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On pseudo projective curvature tensor of a contact metric manifold

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Abstract. The paper deals with extended pseudo projective curvature tensor P^e of contact metric manifolds. We prove that (k, μ)-manifold with vanishing extended pseudo projective curvature tensor P^e is a Sasakian manifold. Several interesting corollaries of this result are drawn. Non-Sasakian (k, μ)-manifold with pseudo projective curvature tensor P satisfying $P(\xi, X) \cdot S = 0$, where S is the Ricci tensor, are classified.

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§1. Introduction

The unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature admits a contact metric structure (φ, ξ, η, g) such that the characteristic vector field ξ belongs to the (k, μ)-nullity distribution for some real numbers k and μ. This means that for any vector fields X and Y the curvature tensor R satisfies the condition

$$R(X, Y)\xi = (kI + \mu h)R_0(X, Y)\xi, \tag{1.1}$$

where

$$R_0(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{1.2}$$

and h denote Lie derivative of the structure tensor field φ in the direction of ξ. The class of contact metric manifolds which satisfies (1.1) has been classified in all dimensions at least locally (see [7] and [8]).

Recently, B.Prasad[15]introduced a new type of curvature tensor which is known as pseudo projective curvature tensor. A K-contact manifold is always a contact metric manifold, but the converse is not true in general. Thus, it is worthwhile to study pseudo projective curvature tensor P and E-pseudo projective curvature tensor P^e in contact metric manifold. Here we prove that a (k,μ)-manifold with vanishing E-pseudo projective curvature tensor is a Sasakian manifold. Then, we draw several corollaries of this result to $N(k)$-contact metric manifolds [16], the unit tangent sphere bundles [7], $N(k)$-contact space forms [10] and (k,μ)-space forms [11].

In [13] and [14] contact metric manifolds satisfying $R(X,\xi)\cdot S = 0$ and in ([1], [2] and [3]) Kenmotsu and 3-dimensional trans-Sasakian manifolds satisfying some curvature conditions are studied. From these studies, we classify non-Sasakian (k,μ)-manifolds with pseudo projective curvature tensor P satisfying $P(\xi,X)\cdot S = 0$ and obtain some interesting results.

§2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is called an almost contact manifold if either its structural group can be reduced to $U(n)\times 1$ or equivalently, there is an almost contact structure (φ,ξ,η) consisting of a $(1,1)$ tensor field φ, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta\otimes\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\circ\varphi = 0. \tag{2.2}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M\times\mathbf{R}$ defined by

$$J\left(X,\lambda\frac{d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t the coordinate of $\mathbf{R}$ and λ a smooth function on $M\times\mathbf{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi,\varphi] + 2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ. Let g be a compatible Riemannian metric with (φ,ξ,η), that is,

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

or equivalently,

$$g(X,\varphi Y) = -g(\varphi X,Y) \quad \text{and} \quad g(X,\xi) = \eta(X)$$

for all vector fields X, Y. Then, M become an almost contact metric manifold equipped with an almost contact metric structure (φ,ξ,η,g).

An almost contact metric structure become a contact metric structure if

$$g(X, \varphi Y) = d\eta(X, Y), \quad \text{for all vector fields } X, \ Y.$$

In a contact metric manifold, the $(1,1)$-tensor field h is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \nabla\xi = -\varphi - \varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0, \tag{2.4}$$

where ∇ is the Levi-Civita connection.

A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$\nabla_X \varphi = R_0(\xi, X), \tag{2.5}$$

while a contact metric manifold M is Sasakian if and only if

$$R(X, Y)\xi = R_0(X, Y)\xi, \quad \text{for all vector fields } X, Y \text{ on } M. \tag{2.6}$$

The (k, μ)-nullity distribution $N(k, \mu)$ of a contact metric manifold M for the pair $(k, \mu) \in \mathbf{R}^2$, is a distribution (see [7] and [13])

$$\begin{aligned} N(k,\mu) \quad &: \quad P \mapsto N_P(k,\mu) \\ &= \quad \{U \in T_P M \mid R(X,Y)U = (kI + \mu h)R_0(X,Y)U, \ \forall \ X, Y \in T_P M\}. \end{aligned}$$

A contact metric manifold with $\xi \in N(k, \mu)$ is called a (k, μ)-manifold. For a (k, μ)-manifold it is known that $h^2 = (k-1)\varphi^2$. This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for (k, μ)-manifold the condition of being Sasakian manifold, K-Contact manifold, $k = 1$ and $h = 0$ are all equivalent. If $\mu = 0$, the (k, μ)-nullity distribution $N(k, \mu)$ is reduced to the k-nullity distribution $N(k)$ (see [16]). Further if ξ belongs to $N(k)$, then we call a contact metric manifold M an $N(k)$-contact metric manifold.
We recall the following theorem due to D.E. Blair [5]:

Theorem 1. *A contact metric manifold M^{2n+1} satisfying $R(X, Y)\xi = 0$ is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

We also need the following definition:

Definition 1. A contact metric manifold M is said to be η-Einstein if the Ricci operator Q satisfies

$$Q = \alpha I + \beta \eta \otimes \xi, \tag{2.7}$$

where α and β are smooth functions on the manifold. In particular if $\beta = 0$, then M is an Einstein manifold.

§3. (k,μ)-manifold with vanishing E-pseudo projective curvature tensor

In [15], pseudo projective curvature tensor in an almost contact metric manifold is defined as follows:

$$P(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{2n+1}\left[\frac{a}{2n} + b\right][g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

where a and b are constants such that a, $b \neq 0$ and r denote scalar curvature of the manifold. For a $(2n+1)$-dimensional (k,μ)-manifold M, we have

$$R(X,Y)\xi = (kI + \mu h)R_0(X,Y)\xi, \tag{3.2}$$

which is equivalent to

$$R(\xi,X) = R_0(\xi,(kI+\mu h)X) = -R(X,\xi). \tag{3.3}$$

In particular, one can get

$$R(\xi,X)\xi = k(\eta(X)\xi - X) - \mu hX = -R(X,\xi)\xi. \tag{3.4}$$

From (3.1), (3.2) and (3.3), it follows that

$$P(X,Y)\xi = \left[(a+2nb)(k - \frac{r}{2n(2n+1)})I + a\mu h\right]R_0(X,Y)\xi, \tag{3.5}$$

$$P(\xi,X) = \left[(a+2nb)(k - \frac{r}{2n(2n+1)})\right]R_0(\xi,X) + a\mu R_0(\xi,hX). \tag{3.6}$$

Consequently, we have

$$P(\xi,X)\xi = \left[(a+2nb)(k - \frac{r}{2n(2n+1)})\right](\eta(X)\xi - X) - a\mu hX, \tag{3.7}$$

$$\eta(P(X,Y)\xi) = 0, \tag{3.8}$$

$$\eta(P(\xi,X)Y) = \left[(a+2nb)(k - \frac{r}{2n(2n+1)})\right][g(X,Y) - \eta(X)\eta(Y)] + a\mu g(hX,Y). \tag{3.9}$$

The E-pseudo projective curvature tensor P^e of pseudo projective curvature tensor P is defined as follows:

$$P^e(X,Y)Z = P(X,Y)Z - \eta(X)P(\xi,Y)Z - \eta(Y)P(X,\xi)Z - \eta(Z)P(X,Y)\xi. \tag{3.10}$$

Let M be a $(2n+1)$-dimensional (k,μ)-manifold. If E-pseudo projective curvature tensor of M vanishes, then from (3.7) and (3.10) we have

$$\begin{aligned} 0 &= P^e(X,\xi)\xi \\ &= \left[(a+2nb)\left(k - \frac{r}{2n(2n+1)}\right)\right](\eta(X)\xi - X) - a\mu hX \\ &= -P(X,\xi)\xi, \end{aligned} \tag{3.11}$$

which in view of $h^2 = (k-1)\varphi^2$, gives

$$h^2 = \frac{a}{a+2nb}\left[\frac{2n(2n+1)}{r - 2nk(2n+1)}\right](k-1)\mu h. \tag{3.12}$$

Taking the trace of (3.12), we obtain

$$\text{trace}(h^2) = 2n(1-k) = 0, \tag{3.13}$$

which gives $k = 1$. Thus M becomes Sasakian. Hence we state the following:

Theorem 2. *A (k,μ)-manifold with vanishing E-pseudo projective curvature tensor is a Sasakian manifold.*

From Theorem 2 we derive

Corollary 1. *An $N(k)$-contact metric manifold with vanishing E-pseudo projective curvature tensor is a Sasakian manifold.*

The unit tangent sphere bundle T_1M equipped with the standard contact metric structure is a (k,μ)-manifold if and only if the base manifold M is of constant curvature c with $k = c(2-c)$ and $\mu = -2c$ ([7]). In case of $c \neq 1$, the unit tangent sphere bundle is non-Sasakian. Denote the unit tangent sphere bundle of a space of constant curvature c with standard contact metric structure as $T_1M(c)$. Applying Theorem 2 to $T_1M(c)$, one can obtain

Corollary 2. *In $T_1M(c)$ if the E-pseudo projective curvature tensor vanishes, then $c = 1$.*

In an almost contact metric manifold if a unit vector X is orthogonal to ξ, then X and φX span a φ-section. And if the sectional curvature $c(X)$ of all φ-sections is independent of X, then M is of pointwise constant φ-sectional curvature. Further an $N(k)$-contact metric manifold M with pointwise constant φ-sectional curvature c is called an $N(k)$-contact space form $M(c)$. The curvature tensor of $M(c)$ is given by [10]:

$$
\begin{aligned}
(3.14)\quad & 4R(X,Y)Z \\
&= (c+3)[g(Y,Z)X - g(X,Z)Y] \\
&\quad + (c-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi \\
&\quad - \eta(X)g(Y,Z)\xi + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z] \\
&\quad + 4(k-1)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi \\
&\quad - \eta(Y)g(X,Z)\xi] + 4[g(hY,Z)X - g(hX,Z)Y + g(Y,Z)hX \\
&\quad - g(X,Z)hY + \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + \eta(Y)g(hX,Z)\xi \\
&\quad - \eta(X)g(hY,Z)\xi] + 2[g(hY,Z)hX - g(hX,Z)hY \\
&\quad + g(\varphi hX,Z)\varphi hY - g(\varphi hY,Z)\varphi hX],
\end{aligned}
$$

for all vector fields X, Y and Z, where c is constant on M if dim $(M) > 3$.

Now, applying Theorem 2 to an $N(k)$-contact space form, we state the following:

Corollary 3. *An $N(k)$-contact space form with vanishing E-pseudo projective curvature tensor is a Sasakian space form.*

Let M be a $(2n+1)$-dimensional (k,μ)-manifold $(n > 1)$. Next, if M has a constant φ-sectional curvature c then it is called a (k,μ)-space form. The curvature tensor of (k,μ)-space form is given by [11]:

(3.15)

$$
\begin{aligned}
& R(X,Y)Z \\
&= \frac{(c+3)}{4}[g(Y,Z)X - g(X,Z)Y] \\
&\quad + \frac{(c-1)}{4}[2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X] \\
&\quad + \frac{(c+3-4k)}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\
&\quad - g(Y,Z)\eta(X)\xi] + \frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\varphi hX,Z)\varphi hY \\
&\quad - g(\varphi hY,Z)\varphi hX] + g(\varphi Y,\varphi Z)hX - g(\varphi X,\varphi Z)hY \\
&\quad + g(hX,Z)\varphi^2 Y - g(hY,Z)\varphi^2 X + \mu[\eta(Y)\eta(Z)hX \\
&\quad - \eta(X)\eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi],
\end{aligned}
$$

for all vector fields X, Y and Z, where $c + 2k = -1 = k - \mu$ if $k < 1$.

Applying Theorem 2 to a (k,μ)-contact space form, we obtain the following:

Corollary 4. *A (k,μ)-contact space form with vanishing E-pseudo projective curvature tensor is a Sasakian space form.*

Remark 1. Theorem 2 and its Corollaries 1 to 4 are valid for vanishing of pseudo projective curvature tensor P also.

§4. (k,μ)-manifold satisfying $P(\xi,X)\cdot S=0$

For a $(2n{+}1)$-dimensional (k,μ)-manifold M, it is well known that

$$S(X,\xi)=2nk\eta(X). \tag{4.1}$$

In view of (3.8) and (3.9), (4.1) gives

$$\begin{aligned} S(P(\xi,X)\xi,Y) = 2nk(a+2nb)\left(k-\frac{r}{2n(2n+1)}\right)\eta(X)\eta(Y) \\ -(a+2nb)\left(k-\frac{r}{2n(2n+1)}\right)S(X,Y) \\ -a\mu S(hX,Y) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} S(P(\xi,X)Y,\xi) = 2nk(a+2nb)\left(k-\frac{r}{2n(2n+1)}\right)[g(X,Y) \\ -\eta(X)\eta(Y)]+2nka\mu g(hX,Y) \end{aligned} \tag{4.3}$$

respectively.

In a $(2n{+}1)$-dimensional (k,μ)-manifold, the condition $P(\xi,X)\cdot S=0$ is equivalent to

$$S(P(\xi,X)Y,\xi)+S(Y,P(\xi,X)\xi)=0. \tag{4.4}$$

Substituting (4.2) and (4.3) in (4.4) followed by a simple calculation gives,

$$\begin{aligned} \left[(a+2nb)\left(k-\frac{r}{2n(2n+1)}\right)\right][S(X,Y)-2nkg(X,Y)] \\ +a\mu[S(hX,Y)-2nkg(hX,Y)]=0. \end{aligned} \tag{4.5}$$

It is well known that in a $(2n{+}1)$-dimensional non-Sasakian (k,μ)-manifold M the Ricci operator Q is given as follows [7]:

$$\begin{aligned} Q=(2(n-1)-n\mu)I+(2(n-1)+\mu)h \\ +(2(1-n)+n(2k+\mu))\eta\otimes\xi. \end{aligned} \tag{4.6}$$

Consequently, the Ricci tensor S and the scalar curvature r are given by

$$\begin{aligned} S(X,Y)=(2(n-1)-n\mu)g(X,Y)+(2(n-1)+\mu)g(hX,Y) \\ +(2(1-n)+n(2k+\mu))\eta(X)\eta(Y), \end{aligned} \tag{4.7}$$

$$r=2n(2n-2+k-n\mu). \tag{4.8}$$

By virtue of (2.3) and (4.7), we also have

$$S(hX,Y) = (2(n-1) - n\mu)g(hX,Y) - (k-1)(2(n-1)+\mu)[g(X,Y) - \eta(X)\eta(Y)], \tag{4.9}$$

where $\eta \circ h = 0$, $h^2 = (k-1)\varphi^2$.

From(2.7) and (4.7), one can see that a non-Sasakian (k,μ)-manifold M is η-Einstein if and only if $\mu = -2(n-1)$. In this case the Ricci tensor is given by

$$S = 2(n^2-1)g - 2(n^2 - nk - 1)\eta \otimes \eta. \tag{4.10}$$

Putting $\mu = -2(n-1)$ in (4.8), we obtain

$$r = 2n(k + 2(n-1)(n+1)). \tag{4.11}$$

Now by considering $\mu = -2(n-1)$ in (4.3), then it takes the form

$$S(P(\xi,X)Y,\xi) = 2nk(a+2nb)\left(k - \frac{r}{2n(2n+1)}\right)[g(X,Y) - \eta(X)\eta(Y)] + 4n(1-n)kag(hX,Y). \tag{4.12}$$

In view of (4.2) and (4.10), we get

$$S(P(\xi,X)\xi,Y) = 4a(n-1)(n^2-1)g(hX,Y) + 2(1-n^2)(a+2nb)\left(k - \frac{r}{2n(2n+1)}\right)[g(X,Y) - \eta(X)\eta(Y)]. \tag{4.13}$$

If M satisfies $P(\xi,X) \cdot S = 0$, from (4.4), (4.12) and (4.13) we get

$$S(P(\xi,X)Y,\xi) + S(Y,P(\xi,X)\xi) = 0,$$

which is equivalent to

$$2(1+nk-n^2)(a+2nb)\left(k - \frac{r}{2n(2n+1)}\right)[g(X,Y) - \eta(X)\eta(Y)] - 4(n-1)(1+nk-n^2)ag(hX,Y) = 0.$$

Contracting the above equation and then by taking account of (2.4), we have

$$4n(1+nk-n^2)(a+2nb)\left(k - \frac{r}{2n(2n+1)}\right) = 0.$$

This implies

$$k - \frac{r}{2n(2n+1)} = 0.$$

Using (4.11) in above, we obtain

$$k = \frac{n^2 - 1}{n}, \tag{4.14}$$

which is equivalent to $(1 + nk - n^2) = 0$. Thus in view of (4.10), M reduces to Einstein manifold. Hence we state the following:

Theorem 3. *In a* $(2n+1)$*-dimensional non-Sasakian* η*-Einstein* (k, μ)*-manifold* M *if the pseudo projective curvature tensor* P *satisfies* $P(\xi, X) \cdot S = 0$*, then* M *reduces to an Einstein manifold.*

From (4.14), we have $k = (n^2 - 1)/n < 1$. So $n = 1$ is the only case. This gives $\mu = 0$ which with $n = 1$ gives $k = 0$. Thus substituting $k = 0 = \mu$ in (1.1), we state the following:

Theorem 4. *In a* $(2n+1)$*-dimensional non-Sasakian* η*-Einstein* (k, μ)*-manifold* M *if the pseudo projective curvature tensor* P *satisfies* $P(\xi, X) \cdot S = 0$*, then* M *is flat and* 3*-dimensional.*

Next, let M be a $(2n+1)$-dimensional (k, μ)-manifold satisfying $P(\xi, X) \cdot S = 0$. Then we have the following four possible cases.

Case-1: Suppose $k = 0 = \mu$.

From (1.1) we have $R(X, Y)\xi = 0$. Thus, in view of Theorem 1, M is flat and 3-dimensional or it is locally isometric to $E^{n+1}(0) \times S^n(4)$.

Case-2: Suppose $k \neq 0 = \mu$.

Using $\mu = 0$ in (4.5), we have $S(X, Y) = 2nkg(X, Y)$. Thus M reduces to an Einstein Sasakian manifold.

Case-3(i): Suppose $k = 0 \neq \mu$ and $n > 1$.

Using $k = 0$ in (4.5), (4.7) and (4.9) we get

$$\begin{aligned} rS(X, Y) &= 2n(2n+1)\left(\frac{a}{a + 2nb}\right)\mu S(hX, Y), \\ S(X, Y) &= (2(n-1) - n\mu)[g(X, Y) - \eta(X)\eta(Y)] \\ &\quad + (2(n-1) + \mu)g(hX, Y) \quad \text{and} \\ S(hX, Y) &= (2(n-1) - n\mu)g(hX, Y) \\ &\quad + (2(n-1) + \mu)[g(X, Y) - \eta(X)\eta(Y)] \end{aligned}$$

respectively. From the above three equations, we get $S(X, Y) = C[g(X, Y) - \eta(X)\eta(Y)]$, for some suitable C. Now in view of Theorem 4, we see that the Case-3(i) is not possible.

Case-3(ii): Suppose $k = 0 \neq \mu$ and $n = 1$.

Using $k = 0$ and $n = 1$ in (4.5), (4.7) and (4.9) we get

$$rS(X,Y) = 6\left(\frac{a}{a+2nb}\right)\mu S(hX,Y),$$
$$S(X,Y) = -\mu[g(X,Y) - \eta(X)\eta(Y)] + \mu g(hX,Y) \text{ and}$$
$$S(hX,Y) = -\mu g(hX,Y) + \mu[g(X,Y) - \eta(X)\eta(Y)]$$

respectively.

From the above three relations, we get $\left[\left(\frac{a+2nb}{a}\right)\left(\frac{r}{6\mu}\right)+1\right]S(X,Y) = 0$. This gives either $\left(\frac{a+2nb}{a}\right)\left(\frac{r}{6\mu}\right)+1 = 0$ or $S(X,Y) = 0$. If $\left(\frac{a+2nb}{a}\right)\left(\frac{r}{6\mu}\right)+1 = 0$, then $r = -6\mu\left(\frac{a}{a+2nb}\right)$. Putting $k = 0$ and $n = 1$ in (4.8), we get $r = -2\mu$. Thus $(\frac{a+2nb}{a})(\frac{r}{6\mu}) + 1 = 0$ is not possible.

If $S(X,Y) = 0$, then taking $X = Y = \xi$ we have

$$S(\xi,\xi) = 2nk = 0,$$

which implies that $k = 0$. Using $k = 0$ in (4.8), we get $n\mu = 2(n-1)$. But we have $n = 1$, this implies $\mu = 0$, which is a contradiction. Thus, Case-3(ii) is also not possible.

Case-4(i): Suppose $k \neq 0$, $\mu \neq 0$ and $n > 1$. After eliminating $g(hX,Y)$ and $S(hX,Y)$ from (4.5), (4.7) and (4.9) we get $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y))$, for some suitable α and β. Thus M reduces to an η-Einstein manifold.

(ii): Suppose $k \neq 0$, $\mu \neq 0$ and $n = 1$.

Putting $n = 1$ in (4.5), (4.7) and (4.9) we get

$$\left(k - \frac{r}{6}\right)S(X,Y) = 2k\left(k - \frac{r}{6}\right)g(X,Y) + \left(\frac{a}{a+2b}\right)2k\mu g(hX,Y)$$
$$- \left(\frac{a}{a+2b}\right)\mu S(hX,Y),$$
$$S(X,Y) = -\mu g(X,Y) + \mu g(hX,Y) + (2k+\mu)\eta(X)\eta(Y) \text{ and}$$
$$S(hX,Y) = -\mu g(hX,Y) - (k-1)\mu g(X,Y)$$
$$+ (k-1)\mu\eta(X)\eta(Y)$$

respectively. Eliminating $g(hX,Y)$ and $S(hX,Y)$ from the above three equations, we have $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, for some suitable α and β. Thus, M is a η-Einstein manifold and in this case $\mu = -2(n-1)$. But $n = 1$, implies $\mu = 0$ which is a contradiction. Hence this case is not possible. Thus from the above four possible cases, we can able to state the following:

Theorem 5. *Let M be a $(2n+1)$-dimensional non-Sasakian (k,μ)-manifold satisfying the condition $P(\xi,X)\cdot S=0$ such that $a+2nb\neq 0$. Then the manifold M is either flat and 3-dimensional or is locally isometric to $E^{n+1}(0)\times S^n(4)$ or is an η-Einstein manifold or is a 3-dimensional Einstein manifold.*

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