

## Motion of charged particles in Sasakian manifolds

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**Abstract.** It is known that the image of a horizontal geodesic under a Riemannian submersion is a geodesic. However, in general the image of a geodesic under a Riemannian submersion is not a geodesic. In this paper, we define a Sasaki-Kähler submersion from a Sasakian manifold onto a Kähler manifold, and show that the image of the motion of a charged particle is the motion of a charged particle. In particular, the image of a geodesic is the motion of a charged particle under a Sasaki-Kähler submersion. A Sasaki-Kähler submersion is a kind of Riemannian submersion.

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### §1. Charged particles and Okumura geodesics

Let  $(M, g)$  be an odd-dimensional Riemannian manifold with the Riemannian metric  $g$ . We denote by  $\nabla$  the Levi-Civita connection of  $M$ . A Sasakian structure on  $M$  is defined by a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$  and 1-form  $\eta$  such that

$$(1.1) \quad \phi^2 = -1 + \eta \otimes \xi,$$

$$(1.2) \quad \eta(\xi) = 1,$$

$$(1.3) \quad g(X, \xi) = \eta(X),$$

$$(1.4) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(1.5) \quad d\eta(X, Y) = \frac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) = g(X, \phi Y),$$

$$(1.6) \quad (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X.$$

(If we adopt the notation in [5], we must replace  $\phi$  here by  $-\phi$ .)

A Riemannian manifold equipped with a Sasakian structure is called a Sasakian manifold.

In this section we assume that  $(M, g, \phi, \eta, \xi)$  is a Sasakian manifold. Then by (1.5),  $\phi$  is skew-symmetric with respect to  $g$ . Further,  $\xi$  is a Killing vector field, which satisfies  $\|\xi\| = 1$ ,  $\phi\xi = 0$  and  $\nabla_X\xi = -\phi X$ . The integral curves of  $\xi$  are geodesics.

For a constant  $r \in \mathbf{R}$ , we define a tensor field  $A$  of type  $(1, 2)$  by

$$A(X)Y = d\eta(X, Y)\xi + r\eta(X)\phi Y + \eta(Y)\phi X.$$

Then  $A(X)$  is skew-symmetric with respect to  $g$ . The Okumura linear connection  $\tilde{\nabla}$  is defined by  $\tilde{\nabla}_X Y = \nabla_X Y + A(X)Y$ , which satisfies  $\tilde{\nabla}g = 0$  and  $\tilde{\nabla}\xi = 0$  (See [5]). We have

$$(1.7) \quad \tilde{\nabla}_X X = \nabla_X X + (r + 1)\eta(X)\phi X.$$

A curve  $x(t)$  in  $M$  is called the motion of a charged particle if  $\nabla_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$  for a constant  $\kappa$ . The constant  $\kappa$  is the charge-to-mass ratio of  $x(t)$  (See [1], [2] and [3] for related topics).

**Proposition 1.1.** (1) *If  $x(t)$  is an Okumura geodesic, that is  $\tilde{\nabla}_{\dot{x}}\dot{x} = 0$ , then  $\eta(\dot{x}(t))$  is a constant.*

(2) *If  $x(t)$  is the motion of a charged particle, then  $\eta(\dot{x}(t))$  is a constant.*

*Proof.* (1) Using (1.3),  $\tilde{\nabla}g = 0$  and  $\tilde{\nabla}\xi = 0$ , we have

$$\frac{d}{dt}\eta(\dot{x}(t)) = \frac{d}{dt}g(\dot{x}(t), \xi) = g(\tilde{\nabla}_{\dot{x}}\dot{x}, \xi) + g(\dot{x}, \tilde{\nabla}_{\dot{x}}\xi) = 0.$$

(2) Using (1.3),  $\nabla g = 0$ , we have

$$\frac{d}{dt}\eta(\dot{x}(t)) = g(\nabla_{\dot{x}}\dot{x}, \xi) + g(\dot{x}, \nabla_{\dot{x}}\xi) = \kappa g(\phi(\dot{x}), \xi) - g(\dot{x}, \phi(\dot{x})) = 0.$$

□

Proposition 1.1 and (1.7) immediately imply the following:

**Proposition 1.2.** (1) *Let  $x(t)$  be an Okumura geodesic. Set  $c = \eta(\dot{x}(t))$ , then  $x(t)$  is the motion of a charged particle of the charge-to-mass ratio  $\kappa = -(r + 1)c$ .*

(2) *Let  $x(t)$  be the motion of a charged particle. Set  $c = \eta(\dot{x}(t))$ .*

(2-1) *When  $c \neq 0$ , then  $x(t)$  is an Okumura geodesic for  $r = -(\frac{\kappa}{c} + 1)$ .*

(2-2) *When  $c = 0$ , then  $\tilde{\nabla}_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$ .*

**Corollary 1.3.** *A curve  $x(t)$  is a geodesic with respect to the Levi-Civita connection if and only if*

(1)  *$x(t)$  is an Okumura geodesic for  $r = -1$  when  $\eta(\dot{x}) \neq 0$ ,*

(2)  *$x(t)$  is an Okumura geodesic for any  $r$  when  $\eta(\dot{x}) = 0$ .*

§2. Sasaki-Kähler submersion

Let  $\pi : \bar{M} \rightarrow M$  be a Riemannian submersion from a Sasakian manifold  $(\bar{M}, g, \phi, \eta, \xi)$  of dimension  $2n + 1$  onto a Kähler manifold  $(M, g, J)$  of real dimension  $2n$ . We call  $\pi$  a Sasaki-Kähler submersion if

- (1)  $\pi^{-1}(y)$  ( $y \in M$ ) is the image of an integral curve of  $\xi$ ,
- (2)  $d\pi\phi X = Jd\pi X$  for any horizontal vector  $X$ , that is  $\eta(X) = 0$ .

For instance, we can construct a Sasaki-Kähler submersion from any Hermitian symmetric space  $M$ .

**Theorem 2.1.** *Let  $\pi : \bar{M} \rightarrow M$  be a Sasaki-Kähler submersion. Assume that  $x(t) \in \bar{M}$  is the motion of a charged particle of the charge-to-mass ratio  $\kappa$ . Define a constant  $c$  by  $c = \eta(\dot{x})$ . Then  $y(t) = \pi(x(t))$  is the motion of a charged particle of the charge-to-mass ratio  $\kappa + 2c$ , that is  $\nabla_{\dot{y}}\dot{y} = (\kappa + 2c)J\dot{y}$ , where  $\nabla$  is the Levi-Civita connection of  $M$ . In particular, if  $x(t)$  is a geodesic, then  $y(t)$  is the motion of a charged particle of the charge-to-mass ratio  $2c$ .*

*Proof.* Since  $\|\dot{x}\|$  is a constant,  $\dot{x}(t) = 0$  for some  $t$  if and only if  $\dot{x}(t) = 0$  for any  $t$ . In this case,  $x(t)$  is a single point. Hence we may assume  $\dot{x}(t) \neq 0$  for any  $t$ . If  $\dot{x}(t)$  is proportional to  $\xi$  for some  $t$ , then  $x(t)$  is an integral curve of  $\xi$ . In this case,  $y(t)$  is a single point. Hence we may assume that  $\dot{x}$  is not proportional to  $\xi$  for any  $t$ . In other words, we may assume  $\dot{y}(t) \neq 0$  for any  $t$ . Hence there exists a (local) vector field  $X$  of  $M$  such that  $X = \dot{y}$ . If we denote by  $\bar{X}$  the horizontal lift of  $X$ , then we have  $\dot{x} = \bar{X} + \eta(\dot{x})\xi = \bar{X} + c\xi$ . Since  $x(t)$  is the motion of a charged particle, we get

$$\kappa\phi\bar{X} = \kappa\phi\dot{x} = \bar{\nabla}_{\dot{x}}\dot{x} = \bar{\nabla}_{\bar{X}+c\xi}(\bar{X} + c\xi) = \bar{\nabla}_{\bar{X}}\bar{X} + c(-2\phi\bar{X} + [\xi, \bar{X}]),$$

where  $\bar{\nabla}$  is the Levi-Civita connection of  $\bar{M}$ . Since  $\xi$  and  $0$  are  $\pi$ -related, and  $\bar{X}$  and  $X$  are  $\pi$ -related, we have  $\pi[\xi, \bar{X}] = [\pi\xi, \pi\bar{X}] = 0$ . Hence  $[\xi, \bar{X}]$  is vertical. Since  $\xi$  is a Killing vector field and  $\bar{X}$  is perpendicular to  $\xi$ , we have  $\eta([\xi, \bar{X}]) = g(\xi, [\xi, \bar{X}]) = \xi(g(\xi, \bar{X})) = 0$ . Hence  $[\xi, \bar{X}] = 0$ , which implies that  $\kappa\phi\bar{X} = \bar{\nabla}_{\bar{X}}\bar{X} - 2c\phi\bar{X}$ . Using [4, p. 212, Lemma 45, (3)], we obtain  $\nabla_{\dot{y}}\dot{y} = \nabla_X X = d\pi(\bar{\nabla}_{\bar{X}}\bar{X}) = (\kappa + 2c)\pi\phi\bar{X} = (\kappa + 2c)J\dot{y}$ . □

References

- [1] O. Ikawa, Hamiltonian dynamics of a charged particle, *Hokkaido Math. J.* **32** (2003) 661–671.
- [2] O. Ikawa, Motion of a charged particles in Kähler  $C$ -spaces, *Yokohama Math. J.* **50** (2003) 31–39.

- [3] O. Ikawa, Motion of a charged particles in homogeneous Kähler and homogeneous Sasakian manifolds, *Far East J. Math. Sci.* **14** (2004), 283–302.
- [4] B. O’Neill, *Semi-Riemannian geometry*, Academic Press, New York (1983).
- [5] T. Takahashi, Sasakian  $\phi$ -symmetric spaces, *Tôhoku Math. Journ.* **29** (1977), 91–113.

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