 1$ for all $h+1 \le i \le h'$, and $|E(b_1^{(i)}, W)| \ge t^2 - \frac{5}{6}t + 2$ for all $h'+1 \le i \le |J_0|$. Let $J'_{0,2} = \{i \mid h+1 \le i \le h'\}$ and $J''_{0,2} = \{i \mid h'+1 \le i \le |J_0|\}$ (if $h' = h$, then $J'_{0,2} = \emptyset$). Recall that $d_H(v) + |J_0| \le t$. Hence it follows from Lemma 5.2 and Lemma 5.1(ii) that for each $i \in J'_{0,2}$,

$$(5.8)\qquad \begin{aligned} |E(b_1^{(i)}, W - \{v\} - N_H(v))| &\ge |E(b_1^{(i)}, W)| - 1 - d_H(v) \\ &\ge |E(b_1^{(i)}, W)| - 1 + |J_0| - t \\ &\ge \Big(\frac{5}{6}t^2 + 3t + 1\Big) - 1 + \Big(\frac{5}{6}t - 1\Big) - t \\ &= \frac{5}{6}t^2 + \frac{17}{6}t - 1, \end{aligned}$$

and it follows from Lemma 5.2 and the choice of h' that for each $i \in J''_{0,2}$,

$$(5.9)\qquad \begin{aligned} |E(b_1^{(i)}, W - \{v\} - N_H(v))| &\ge |E(b_1^{(i)}, W)| - 1 - d_H(v) \\ &\ge |E(b_1^{(i)}, W)| - 1 + |J_0| - t \\ &\ge \Big(t^2 - \frac{5}{6}t + 2\Big) - 1 + (\frac{5}{6}t - 1) - t \\ &= t^2 - t. \end{aligned}$$

We now argue as in the proof of Lemma 5.2. For each i $(1 \le i \le |J_0|)$, we can choose a subset $X^{(i)} \subset N(v) \cap V(C^{(i)})$ so that

$$|X^{(i)}| \le |E(V(C^{(i)}), v)| - 1, \quad \sum_{i=1}^{|J_0|} |X^{(i)}| = t - d_H(v)$$

and

$$a^{(i)} \notin X^{(i)} \text{ for } 1 \le i \le h, \quad b_1^{(i)} \notin X^{(i)} \text{ for } h+1 \le i \le |J_0|.$$

Then we can find a t-claw with center v in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$. Note that by the condition in Lemma 5.1(ii), we have $a^{(i)} \notin X^{(i)}$ also for $h+1 \le i \le |J_0|$.

We define $Y^{(i)} = V(C^{(i)}) - X^{(i)}$ for $1 \le i \le h$ and $Y^{(i)} = \{a^{(i)}, b_1^{(i)}\}$ for $h+1 \le i \le |J_0|$. We take disjoint subsets $Z^{(i)}$ of $V(H) - \{v\} - N_H(v)$ for $1 \le i \le |J_0|$ so that

$$\begin{aligned} &Z^{(i)} \subset N(a^{(i)}),\ \ |Z^{(i)}| = |X^{(i)}| \ \text{ for } \ 1 \le i \le h, \\ &Z^{(i)} \subset N(b_1^{(i)}),\ \ |Z^{(i)}| = t - 1 \ \text{ for } \ h+1 \le i \le |J_0|. \end{aligned}$$

This can be done by determining $Z^{(i)}$ from $i = 1$ up to $|J_0|$, because for $1 \leq i \leq h$,

$$\sum_{j=1}^{i} |X^{(j)}| \leq \sum_{j=1}^{|J_0|} |X^{(j)}| = t - d_H(v) \leq |E(a^{(i)}, V(H) - \{v\} - N_H(v))|$$

by Lemma 5.1(i), and for $h+1 \leq i \leq h'$, since $|J'_{0,2}| = h'-h \leq h' \leq t - \left\lfloor \frac{1}{6}t \right\rfloor - 1$, we have

$$\begin{aligned}
\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) &\leq \sum_{j=1}^{h} (|E(V(C^{(j)}), v)| - 1) + (t-1)|J'_{0,2}| \\
&\leq (t - |J_{0,2}|) + (t-1)\Big(t - \Big\lfloor \frac{1}{6}t \Big\rfloor - 1\Big) \\
&\leq \frac{1}{6}t + 1 + (t-1)\Big(t - \Big\lfloor \frac{1}{6}t \Big\rfloor - 1\Big) \\
&\leq \frac{1}{6}t + 1 + \frac{5}{6}t(t-1) = \frac{5}{6}t^2 - \frac{4}{6}t + 1 \\
&\leq |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|
\end{aligned}$$

by (5.3), Lemma 5.2 and (5.8) and, for $h' + 1 \leq i \leq |J_0|$,

$$\begin{aligned}
\sum_{j=1}^{h} |X^{(j)}| + (t-1)(i-h) &\leq (t-1)i \\
&\leq (t-1)|J_0| \\
&\leq (t-1)t \\
&\leq |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|
\end{aligned}$$

by (5.9). Then for each i with $1 \leq i \leq |J_0|$, $\langle Y^{(i)} \cup Z^{(i)} \rangle$ contains a t-claw with center $a^{(i)}$ or $b_1^{(i)}$ depending on whether $i \leq h$ or $i \geq h+1$. Obviously, these t-claws and the t-claw in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$ are pairwise vertex-disjoint. This contradicts the assumption that G is a counterexample, and completes the proof of Lemma 5.3. □

Lemma 5.4.

$$\sum_{i \in J_{0,2}} |E(V(C^{(i)}), W)| \leq \frac{13}{6}t^3 - 3t^2 + 2t.$$

Proof. For each $i \in J_{0,2}$, we have $|E(b, W)| \leq t - 2$ for every $b \in B^{(i)} - \{b_1^{(i)}\}$ by Lemma 2.2. Also $E(V(C^{(i)}), W) = E(B^{(i)}, W)$ for each $i \in J_{0,2}$ by the first assertion of Lemma 5.1(ii). Hence by Lemma 5.3,

$$\begin{aligned}\sum_{i\in J_{0,2}} |E(V(C^{(i)}), W)| &= \sum_{i\in J_{0,2}} |E(b_1^{(i)}, W)| + \sum_{i\in J_{0,2}} |E(B^{(i)} - \{b_1^{(i)}\}, W)| \\ &\le \frac{7}{6}t^3 + |J_{0,2}|(t-1)(t-2) \\ &\le \frac{7}{6}t^3 + t(t-1)(t-2) \\ &= \frac{13}{6}t^3 - 3t^2 + 2t,\end{aligned}$$

as desired. □

We now consider edges among the $C^{(i)}$.

Lemma 5.5. *For each $i \in J_{0,2}$, there exists $l \in \{1, 2, \ldots, k-1\} - \{i\}$ such that $|E(a^{(i)}, V(C^{(l)}))| \ge 2$.*

Proof. By the definition of $J_{0,2}$, we have $E(a^{(i)}, V(H)) = \emptyset$. Hence from the assumption that $\delta(G) \ge k+t-1$, it follows that

$$\sum_{j\in\{1,\ldots,k-1\}-\{i\}} |E(a^{(i)}, V(C^{(j)}))| \ge (k+t-1) - t = k-1,$$

which immediately implies the desired conclusion. □

Having Lemma 5.5 in mind, we define

$$L = \{l \mid 1 \le l \le k-1,\ \ |E(a^{(i)}, V(C^{(l)}))| \ge 2 \ \text{ for some } \ i \in J_{0,2} - \{l\}\}.$$

Lemma 5.6. *We have $|E(V(C^{(l)}), V(H))| \le t(t-2)+1$ for each $l \in L$; in particular $L \cap J = \emptyset$.*

Proof. Let $l \in L$. By the definition of L, there exists $i \in J_{0,2} - \{l\}$ such that $|E(a^{(i)}, V(C^{(l)}))| \ge 2$. Then $N(a^{(i)}) \cap B^{(l)} \ne \emptyset$. Take $b_p^{(l)} \in N(a^{(i)}) \cap B^{(l)}$.

Claim 5.6.1 $E(a^{(l)}, V(H)) = \emptyset$.

Proof. Soppose that $E(a^{(l)}, V(H)) \ne \emptyset$. Take $x \in N(a^{(l)}) \cap V(H)$. Then each of $\langle\{a^{(i)}, b_p^{(l)}\} \cup (B^{(i)} - \{b_1^{(i)}\})\rangle$ and $\langle\{a^{(l)}, x\} \cup (B^{(l)} - \{b_p^{(l)}\})\rangle$ contains a t-claw and, since $|E(b_1^{(i)}, W)| \ge \frac{5}{6}t^2+3t+1$ by the definition of $J_{0,2}$, $\langle\{b_1^{(i)}\}\cup(N(b_1^{(i)})\cap (V(H) - \{x\}))\rangle$ also contains a t-claw, a contradiction. □

Claim 5.6.2 $|E(b_p^{(l)}, V(H))| \le t-1$.

Proof. Suppose that $|E(b_p^{(l)}, V(H))| \geq t$. Since $|E(V(C^{(l)}), a^{(i)})| \geq 2$, there exists $x \in N(a^{(i)}) \cap V(C^{(l)})$ with $x \neq b_p^{(l)}$. Also take a subset X of $N(b_p^{(l)}) \cap V(H)$ with $|X| = t$. Then each of $\langle \{a^{(i)}, x\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$, $\langle \{b_p^{(l)}\} \cup X \rangle$ and $\langle \{b_1^{(i)}\} \cup (N(b_1^{(i)}) \cap (V(H) - X)) \rangle$ contains a t-claw , a contradiction. □

Claim 5.6.3 $|E(b, V(H))| \leq t - 2$ *for every* $b \in B^{(l)} - \{b_p^{(l)}\}$.

Proof. Suppose that there exists $b_q^{(l)} \in B^{(l)} - \{b_p^{(l)}\}$ such that $|E(b_q^{(l)}, V(H))| \geq t - 1$. Take a subset X of $N(b_q^{(l)}) \cap V(H)$ with $|X| = t - 1$. Then each of $\langle \{a^{(i)}, b_p^{(l)}\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$, $\langle \{a^{(l)}, b_q^{l}\} \cup X \rangle$ and $\langle \{b_1^{(i)}\} \cup (N(b_1^{(i)}) \cap (V(H) - X)) \rangle$ contains a t-claw, a contradiction. □

Combining Claims 5.6.1 through 5.6.3, we obtain $|E(V(C^{(l)}), V(H))| \leq t - 1 + (t-1)(t-2) = t(t-2) + 1$. Since $|E(V(C^{(j)}), V(H))| \geq \frac{11}{6}t^2 + 3$ for each $j \in J$ by the definition of J, we also get $l \notin J$. This completes the proof of Lemma 5.6. □

Lemma 5.7. *Let* $l \in L$, *and let* $i \in J_{0,2}$ *be an index such that* $|E(a^{(i)}, V(C^{(l)}))| \geq 2$. *Then* $E(a^{(j)}, V(C^{(l)})) = \emptyset$ *for every* $j \in J_{0,2} - \{i\}$.

Proof. Suppose that there exists $j \in J_{0,2} - \{i\}$ such that $E(a^{(j)}, V(C^{(l)})) \neq \emptyset$, and take $y \in N(a^{(j)}) \cap V(C^{(l)})$. Since $|E(a^{(i)}, V(C^{(l)}))| \geq 2$, there exists $x \in N(a^{(i)}) \cap V(C^{(l)})$ with $x \neq y$. Then each of $\langle \{a^{(i)}, x\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$ and $\langle \{a^{(j)}, y\} \cup (B^{(j)} - \{b_1^{(j)}\}) \rangle$ contains a t-claw. Since $i, j \in J_{0,2}$, we can take disjoint subsets $X^{(i)}$ and $X^{(j)}$ of $V(H)$ such that

$$|X^{(i)}| = |X^{(j)}| = t, \quad X^{(i)} \subset N(b_1^{(i)}), \quad X^{(j)} \subset N(b_1^{(j)}).$$

Then each of $\langle \{b_1^{(i)}\} \cup X^{(i)} \rangle$ and $\langle \{b_1^{(j)}\} \cup X^{(j)} \rangle$ contains a t-claw, and thus we get a contradiction. □

Note that it follows from Lemmas 5.2, 5.5 and 5.7 that

$$|L| \geq |J_{0,2}| \geq \frac{5}{6}t - 1. \tag{5.10}$$

§6. Another counting argument

In this section, we complete the proof for Case 1. Let $J_0, J_{0.1}, J_{0,2}, L$ be as in the preceding section. Set $I' = I - L$ and $J' = J - J_{0,2}$. Thus

$$\{1, \ldots, k-1\} = I' \cup J' \cup J_{0,2} \cup L \cup (\{1, \ldots, k-1\} - I - J - L)$$
(disjoint union).

Lemma 6.1. *There exists $v' \in W$ such that*

$$d_H(v') + \sum_{i\in J'} |E(V(C^{(i)}), v')| \geq |J'| + t.$$

Proof. We argue as in the proof of Lemma 4.1. Suppose that

$$d_H(v') + \sum_{i\in J'} |E(V(C^{(i)}), v')| \leq |J'| + t - 1 \text{ for all } v' \in W. \tag{6.1}$$

As in the proof of Lemma 4.1, we have $|E(V(C^{(i)}), U)| \leq t(t+1)$ for $i \in I'$. Hence

$$\sum_{i\in I'} |E(V(C^{(i)}), U)| \leq t(t+1)|I'|. \tag{6.2}$$

For $i \in J'$, since $E(V(C^{(i)}), W) \neq \emptyset$, it follows from Lemma 2.7 that $|E(V(C^{(i)}), U)| \leq ts$. Hence

$$\sum_{i\in J'} |E(V(C^{(i)}), U)| \leq ts|J'|. \tag{6.3}$$

By (5.4) and Lemma 5.1(ii),

$$\sum_{i\in J_{0,2}} |E(V(C^{(i)}), U)| = 0. \tag{6.4}$$

By the definition of I,

$$\sum_{i\in I'} |E(V(C^{(i)}), W)| = 0. \tag{6.5}$$

By (6.1),

$$\sum_{v'\in W} (d_H(v') + \sum_{i\in J'} |E(V(C^{(i)}), v')|) \leq (|J'| + t - 1)(n - ts). \tag{6.6}$$

By Lemma 5.6,

$$\sum_{i\in L} |E(V(C^{(i)}), V(H))| \leq (t^2 - 2t + 1)|L|. \tag{6.7}$$

For $i \notin I' \cup J' \cup J_{0,2} \cup L$, we have $|E(V(C^{(i)}), V(H))| \leq n - s$ by the definition of J. Hence

$$\sum_{i\notin I'\cup J'\cup J_{0,2}\cup L} |E(V(C^{(i)}), V(H))| \leq (n-s)(k - 1 - |I'| - |J'| - |J_{0,2}| - |L|). \tag{6.8}$$

Now since $\delta(G) \geq k+t-1$,

$$\frac{t-1}{t}\sum_{u\in U} d_G(u) + \sum_{v'\in W} d_G(v') \geq (k+t-1)\Big(\frac{t-1}{t}|U| + |W|\Big) \tag{6.9}$$
$$= (k+t-1)(n-s).$$

On the other hand, by (6.2) through (6.8), Lemma 5.4 and (5.10) ,

$$\begin{aligned}
&\frac{t-1}{t}\sum_{u\in U} d_G(u) + \sum_{v'\in W} d_G(v')\\
&= \frac{t-1}{t}\sum_{u\in U}\big(d_H(u) + \sum_{i=1}^{k-1}|E(V(C^{(i)}),u)|\big)\\
&\qquad + \sum_{v'\in W}\big(d_H(v') + \sum_{i=1}^{k-1}|E(V(C^{(i)}),v')|\big)\\
&= \frac{t-1}{t}\Bigg(\sum_{u\in U} d_H(u) + \Big(\sum_{i\in I'} + \sum_{i\in J'} + \sum_{i\in L} + \sum_{i\notin I'\cup J'\cup J_{0,2}\cup L}\Big)|E(V(C^{(i)}),U)|\Bigg)\\
&\qquad + \sum_{v'\in W}\big(d_H(v') + \sum_{i\in J'}|E(V(C^{(i)}),v')|\big)\\
&\qquad + \sum_{i\in J_{0,2}}|E(V(C^{(i)}),W)| + \sum_{i\in L}|E(V(C^{(i)}),W)|\\
&\qquad + \sum_{i\notin I'\cup J'\cup J_{0,2}\cup L}|E(V(C^{(i)}),W)|\\
&\leq \frac{t-1}{t}\Bigg(\sum_{u\in U} d_H(u) + \sum_{i\in I'}|E(V(C^{(i)}),U)| + \sum_{i\in J'}|E(V(C^{(i)}),U)|\Bigg)\\
&\qquad + \sum_{v'\in W}\Bigg(d_H(v') + \sum_{i\in J'}|E(V(C^{(i)}),v')|\Bigg) + \sum_{i\in J_{0,2}}|E(V(C^{(i)}),W)|\\
&\qquad + \sum_{i\in L}|E(V(C^{(i)}),V(H))| + \sum_{i\notin I'\cup J'\cup J_{0,2}\cup L}|E(V(C^{(i)}),V(H))|\\
&\leq \frac{t-1}{t}(t(t-1)s + t(t+1)|I'| + ts|J'|) + (|J'|+t-1)(n-ts)\\
&\qquad + \Big(\frac{13}{6}t^3 - 3t^2 + 2t\Big) + (t^2-2t+1)|L|\\
&\qquad + (n-s)(k-1-|I'|-|J'|-|J_{0,2}|-|L|)\\
&= (k+t-1)(n-s) - (n-s-t^2+1)|I'| - (n-s)\\
&\qquad - (n-s)(|J_{0,2}|+|L|) + \Big(\frac{13}{6}t^3 - 3t^2 + 2t\Big) + (t^2-2t+1)|L|
\end{aligned}$$

$$\begin{aligned}
&\le (k+t-1)(n-s) - \Big(\frac{11}{6}t^2+2\Big) - \Big(\frac{11}{6}t^2+2\Big)(|J_{0,2}|+|L|)\\
&\qquad + \Big(\frac{13}{6}t^3 - 3t^2 + 2t\Big) + (t^2-2t+1)|L|\\
&= (k+t-1)(n-s) + \frac{13}{6}t^3 - \frac{29}{6}t^2 + 2t - 2\\
&\qquad - \Big(\frac{11}{6}t^2+2\Big)|J_{0,2}| - \Big(\frac{11}{6}t^2 + 2 - t^2 + 2t - 1\Big)|L|\\
&\le (k+t-1)(n-s) + \frac{13}{6}t^3 - \frac{29}{6}t^2 + 2t - 2\\
&\qquad - \Big(\frac{11}{6}t^2+2\Big)\Big(\frac{5}{6}t-1\Big) - \Big(\frac{5}{6}t^2+2t+1\Big)\Big(\frac{5}{6}t-1\Big)\\
&= (k+t-1)(n-s) - \frac{2}{36}t^3 - \frac{23}{6}t^2 + \frac{9}{6}t + 1.
\end{aligned}$$

This contradicts (6.9), which completes the proof of Lemma 6.1. □

Now by Lemma 6.1, there exists $v' \in W$ such that $d_H(v') + \sum_{i\in J'} |E(V(C^{(i)}), v')| \ge |J'| + t$, i.e., $\sum_{i\in J'}(|E(V(C^{(i)}), v')| - 1) \ge t - d_H(v')$. Then since $W_1 = \emptyset$, there exists $J_0' \subset J'$ with $2 \le |J_0'| \le t - d_H(v') \le t$ such that $\sum_{i\in J_0'}(|E(V(C^{(i)}), v')| - 1) \ge t - d_H(v')$. We choose J_0' so that $|J_0'|$ is as small as possible. Arguing as in the proof of Lemmas 5.1 and 5.2, we see that there exist at least $\frac{5}{6}t - 1$ indices $i \in J_0'$ such that $E(a^{(i)}, V(H)) = \emptyset$ and $|E(b_p^{(i)}, W)| \ge \frac{5}{6}t^2 + 3t + 1$ for some $b_p^{(i)} \in B^{(i)}$. Set

$$J_{0,2}' = \Big\{ i \in J_0' \;\Big|\; E(a^{(i)}, V(H)) = \emptyset, \text{ and there exists } b_p^{(i)} \in B^{(i)} \text{ such that } |E(b_p^{(i)}, W)| \ge \frac{5}{6}t^2 + 3t + 1 \Big\}.$$

Thus $|J_{0,2}'| \ge \frac{5}{6}t - 1$. For $i \in J_{0,2}'$ we may assume that $b_1^{(i)}$ satisfies the condition $|E(b_1^{(i)}, W)| \ge \frac{5}{6}t^2 + 3t + 1$. By the definition of J', $J_{0,2} \cap J_{0,2}' = \emptyset$. Hence $|J_{0,2} \cup J_{0,2}'| \ge \frac{10}{6}t - 2 \ge t + 1$. Let K be a subset of $J_{0,2} \cup J_{0,2}'$ such that $|K| = t + 1$.

Lemma 6.2. *For each $i \in K$, $|E(a^{(i)}, V(C^{(j)}))| \le 1$ for every $j \in K - \{i\}$.*

Proof. Suppose that $|E(a^{(i)}, V(C^{(j)}))| \ge 2$. Take $x \in N(a^{(i)}) \cap V(C^{(j)})$ with $x \ne b_1^{(j)}$. Then $\langle \{a^{(i)}, x\} \cup (B^{(i)} - \{b_1^{(i)}\}) \rangle$ contains a t-claw. Since $|E(b_1^{(i)}, W)| \ge \frac{5}{6}t^2 + 3t + 1$ and $|E(b_1^{(j)}, W)| \ge \frac{5}{6}t^2 + 3t + 1$, we can take disjoint subsets $X^{(i)}$ and $X^{(j)}$ of $V(H)$ such that $|X^{(i)}| = |X^{(j)}| = t$, $X^{(i)} \subset N(b_1^{(i)})$, $X^{(j)} \subset N(b_1^{(j)})$. Then each of $\langle \{b_1^{(i)}\} \cup X^{(i)} \rangle$ and $\langle \{b_1^{(j)}\} \cup X^{(j)} \rangle$ contains a t-claw, a contradiction. □

We are now in a position to complete the proof for Case 1. Set $K' = \{1, 2, \ldots, k-1\} - K$. Then $|K'| = k - 1 - (t+1) = k - t - 2$. Since $E(a^{(i)}, V(H)) = \emptyset$ for each $i \in K$ and $\delta(G) \geq k + t - 1$, it follows from Lemma 6.2 that

$$\begin{aligned}\left|E(\{a^{(i)} \mid i \in K\}, \bigcup_{j \in K'} V(C^{(j)}))\right| &= \sum_{i \in K} \left|E(a^{(i)}, \bigcup_{j \in K'} V(C^{(j)}))\right| \\ &\geq |K|(k + t - 1 - t - t) \\ &= |K|(k - t - 1) \\ &= |K|(|K'| + 1) \\ &= (|K'| + 1)(t + 1).\end{aligned}$$

Hence there exists an index $j \in K'$ such that $|E(\{a^{(i)} \mid i \in K\}, V(C^{(j)}))| > t + 1$. This implies that there exist two edges $xa^{(l)}$ and $ya^{(m)}$ joining $V(C^{(j)})$ and $\{a^{(i)} \mid i \in K\}$ with $x, y \in V(C^{(j)})$, $x \neq y$ and $l \neq m$. Then each of $\langle \{a^{(l)}, x\} \cup (B^{(l)} - \{b_1^{(l)}\}) \rangle$ and $\langle \{a^{(m)}, y\} \cup (B^{(m)} - \{b_1^{(m)}\}) \rangle$ contains a t-claw. Since $l, m \in K \subset J_{0,2} \cup J'_{0,2}$, we can take disjoint subsets $X^{(l)}$ and $X^{(m)}$ of $V(H)$ such that $|X^{(l)}| = |X^{(m)}| = t$, $X^{(l)} \subset N(b_1^{(l)})$, $X^{(m)} \subset N(b_1^{(m)})$. Then each of $\langle \{b_1^{(l)}\} \cup X^{(l)} \rangle$ and $\langle \{b_1^{(m)}\} \cup X^{(m)} \rangle$ contains a t-claw. This contradicts the assumption that G is a counterexample. This concludes the discussion for Case 1.

§7. Proof of the main theorem

In this section, we consider the case where $W_1 \neq \emptyset$.

Case 2: $W_1 \neq \emptyset$.

Let $v \in W_1$. By the definition of W_1, we can take a t-claw $C = C^{(i)}$ with $i \in J$ such that $d_H(v) + |E(V(C), v)| \geq t + 1$. By Lemma 2.5, we have $E(V(C), U) = \emptyset$, and hence

$$|E(V(C), W)| = |E(V(C), V(H))| = n - s + 1 \geq \frac{11}{6}t^2 + 3. \tag{7.1}$$

Lemma 7.1. *$E(a, W) = \emptyset$.*

Proof. Suppose that $E(a, W) \neq \emptyset$. Then by Lemma 2.2, $|E(b, W)| \leq t$ for each $b \in B$. On the other hand, we see from Lemma 2.1 that $|E(a, W)| = |E(a, W - \{v\} - N_H(v))| + |E(a, \{v\} \cup N_H(v))| \leq (t - 1 - d_H(v)) + (1 + |N_H(v)|) = t$. Hence $|E(V(C), W)| = |E(a, W)| + |E(B, W)| \leq t + t^2$, which contradicts (7.1). □

Note that it follows from Lemma 7.1 that

$$d_H(v) + |E(B, v)| \geq t + 1. \tag{7.2}$$

Hence by Lemma 2.4,

$$|E(b, W - \{v\} - N_H(v))| \leq t - 2 \text{ for every } b \in B, \tag{7.3}$$

which implies that

$$|E(b, W)| \leq 2t - 2 \text{ for every } b \in B. \tag{7.4}$$

By (7.1) and Lemma 7.1, we also have

$$|E(B, W)| \geq \frac{11}{6}t^2 + 3. \tag{7.5}$$

Hence $|E(B, W - \{v\} - N_H(v))| \geq \frac{11}{6}t^2 + 3 - |B|(1 + d_H(v)) \geq \frac{5}{6}t^2 + 3$, which together with (7.3) implies that

$$E(B - \{b\}, W - \{v\} - N_H(v)) \neq \emptyset \text{ for every } b \in B. \tag{7.6}$$

Set $S = \{b \in B \mid |E(b, W)| \geq \frac{11}{6}t\}$. Note that $S \neq \emptyset$ by (7.5).

Case 2.1: $d_H(v) \leq \lfloor \frac{5}{6}t \rfloor$.

Take $b_p \in S$. By (7.2), $\langle \{v\} \cup N_H(v) \cup (N(v) \cap (B - \{b_p\})) \rangle$ contains a t-claw. Since $d_H(v) \leq \lfloor \frac{5}{6}t \rfloor$, it follows from the definition of S that

$$\begin{aligned} |E(b_p, W - \{v\} - N_H(v))| &\geq \frac{11}{6}t - \left(1 + \left\lfloor \frac{5}{6}t \right\rfloor\right) \\ &= t - 1. \end{aligned}$$

Hence $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{v\} - N_H(v))) \rangle$ also contains a t-claw, a contradiction.

Case 2.2: $d_H(v) \geq \lfloor \frac{5}{6}t \rfloor + 1$.

Write $d_H(v) = \lfloor \frac{5}{6}t \rfloor + 1 + h$. Since $d_H(v) \leq t - 1$, we have $0 \leq h \leq \lceil \frac{1}{6}t \rceil - 2$. By (7.2),

$$\begin{aligned} |E(B, v)| &\geq t + 1 - \left(\left\lfloor \frac{5}{6}t \right\rfloor + 1 + h\right) \\ &= \left\lceil \frac{1}{6}t \right\rceil - h. \end{aligned} \tag{7.7}$$

Set $T = \{w \in N_H(v) \mid d_H(w) \geq \lfloor \frac{5}{6}t \rfloor + 1\}$.

Lemma 7.2. *We have* $|E(B,w)| \le \lceil \frac{1}{6}t \rceil$ *for all* $w \in T \cup \{v\}$.

Proof. Suppose that there exists $w \in T \cup \{v\}$ such that $|E(B,w)| \ge \lceil \frac{1}{6}t \rceil + 1$. Since $d_H(w) \ge \lfloor \frac{5}{6}t \rfloor + 1$, we can take a subset X of $N_H(w)$ such that $|X| = \lfloor \frac{5}{6}t \rfloor$. Take $b_p \in S$. Then $|E(B - \{b_p\}, w)| \ge \lceil \frac{1}{6}t \rceil$. Hence $\langle \{w\} \cup X \cup (N(w) \cap (B - \{b_p\})) \rangle$ contains a t-claw. By the definition of S, $|E(b_p, W - \{w\} - X)| \ge |E(b_p, W)| - 1 - |X| \ge \frac{11}{6}t - 1 - \lfloor \frac{5}{6}t \rfloor \ge t - 1$. Hence $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{w\} - X)) \rangle$ contains a t-claw, a contradiction. □

Lemma 7.3. *We have* $d_H(w) + |E(B,w)| \le t$ *for all* $w \in N_H(v) - T$.

Proof. Suppose that there exists $w \in N_H(v) - T$ such that $d_H(w) + |E(B,w)| \ge t + 1$. Take $b_p \in S$. Then $\langle \{w\} \cup N_H(w) \cup (N(w) \cap (B - \{b_p\})) \rangle$ contains a t-claw. Since $w \notin T$, $d_H(w) \le \lfloor \frac{5}{6}t \rfloor$. Hence it follows from the definition of S that

$$\begin{aligned} |E(b_p, W - \{w\} - N_H(w))| &\ge \frac{11}{6}t - \left(1 + \left\lfloor \frac{5}{6}t \right\rfloor\right) \\ &= t - 1. \end{aligned}$$

Consequently $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{w\} - N_H(w))) \rangle$ contains a t-claw, a contradiction. □

By Lemmas 7.2 and 7.3,

$$\begin{aligned} &|E(B, \{v\} \cup N_H(v))| \\ &= |E(B,v)| + \sum_{w \in T} |E(B,w)| + \sum_{w \in N_H(v) - T} |E(B,w)| \\ &\le \left\lceil \frac{1}{6}t \right\rceil + |T| \left\lceil \frac{1}{6}t \right\rceil + \sum_{w \in N_H(v) - T} (t - d_H(w)) \\ &\le \frac{1}{6}t + 1 + |T| \left(\frac{1}{6}t + 1 \right) + |N_H(v) - T| t - \sum_{w \in N_H(v) - T} d_H(w) \\ &= \frac{1}{6}t + 1 + \frac{1}{6}t|T| + |T| + |N_H(v)|t - t|T| - \sum_{w \in N_H(v) - T} d_H(w) \\ &= \frac{1}{6}t + 1 - \frac{5}{6}t|T| + |T| + \left(\left\lfloor \frac{5}{6}t \right\rfloor + 1 + h \right) t - \sum_{w \in N_H(v) - T} d_H(w) \\ &\le \frac{5}{6}t^2 + ht + \frac{7}{6}t - \frac{5}{6}t|T| + |T| + 1 - \sum_{w \in N_H(v) - T} d_H(w). \end{aligned}$$

Hence by (7.5),

(7.8)

$$
\begin{aligned}
&|E(B, W-\{v\}-N_H(v))|\\
&\quad = |E(B,W)| - |E(B,\{v\}\cup N_H(v))|\\
&\quad \geq \frac{11}{6}t^2+3-\Big(\frac{5}{6}t^2+ht+\frac{7}{6}t-\frac{5}{6}t|T|+|T|+1-\sum_{w\in N_H(v)-T} d_H(w)\Big)\\
&\quad = t^2-\frac{7}{6}t-ht+\frac{5}{6}t|T|-|T|+2+\sum_{w\in N_H(v)-T} d_H(w).
\end{aligned}
$$

By (7.7), we can take a subset B' of $N(v)\cap B$ with $|B'| = \lceil \frac{1}{6}t\rceil - h - 1$. Then since $|T| \leq d_H(v) \leq t-1$, $h \leq \lceil \frac{1}{6}t\rceil - 2$ and $t \geq 24$, it follows from (7.3) and (7.8) that

(7.9)

$$
\begin{aligned}
&|E(B-B', W-\{v\}-N_H(v))|\\
&\quad \geq t^2-\frac{7}{6}t-ht+\frac{5}{6}t|T|-|T|+2+\sum_{w\in N_H(v)-T} d_H(w)\\
&\qquad\qquad -\Big(\Big\lceil\frac{1}{6}t\Big\rceil-h-1\Big)(t-2)\\
&\quad \geq t^2-\frac{7}{6}t-ht+\frac{5}{6}t|T|-|T|+2+\sum_{w\in N_H(v)-T} d_H(w)-\Big(\frac{1}{6}t-h\Big)(t-2)\\
&\quad = \frac{5}{6}t^2-\frac{5}{6}t-\frac{1}{6}t|T|+2-2h+|T|(t-1)+\sum_{w\in N_H(v)-T} d_H(w)\\
&\quad \geq \frac{5}{6}t^2-\frac{5}{6}t-\frac{1}{6}t(t-1)+2-2h+|T|(t-1)+\sum_{w\in N_H(v)-T} d_H(w)\\
&\quad = \frac{4}{6}t^2-\frac{4}{6}t+2-2h+|T|(t-1)+\sum_{w\in N_H(v)-T} d_H(w)\\
&\quad \geq \frac{2}{3}t^2-\frac{2}{3}t+2-2\Big(\Big\lceil\frac{1}{6}t\Big\rceil-2\Big)+|T|(t-1)+\sum_{w\in N_H(v)-T} d_H(w)\\
&\quad \geq \frac{2}{3}t^2-t+4+|T|(t-1)+\sum_{w\in N_H(v)-T} d_H(w)\\
&\quad > |T|(t-1)+\sum_{w\in N_H(v)-T} d_H(w).
\end{aligned}
$$

On the other hand,

$$(7.10) \quad \begin{aligned} &|E(\langle \{v\} \cup N_H(v) \rangle)| + |E(N_H(v), W - \{v\} - N_H(v))| \\ &\leq \sum_{w \in N_H(v)} d_H(w) = \sum_{w \in T} d_H(w) + \sum_{w \in N_H(v) - T} d_H(w) \\ &\leq |T|(t-1) + \sum_{w \in N_H(v) - T} d_H(w). \end{aligned}$$

Replace C by the t-claw with center v contained in $\langle \{v\} \cup N_H(v) \cup B' \rangle$. Let $H' = \langle (V(H) - \{v\} - N_H(v)) \cup (V(C) - B') \rangle$, and let U' be the union of the vertex sets of the K_t components of H'. Also set $W' = V(H') - U'$. Then by Lemma 7.1 (and by (7.6) if $|B'| = 1$), $\{a\} \cup (B - B')$ is not contained in a K_t components of H', which means $U' = U$. Therefore by (7.9) and (7.10), $|E(\langle W' \rangle)| + \frac{2}{t}|E(\langle U' \rangle)| > |E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$. This contradicts the maximality of $|E(\langle W \rangle)| + \frac{2}{t}|E(\langle U \rangle)|$, which completes the proof for Case 2.2.

This completes the proof of the main theorem.

References

[1] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar. **14** (1963), 423–443.

[2] Y. Egawa, S. Fujita and K. Ota, $K_{1,3}$-factors in graphs, Discrete Math, in press.

[3] Y. Egawa and K. Ota, Vertex-disjoint claws in graphs, Discrete Math. **197/198** (1999), 225–246.

[4] H. Enomoto, A. Kaneko and Zs. Tuza, P_3-factors and covering cycles in graphs of minimum degree $n/3$, Colloq. Math, Soc. János Bolyai **52** (1987), 213–220.

[5] S. Fujita, Vertex-Disjoint $K_{1,t}$'s in Graphs, Ars Combin. **64** (2002), 211–223.

[6] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős, Colloq. Math, Soc. János Bolyai **4** (1970), 601–623.

[7] R. Johansson, An El-Zahar type condition ensuring path-factors, J. Graph Theory **28** (1998), 39–42.

[8] K. Ota, Vertex-disjoint stars in graphs, Discuss. Math. Graph Theory **21** (2001), 179–185.

Shuya Chiba
Department of Mathematical Information Science, Tokyo University of Science
Shinjuku-ku, Tokyo, 162-8601 Japan