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On Weakly Quasi-Conformally Symmetric Manifolds

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Abstract. The object of the present paper is to study weakly quasi-conformally symmetric Riemannian manifolds. Among others we obtain various sufficient conditions for such a manifold to be of weakly symmetric. The decomposable weakly quasi-conformally symmetric manifolds are studied and classified regorously. The existence of a weakly quasi-conformally symmetric and decomposable weakly quasi-conformally symmetric manifolds have been ensured by several non-trivial examples.

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§1. Introduction

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamássy and Binh [8] and later Binh [1] studied decomposable weakly symmetric manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$
\begin{aligned}
(1.1)\quad (\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\
&\quad +\gamma(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\
&\quad +\sigma(V)R(Y,Z,U,X)
\end{aligned}
$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (not simultaneously zero), $\chi(M^n)$ is the set of all smooth vector fields over the manifold and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. In 1999 U. C. De and S. Bandyopadhyay [3] established the existence of a $(WS)_n$ by an example and proved that in a $(WS)_n$, the associated 1-forms $\beta = \gamma$ and $\delta = \sigma$. Hence (1.1) reduces to the following:

$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\
&\quad +\beta(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\
&\quad +\delta(V)R(Y,Z,U,X).
\end{aligned} \tag{1.2}
$$

Also De and Bandyopadhyay [4] studied weakly conformally symmetric manifolds. In this connection it may be noted that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric manifold introduced by Chaki [2], but the defining condition of a $(WS)_n$ is little weaker than that of a generalized pseudo-symmetric manifold. That is, if in (1.1) the 1-form α is replaced by 2α and σ is replaced by α then the manifold will be a generalized pseudo-symmetric manifold [2]. In 1968 Yano and Sawaki [9] defined and studied a tensor field W on a Riemannian manifold of dimension n which includes both the conformal curvature tensor C and the concircular curvature tensor $\widetilde{C}$ as special cases. This tensor field W is known as quasi-conformal curvature tensor given by

$$
\begin{aligned}
W(X,Y,Z,U) &= -(n-2)bC(X,Y,Z,U) \\
&\quad +[a+(n-2)b]\widetilde{C}(X,Y,Z,U),
\end{aligned} \tag{1.3}
$$

where a and b are arbitrary constants not simultaneously zero, C and $\widetilde{C}$ are the conformal curvature tensor and the concircular curvature tensor of type (0, 4) respectively. The present paper deals with a non-quasi-conformally flat Riemannian manifold $(M^n, g)(n > 3)$ [the condition $(n > 3)$ is assumed throughout this paper as the conformal curvature tensor vanishes for $n = 3$] whose quasi-conformal curvature tensor W satisfies the condition

$$
\begin{aligned}
(\nabla_X W)(Y,Z,U,V) &= \alpha(X)W(Y,Z,U,V) + \beta(Y)W(X,Z,U,V) \\
&\quad +\gamma(Z)W(Y,X,U,V) + \delta(U)W(Y,Z,X,V) \\
&\quad +\sigma(V)W(Y,Z,U,X),
\end{aligned} \tag{1.4}
$$

where $\alpha, \beta, \gamma, \delta$ and σ are 1-forms (not simultaneously zero). Such a manifold will be called a weakly quasi-conformally symmetric manifold and denoted by $(WQCS)_n$, where the first 'W' stands for *'weakly'* and 'QC' stands for *'quasi-conformal curvature tensor'* as the 'weakly conformally symmetric manifold' was denoted by $(WCS)_n$ [4]. In particular, if $a = 1$ and $b = -\frac{1}{n-2}$ then a $(WQCS)_n$ reduces to a $(WCS)_n$. The manifold $(WQCS)_n$ is introduced and studied by the first author and K. K. Baishya [7]. It is also shown in [7] that in a $(WQCS)_n$, the 1-forms $\beta = \gamma$ and $\delta = \sigma$ and hence (1.4) reduces to the following form:

$$(1.5)\quad \begin{aligned}(\nabla_X W)(Y,Z,U,V) &= \alpha(X)W(Y,Z,U,V) + \beta(Y)W(X,Z,U,V)\\ &+\beta(Z)W(Y,X,U,V) + \delta(U)W(Y,Z,X,V)\\ &+\delta(V)W(Y,Z,U,X),\end{aligned}$$

where α, β and δ are 1-forms (not simultaneously zero).

Section 2 is concerned with some basic results of $(WQCS)_n$. It is shown that if in a $(WQCS)_n$ the Ricci tensor is of Codazzi [5] type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P defined by $g(X,P) = \lambda(X)$, where r is the scalar curvature of the manifold. Also it is proved that if a $(WQCS)_n$ is of constant scalar curvature then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L_1 defined by $g(X,L_1) = \alpha(X)$. Section 3 is devoted to the decomposable $(WQCS)_n$, which is generally called the product $(WQCS)_n$ and it is shown that in such a manifold satisfying certain conditions one of the decomposition is locally symmetric and the other is quasi-conformally flat. Also we obtain some other illuminating results on a decomposable $(WQCS)_n$. Section 4 is devoted to the $(WQCS)_n$ satisfying certain conditions and obtained several interesting results for such a manifold to be a $(WS)_n$.

The last section deals with several non-trivial examples of $(WQCS)_n$ and also of decomposable $(WQCS)_n$.

§2. Some basic results of $(WQCS)_n$

In this section we deduce some basic results of a $(WQCS)_n$. The conformal curvature tensor field C of type (0, 4) and the concircular curvature tensor field $\widetilde{C}$ of type (0, 4) are respectively given by

$$\begin{aligned}C(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{n-2}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\\ &+g(Y,Z)S(X,U) - g(X,Z)S(Y,U)]\\ &+\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]\end{aligned}$$

and

$$\begin{aligned}\widetilde{C}(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{r}{n(n-1)}[g(Y,Z)g(X,U)\\ &-g(X,Z)g(Y,U)],\end{aligned}$$

for all vector fields $X,Y,Z,U \in \chi(M^n)$, where R, S, r are the Riemannian curvature tensor of type (0, 4), the Ricci tensor of type (0, 2) and the scalar curvature respectively of the manifold. The Riemannian curvature tensor R of type (0, 4) on a Riemannian manifold is defined as a quadrilinear mapping $R : \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) \to C^{\infty}(M)$ and is given by $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ for all $X, Y, Z, U \in \chi(M^n)$, where we have used the same symbol R of the curvature tensor of type (1, 3) as well as of type (0, 4) and $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, ∇ being the Levi-Civita connection and $C^{\infty}(M)$ is the set of all smooth functions over the manifold M. The Ricci tensor field S is the covariant tensor field of degree 2 defined by $S(Y, Z) = Tr.[X \to R(X, Y)Z]$ and the scalar curvature r is defined as the trace of the (1, 1) Ricci tensor Q i.e., $r = Tr.Q$ where $S(X, Y) = g(QX, Y)$ for all $X, Y \in \chi(M)$. Using the above expressions of Weyl conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ in (1.3) one can easily obtain

$$
\begin{aligned}
(2.1)\quad W(X, Y, Z, U) &= aR(X, Y, Z, U) + b[S(Y, Z)g(X, U) \\
&\quad -S(X, Z)g(Y, U) + g(Y, Z)S(X, U) \\
&\quad -g(X, Z)S(Y, U)] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y, Z)g(X, U) \\
&\quad -g(X, Z)g(Y, U)].
\end{aligned}
$$

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S of type (0, 2) and the scalar curvature r are given by the following

$$
\begin{aligned}
S(X, Y) &= \sum_{i=1}^{n} R(e_i, X, Y, e_i) \\
\text{and}\quad r &= \sum_{i=1}^{n} S(e_i, e_i) = \sum_{i=1}^{n} g(Qe_i, e_i).
\end{aligned}
$$

Again from (2.1) we can obtain

$$
\begin{aligned}
(2.2)\qquad \sum_{i=1}^{n} W(e_i, Y, Z, e_i) &= \sum_{i=1}^{n} W(Y, e_i, e_i, Z) \\
&= \{a + (n-2)b\}[S(Y, Z) - \frac{r}{n} g(Y, Z)].
\end{aligned}
$$

Differentiating (2.1) covariantly and then taking cyclic sum with respect to X, Y, Z we obtain by virtue of Bianchi identity that

$$
\begin{aligned}
(2.3)\qquad & (\nabla_X W)(Y, Z, U, V) + (\nabla_Y W)(Z, X, U, V) + (\nabla_Z W)(X, Y, U, V) \\
& = b[\{(\nabla_X S)(Z, U) - (\nabla_Z S)(X, U)\}g(Y, V) + \{(\nabla_Y S)(X, U) \\
& -(\nabla_X S)(Y, U)\}g(Z, V) + \{(\nabla_Z S)(Y, U) - (\nabla_Y S)(Z, U)\}g(X, V) \\
& +\{(\nabla_X S)(Y, V) - (\nabla_Y S)(X, V)\}g(Z, U) + \{(\nabla_Z S)(X, V)
\end{aligned}
$$

$$-(\nabla_X S)(Z,V)\}g(Y,U) + \{(\nabla_Y S)(Z,V) - (\nabla_Z S)(Y,V)\}g(X,U)]$$
$$-\frac{1}{n}(\frac{a}{n-1} + 2b)[dr(X)\{g(Z,U)g(Y,V) - g(Z,V)g(Y,U)\}$$
$$+dr(Y)\{g(Z,V)g(X,U) - g(Z,U)g(X,V)\}$$
$$+dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}].$$

We now suppose that in a Riemannian manifold the Ricci tensor is of Codazzi type [5]. Then we have

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) = (\nabla_Z S)(X,Y)$$

for all vector fields $X,\ Y,\ Z$ on the manifold. This implies that

$$dr(X) = 0 \text{ for all } \ X.$$

Therefore (2.3) yields

$$(2.4)\ \ (\nabla_X W)(Y,Z,U,V) + (\nabla_Y W)(Z,X,U,V) + (\nabla_Z W)(X,Y,U,V) = 0.$$

Hence if the Ricci tensor is of Codazzi type then in a Riemannian manifold the relation (2.4) holds. Again if a Riemannian manifold (M,g) satisfies the relation (2.4), then (2.3) yields

$$b[\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\}g(Y,V) + \{(\nabla_Y S)(X,U)$$
$$-(\nabla_X S)(Y,U)\}g(Z,V) + \{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\}g(X,V)$$
$$+\{(\nabla_X S)(Y,V) - (\nabla_Y S)(X,V)\}g(Z,U) + \{(\nabla_Z S)(X,V)$$
$$-(\nabla_X S)(Z,V)\}g(Y,U) + \{(\nabla_Y S)(Z,V) - (\nabla_Z S)(Y,V)\}g(X,U)]$$
$$-\frac{1}{n}(\frac{a}{n-1} + 2b)[dr(X)\{g(Z,U)g(Y,V) - g(Z,V)g(Y,U)\}$$
$$+dr(Y)\{g(Z,V)g(X,U) - g(Z,U)g(X,V)\}$$
$$+dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}] = 0.$$

Setting $Y = V = e_i$ in the above relation and then taking summation over i, $1 \leq i \leq n$ we get

$$(n-3)b[(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)] - \{\frac{(n-2)a}{n(n-1)}$$
$$+\frac{(3n-8)b}{2n}\}[dr(X)g(Z,U) - dr(Z)g(X,U)] = 0,$$

which yields on contraction over Z and U that $dr(X) = 0$ for all X provided $a + (n-2)b \neq 0$ and consequently the last relation reduces to

$$(\nabla_X S)(Z,U) = (\nabla_Z S)(X,U)$$
$$\text{for all } \ X, Z, U \in \chi(M) \ \text{ provided } \ b \neq 0.$$

Hence the Ricci tensor is of Codazzi type. Thus we can state the following:

Proposition 2.1. *In a Riemannian manifold* (M^n, g) *with* $b \neq 0$ *and* $a+(n-2)b \neq 0$*, the Ricci tensor is of Codazzi type if and only if the relation* (2.4) *holds.*

In view of (1.5), the relation (2.4) reduces to

$$\lambda(X)W(Y,Z,U,V) + \lambda(Y)W(Z,X,U,V) + \lambda(Z)W(X,Y,U,V) = 0, \tag{2.5}$$

where $\lambda(X) = \alpha(X) - 2\beta(X)$ for all X. By virtue of (2.1), (2.5) takes the form

$$\begin{aligned}
&a[\lambda(X)R(Y,Z,U,V) + \lambda(Y)R(Z,X,U,V) + \lambda(Z)R(X,Y,U,V)] \\
&+b[\lambda(X)\{S(Z,U)g(Y,V) - S(Y,U)g(Z,V) + S(Y,V)g(Z,U) \\
&-S(Z,V)g(Y,U)\} + \lambda(Y)\{S(X,U)g(Z,V) - S(Z,U)g(X,V) \\
&+S(Z,V)g(X,U) - S(X,V)g(Z,U)\} + \lambda(Z)\{S(Y,U)g(X,V) \\
&-S(X,U)g(Y,V) + S(X,V)g(Y,U) - S(Y,V)g(X,U)\}] \\
&-\frac{r}{n}(\frac{a}{n-1} + 2b)[\lambda(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} \\
&+\lambda(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\} \\
&+\lambda(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}] = 0.
\end{aligned} \tag{2.6}$$

Setting $Y = V = e_i$ in (2.6) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}
&\{a + (n-3)b\}[\lambda(X)S(Z,U) - \lambda(Z)S(X,U)] + a\lambda(R(Z,X)U) \\
&+b[\lambda(QZ)g(X,U) - \lambda(QX)g(Z,U)] - \frac{r}{n}\{\frac{(n-2)a}{n-1} \\
&+(n-4)b\}[\lambda(X)g(Z,U) - \lambda(Z)g(X,U)] = 0.
\end{aligned} \tag{2.7}$$

Again putting $X = U = e_i$ in (2.7) and taking summation over i, $1 \leq i \leq n$, we obtain

$$\begin{aligned}
&\{a + (n-2)b\}[\lambda(QZ) - \frac{r}{n}\lambda(Z)] = 0, \quad \text{which yields} \\
&S(Z,P) = \frac{r}{n}g(Z,P),
\end{aligned}$$

provided that $a + (n-2)b \neq 0$ where $\lambda(X) = \alpha(X) - 2\beta(X)$ and $g(X,P) = \lambda(X)$. This leads to the following:

Proposition 2.2. *If in a* $(WQCS)_n$ *the Ricci tensor is of Codazzi type then* $\frac{r}{n}$ *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* P*, defined by* $g(X,P) = \lambda(X)$*, provided that* $a + (n-2)b \neq 0$.

Next in view of (1.5), the relation (2.3) takes the form

$$\begin{aligned}(2.8)\quad & b[\{(\nabla_X S)(Z,U)-(\nabla_Z S)(X,U)\}g(Y,V)+\{(\nabla_Y S)(X,U)\\ &-(\nabla_X S)(Y,U)\}g(Z,V)+\{(\nabla_Z S)(Y,U)-(\nabla_Y S)(Z,U)\}g(X,V)\\ &+\{(\nabla_X S)(Y,V)-(\nabla_Y S)(X,V)\}g(Z,U)+\{(\nabla_Z S)(X,V)\\ &-(\nabla_X S)(Z,V)\}g(Y,U)+\{(\nabla_Y S)(Z,V)-(\nabla_Z S)(Y,V)\}g(X,U)]\\ &-\frac{1}{n}(\frac{a}{n-1}+2b)[dr(X)\{g(Z,U)g(Y,V)-g(Z,V)g(Y,U)\}\\ &+dr(Y)\{g(Z,V)g(X,U)-g(Z,U)g(X,V)\}\\ &+dr(Z)\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}]\\ &=\lambda(X)W(Y,Z,U,V)+\lambda(Y)W(Z,X,U,V)+\lambda(Z)W(X,Y,U,V),\end{aligned}$$

where $\lambda(X)=\alpha(X)-2\beta(X)$ for all X. Setting $Y=V=e_i$ in (2.8) and taking summation over i, $1\leq i\leq n$, we obtain by virtue of (2.1) and (2.2) that

$$\begin{aligned}(2.9)\quad & (n-3)b[(\nabla_X S)(Z,U)-(\nabla_Z S)(X,U)]\\ &-[\frac{a(n-2)}{n(n-1)}-\frac{b(3n-8)}{2n}][dr(X)g(Z,U)-dr(Z)g(X,U)]\\ &=[a+(n-3)b][\lambda(X)S(Z,U)-\lambda(Z)S(X,U)]\\ &+a\lambda(R(Z,X)U)+b[\lambda(QZ)g(X,U)-\lambda(QX)g(Z,U)]\\ &-\frac{r}{n}[\frac{(n-2)a}{n-1}+(n-4)b][\lambda(X)g(Z,U)-\lambda(Z)g(X,U)].\end{aligned}$$

Putting $X=U=e_i$ in (2.9) and taking summation over i, $1\leq i\leq n$, we get

$$(2.10)\qquad \frac{n-2}{2n}dr(Z)=\lambda(QZ)-\frac{r}{n}\lambda(Z)\quad\text{for}\quad a+(n-2)b\neq 0.$$

If the manifold under consideration is of constant scalar curvature then (2.10) yields

$$(2.11)\qquad \lambda(QZ)=\frac{r}{n}\lambda(Z)\quad\text{for}\quad a+(n-2)b\neq 0.$$

If P is the vector field associated with λ such that $g(X,P)=\lambda(X)=\alpha(X)-2\beta(X)$ then (2.11) can be written as

$$(2.12)\qquad S(Z,P)=\frac{r}{n}g(Z,P)\quad\text{for}\quad a+(n-2)b\neq 0.$$

Thus we can state the following:

Proposition 2.3. *If a $(WQCS)_n$ is of constant scalar curvature with $a+(n-2)b\neq 0$ then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P, defined by $g(X,P)=\lambda(X)$ for all X.*

Now using (2.1) in (1.5) we obtain

$$
\begin{aligned}
& a(\nabla_X R)(Y,Z,U,V) + b[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V) \\
& +(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)] \\
& -\frac{1}{n}dr(X)(\frac{a}{n-1} + 2b)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)] \\
& = a[\alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) + \beta(Z)R(Y,X,U,V) \\
& +\delta(U)R(Y,Z,X,V) + \delta(V)R(Y,Z,U,X)] + b[\alpha(X)\{S(Z,U)g(Y,V) \\
& -S(Y,U)g(Z,V) + S(Y,V)g(Z,U) - S(Z,V)g(Y,U)\} \\
& +\beta(Y)\{S(Z,U)g(X,V) - S(X,U)g(Z,V) + S(X,V)g(Z,U) \\
& -S(Z,V)g(X,U)\} + \beta(Z)\{S(X,U)g(Y,V) - S(Y,U)g(X,V) \\
& +S(Y,V)g(X,U) - S(X,V)g(Y,U)\} + \delta(U)\{S(Z,X)g(Y,V) \\
& -S(X,Y)g(Z,V) + S(Y,V)g(Z,X) - S(Z,V)g(X,Y)\} \\
& +\delta(V)\{S(Z,U)g(X,Y) - S(Y,U)g(Z,X) + S(X,Y)g(Z,U) \\
& -S(Z,X)g(Y,U)\}] - \frac{r}{n}(\frac{a}{n-1} + 2b)[\alpha(X)\{g(Z,U)g(Y,V) \\
& -g(Y,U)g(Z,V)\} + \beta(Y)\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\} \\
& +\beta(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\} \\
& +\delta(U)\{g(Z,X)g(Y,V) - g(X,Y)g(Z,V)\} \\
& +\delta(V)\{g(Z,U)g(X,Y) - g(Y,U)g(Z,X)\}].
\end{aligned} \tag{2.13}
$$

Setting $Y = V = e_i$ in (2.13) and taking summation over i, $1 \le i \le n$, we get

$$
\begin{aligned}
& \{a + (n-2)b\}[(\nabla_X S)(Z,U) - \frac{1}{n}dr(X)g(Z,U)] \\
& = \{a + (n-2)b\}[\alpha(X)\{S(Z,U) - \frac{r}{n}g(Z,U)\} \\
& +\beta(Z)\{S(X,U) - \frac{r}{n}g(X,U)\} \\
& +\delta(U)\{S(Z,X) - \frac{r}{n}g(Z,X)\}] + a[\beta(R(X,Z)U) \\
& +\delta(R(X,U)Z)] + b[\beta(X)S(Z,U) - \beta(Z)S(X,U) \\
& +\delta(X)S(Z,U) - \delta(U)S(Z,X) \\
& +\beta(QX)g(Z,U) - \beta(QZ)g(X,U) + \delta(QX)g(Z,U) \\
& -\delta(QU)g(Z,X)] - \frac{r}{n}(\frac{a}{n-1} + 2b)[\beta(X)g(Z,U) \\
& -\beta(Z)g(X,U) + \delta(X)g(Z,U) - \delta(U)g(Z,X)].
\end{aligned} \tag{2.14}
$$

Again contracting (2.14) over Z and U we obtain

$$
\beta(QX) + \delta(QX) = \frac{r}{n}[\beta(X) + \delta(X)], \quad \text{for} \quad a + (n-2)b \neq 0. \tag{2.15}
$$

Also contracting (2.14) over X and U we have

$$\frac{n-2}{2n}dr(Z) = \alpha(QZ) - \beta(QZ) + \delta(QZ) - \frac{r}{n}[\alpha(Z) - \beta(Z) + \delta(Z)], \tag{2.16}$$

for $a + (n-2)b \neq 0$. Furthermore, contracting (2.14) over X and Z we obtain

$$\frac{n-2}{2n}dr(U) = \alpha(QU) + \beta(QU) - \delta(QU) - \frac{r}{n}[\alpha(U) + \beta(U) - \delta(U)], \tag{2.17}$$

provided that $a + (n-2)b \neq 0$. Replacing U by Z in (2.17) yields

$$\frac{n-2}{2n}dr(Z) = \alpha(QZ) + \beta(QZ) - \delta(QZ) - \frac{r}{n}[\alpha(Z) + \beta(Z) - \delta(Z)]. \tag{2.18}$$

From (2.16) and (2.18) it follows that

$$\beta(QZ) - \delta(QZ) = \frac{r}{n}[\beta(Z) - \delta(Z)], \quad \text{for} \quad a + (n-2)b \neq 0. \tag{2.19}$$

In view of (2.15) and (2.19), we obtain

$$\beta(QZ) = \frac{r}{n}\beta(Z) \tag{2.20}$$

and

$$\delta(QZ) = \frac{r}{n}\delta(Z), \quad \text{for} \quad a + (n-2)b \neq 0. \tag{2.21}$$

This leads to the following:

Proposition 2.4. *In a $(WQCS)_n$ with $a + (n-2)b \neq 0$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvectors L_2 and L_3 defined by $g(X, L_2) = \beta(X)$ and $g(X, L_3) = \delta(X)$ respectively, for all X.*

Using (2.20) and (2.21) in (2.18) we get

$$\frac{n-2}{2n}dr(Z) = \alpha(QZ) - \frac{r}{n}\alpha(Z), \text{ for } a + (n-2)b \neq 0. \tag{2.22}$$

If the manifold is of constant scalar curvature then (2.22) yields

$$\alpha(QZ) = \frac{r}{n}\alpha(Z), \quad \text{for} \quad a + (n-2)b \neq 0. \tag{2.23}$$

Thus we can state the following:

Proposition 2.5. *If a $(WQCS)_n$ is of constant scalar curvature with $a+(n-2)b \neq 0$, then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L_1 defined by $g(X, L_1) = \alpha(X)$ for all X.*

Using (2.20) and (2.21) in (2.14) we obtain

$$
\begin{aligned}
&\{a+(n-2)b\}[(\nabla_X S)(Z,U) - \frac{1}{n}dr(X)g(Z,U)] \\
&= \{a+(n-2)b\}[\alpha(X)\{S(Z,U) - \frac{r}{n}g(Z,U)\} + \beta(Z)\{S(X,U) \\
&-\frac{r}{n}g(X,U)\} + \delta(U)\{S(Z,X) - \frac{r}{n}g(Z,X)\}] \\
&+a[\beta(R(X,Z)U) + \delta(R(X,U)Z)] + b[\beta(X)S(Z,U) \\
&-\beta(Z)S(X,U) + \delta(X)S(Z,U) - \delta(U)S(Z,X)] \\
&-\frac{r}{n}(\frac{a}{n-1} + b)[\beta(X)g(Z,U) - \beta(Z)g(X,U) \\
&+\delta(X)g(Z,U) - \delta(U)g(Z,X)].
\end{aligned} \tag{2.24}
$$

The above results will be used in the later sections.

§3. Decomposable $(WQCS)_n$

A Riemannian manifold (M^n, g) is said to be decomposable or product manifold [6] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$.

Let (M^n, g) be a Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$. We assume that M is a weakly quasi-conformally symmetric manifold, that is, for $X, Y, Z, U, V \in \chi(M)$

$$
\begin{aligned}
(\nabla_X W)(Y,Z,U,V) &= \alpha(X)W(Y,Z,U,V) + \beta(Y)W(X,Z,U,V) \\
&+\beta(Z)W(Y,X,U,V) + \delta(U)W(Y,Z,X,V) \\
&+\delta(V)W(Y,Z,U,X),
\end{aligned}
$$

where α, β and δ are (not simultaneously zero) 1-forms on M. Then we find

$$
\begin{aligned}
(\nabla_{\bar{X}} W)(\bar{Y},\bar{Z},\bar{U},\bar{V}) &= \alpha(\bar{X})W(\bar{Y},\bar{Z},\bar{U},\bar{V}) + \beta(\bar{Y})W(\bar{X},\bar{Z},\bar{U},\bar{V}) \\
&+\beta(\bar{Z})W(\bar{Y},\bar{X},\bar{U},\bar{V}) + \delta(\bar{U})W(\bar{Y},\bar{Z},\bar{X},\bar{V}) \\
&+\delta(\bar{V})W(\bar{Y},\bar{Z},\bar{U},\bar{X}),
\end{aligned} \tag{3.1}
$$

$$\alpha(\overset{*}{X})W(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0, \tag{3.2}$$

$$\beta(\overset{*}{Y})W(\bar{X},\bar{Z},\bar{U},\bar{V}) = 0, \tag{3.3}$$

$$\delta(\overset{*}{U})W(\bar{Y},\bar{Z},\bar{X},\bar{V}) = 0, \tag{3.4}$$

$$(3.5)\quad \beta(\bar{Z})W(\overset{*}{X},\overset{*}{Y},\bar{U},\bar{V})+\delta(\bar{U})W(\overset{*}{X},\bar{V},\bar{Z},\overset{*}{Y})-\delta(\bar{V})W(\overset{*}{X},\bar{U},\bar{Z},\overset{*}{Y})=0,$$

$$(3.6)\quad \beta(\bar{Y})W(\overset{*}{X},\bar{Z},\bar{V},\overset{*}{U})-\beta(\bar{Z})W(\overset{*}{X},\bar{Y},\bar{V},\overset{*}{U})+\delta(\bar{V})W(\overset{*}{X},\overset{*}{U},\bar{Y},\bar{Z})=0,$$

$$(3.7)\quad (\nabla_{\bar{X}}W)(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V}) = \alpha(\bar{X})W(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V})+\beta(\bar{Z})W(\overset{*}{Y},\bar{X},\bar{U},\overset{*}{V}) +\delta(\bar{U})W(\overset{*}{Y},\bar{Z},\bar{X},\overset{*}{V}),$$

$$(3.8)\quad (\nabla_{\overset{*}{X}}W)(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V}) = \alpha(\overset{*}{X})W(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V})+\beta(\overset{*}{Y})W(\overset{*}{X},\bar{Z},\bar{U},\overset{*}{V}) +\delta(\overset{*}{V})W(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{X}),$$

$$(3.9)\quad \beta(\overset{*}{Z})W(\bar{X},\bar{Y},\overset{*}{U},\overset{*}{V})+\delta(\overset{*}{U})W(\overset{*}{Z},\bar{Y},\bar{X},\overset{*}{V})-\delta(\overset{*}{V})W(\overset{*}{Z},\bar{Y},\bar{X},\overset{*}{U})=0,$$

$$(3.10)\quad \beta(\overset{*}{Y})W(\overset{*}{Z},\bar{X},\bar{U},\overset{*}{V})-\beta(\overset{*}{Z})W(\overset{*}{Y},\bar{X},\bar{U},\overset{*}{V})+\delta(\overset{*}{V})W(\overset{*}{Y},\overset{*}{Z},\bar{X},\bar{U})=0,$$

$$(3.11)\quad \alpha(\bar{X})W(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V})=0,$$

$$(3.12)\quad \beta(\bar{Y})W(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V})=0,$$

$$(3.13)\quad \delta(\bar{U})W(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V})=0,$$

$$(3.14)\quad (\nabla_{\overset{*}{X}}W)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = \alpha(\overset{*}{X})W(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V})+\beta(\overset{*}{Y})W(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) +\beta(\overset{*}{Z})W(\overset{*}{Y},\overset{*}{X},\overset{*}{U},\overset{*}{V})+\delta(\overset{*}{U})W(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) +\delta(\overset{*}{V})W(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{X})$$

for $\bar{X},\bar{Y},\bar{Z},\bar{U},\bar{V}\in\chi(M_1)$ and $\overset{*}{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}\in\chi(M_2)$. From (3.2)-(3.4), we have two cases, namely,

(1) $\alpha=0,\beta=0,\delta=0$ on M_2,

(2) M_1 is a quasi-conformally flat.

At first, we consider the case (1). Then from (3.8) it follows that

$$(\nabla_{\overset{*}{X}}W)(\overset{*}{Y},\bar{Z},\bar{U},\overset{*}{V})=0, \quad \text{which implies that}$$

$$(3.15)\quad b(\nabla_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{V})=\frac{\overset{*}{X}\,r}{n}\left(\frac{a}{n-1}+2b\right)g(\overset{*}{Y},\overset{*}{V}).$$

Also from (3.14), we obtain

$$(\nabla_{\overset{*}{X}}W)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V})=0, \text{ that is,}$$

$$(3.16)\quad \begin{aligned} & a(\nabla_{\overset{*}{X}} R)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) \\ & +b\{(\nabla_{\overset{*}{X}} S)(\overset{*}{Z},\overset{*}{U})g(\overset{*}{Y},\overset{*}{V}) - (\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{U})g(\overset{*}{Z},\overset{*}{V}) \\ & +g(\overset{*}{Z},\overset{*}{U})(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) - g(\overset{*}{Y},\overset{*}{U})(\nabla_{\overset{*}{X}} S)(\overset{*}{Z},\overset{*}{V})\} \\ & -\frac{\overset{*}{X}\overset{*}{r}}{n}(\frac{a}{n-1}+2b)\{g(\overset{*}{Z},\overset{*}{U})g(\overset{*}{Y},\overset{*}{V}) - g(\overset{*}{Y},\overset{*}{U})g(\overset{*}{Z},\overset{*}{V})\} = 0, \end{aligned}$$

which yields that

$$(3.17)\quad \begin{aligned} & \{a+(n-p-2)b\}(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) \\ & = \frac{\overset{*}{X}\overset{*}{r}}{n}\{\frac{n-p-1}{n-1}a+(n-2p-2)b\}g(\overset{*}{Y},\overset{*}{V}), \end{aligned}$$

where we denote the scalar curvature on M_2 by $\overset{*}{r}$. It is easy to see from (3.15), (3.17) and $\overset{*}{X}\overset{*}{r}=\overset{*}{X}\, r$ that

$$\{a+(n-1)b\}\{a+(n-2)b\}\ \overset{*}{X}\, r = 0.$$

Thus we have the following three cases:

(1-1) $a+(n-1)b=0$;
(1-2) $a+(n-2)b=0$;
(1-3) $\overset{*}{X}\, r = 0$.

In the case of (1-1), we find from (3.15) and $b \neq 0$

$$(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) = \frac{\overset{*}{X}\, r}{n} g(\overset{*}{Y},\overset{*}{V}),$$

which implies that $\overset{*}{X}\, r = 0$.

Thus we have $(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) = 0$. Similarly, if the case (1-2) holds, then we get $(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) = 0$. By virtue of (3.15) and (3.17), when (1-3) holds, we have $(\nabla_{\overset{*}{X}} S)(\overset{*}{Y},\overset{*}{V}) = 0$ if $a \neq 0$ or $b \neq 0$. Moreover, from (3.16) we find

$$(\nabla_{\overset{*}{X}} R)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0$$

if $a \neq 0$.

Secondly, we discuss the case of (2). From $W=0$ on M_1, we find

$$(3.18)\quad \begin{aligned} & aR(\bar{X},\bar{Y},\bar{Z},\bar{U}) + b[S(\bar{Y},\bar{Z})g(\bar{X},\bar{U}) - S(\bar{X},\bar{Z})g(\bar{Y},\bar{U}) \\ & +g(\bar{Y},\bar{Z})S(\bar{X},\bar{U}) - g(\bar{X},\bar{Z})S(\bar{Y},\bar{U})] \\ & -\frac{r}{n}(\frac{a}{n-1}+2b)\{g(\bar{Y},\bar{Z})g(\bar{X},\bar{U}) - g(\bar{X},\bar{Z})g(\bar{Y},\bar{U})\} = 0, \end{aligned}$$

which implies that

$$\{a+(p-2)b\}S(\bar{Y},\bar{Z})+\{b\bar{r}-\frac{(p-1)r}{n}(\frac{a}{n-1}+2b)\}g(\bar{Y},\bar{Z})=0, \tag{3.19}$$

where $\bar{r}$ is the scalar curvature on M_1. Thus we find

$$b\bar{r}-\frac{(p-1)r}{n}(\frac{a}{n-1}+2b)=-\frac{\bar{r}}{p}\{a+(p-2)b\}. \tag{3.20}$$

Using (3.20) in (3.19) we obtain

$$\{a+(p-2)b\}\{S(\bar{Y},\bar{Z})-\frac{\bar{r}}{p}g(\bar{Y},\bar{Z})\}=0.$$

Therefore we can consider the following two cases:
(2-1) $a+(p-2)b=0$;
(2-2) $a+(p-2)b\neq 0$.
In the case of (2-1), we get from (3.18), (3.20) and $b\neq 0$

$$\begin{aligned}(p-2)R(\bar{X},\bar{Y},\bar{Z},\bar{U})&-\{S(\bar{Y},\bar{Z})g(\bar{X},\bar{U})-S(\bar{X},\bar{Z})g(\bar{Y},\bar{U})\\&+g(\bar{Y},\bar{Z})S(\bar{X},\bar{U})-g(\bar{X},\bar{Z})S(\bar{Y},\bar{U})\}\\&+\frac{\bar{r}}{p-1}\{g(\bar{Y},\bar{Z})g(\bar{X},\bar{U})-g(\bar{X},\bar{Z})g(\bar{Y},\bar{U})\}=0.\end{aligned} \tag{3.21}$$

Thus M_1 is conformally flat if $p\neq 2$. Also, in the case of (2-2), equation (3.18) is rewritten as follows:

$$R(\bar{X},\bar{Y},\bar{Z},\bar{U})=\frac{\bar{r}}{p(p-1)}\{g(\bar{Y},\bar{Z})g(\bar{X},\bar{U})-g(\bar{X},\bar{Z})g(\bar{Y},\bar{U})\}, \tag{3.22}$$

if $a\neq 0$. Hence we have

Theorem 3.1. *Let (M^n,g) be a Riemannian manifold such that $M=M_1^p\times M_2^{n-p}$ $(2\leq p\leq n-2)$. If M is a $(WQCS)_n$, then we get*
(1) in the case of $\alpha=0$, $\beta=0$, $\delta=0$ on M_2, M_2 is a locally symmetric manifold for $a\neq 0$,
(2) when M_1 is a quasi-conformally flat,
(i) if $a+(p-2)b=0$ and $p\geq 3$, then M_1 is conformally flat,
(ii) if $a\neq 0, a+(p-2)b\neq 0$ and $p\geq 3$, then M_1 is a manifold of constant curvature.

Similarly we have from (3.11)–(3.13)

Theorem 3.2. *Let (M^n,g) be a Riemannian manifold such that $M=M_1^p\times M_2^{n-p}$ $(2\leq p\leq n-2)$. If M is a $(WQCS)_n$, then we get*
(1) in the case of $\alpha=0$, $\beta=0$, $\delta=0$ on M_1, M_1 is a locally symmetric

manifold for $a \neq 0$,
(2) *when* M_2 *is a quasi-conformally flat,*
(i) *if* $a + (p-2)b = 0$ *and* $p \leq n-3$, *then* M_2 *is conformally flat,*
(ii) *if* $a \neq 0, a + (p-2)b \neq 0$ *and* $p \leq n-3$, *then* M_2 *is of constant curvature.*

Next, we consider the contraction with respect to $\overset{*}{X}$ and $\overset{*}{U}$ in (3.6) and obtain

$$\beta(\bar{Y})[b\{\overset{*}{r}\, g(\bar{Z},\bar{V}) + (n-p)S(\bar{Z},\bar{V})\} - \frac{(n-p)r}{n}(\frac{a}{n-1} + 2b)g(\bar{Z},\bar{V})]$$
$$-\beta(\bar{Z})[b\{\overset{*}{r}\, g(\bar{Y},\bar{V}) + (n-p)S(\bar{Y},\bar{V})\} - \frac{(n-p)r}{n}(\frac{a}{n-1} + 2b)g(\bar{Y},\bar{V})]$$
$$= 0,$$

which yields that

$$b(n-p)\beta(Q\bar{Y}) = -\frac{r_1}{n}\beta(\bar{Y}), \tag{3.23}$$

where we put

$$r_1 = (n-p)\{\frac{p-1}{n-1}a - (n-2p+2)b\}\bar{r} + (p-1)\{\frac{n-p}{n-1}a + (n-2p)b\}\ \overset{*}{r}\ .$$

Similarly, we have from (3.5)

$$b(n-p)\delta(Q\bar{U}) = -\frac{r_1}{n}\delta(\bar{U}). \tag{3.24}$$

If $b = 0$, that is, $W = a\widetilde{C}$ on M, then from (3.23) and (3.24) we get $r\beta(\bar{Y}) = 0$ and $r\delta(\bar{U}) = 0$. Thus we can consider the two cases:
(3) $r = 0$,
(4) $r \neq 0$, namely, $\beta = 0, \delta = 0$ on M_1.

If $r = 0$, then M is a weakly symmetric manifold. When the case of (4) holds, we obtain from (3.7) that

$$\alpha(\bar{X}) = -\bar{X}\log|r|. \tag{3.25}$$

It is clear from (3.1) that

$$(\nabla_{\bar{X}}\widetilde{C})(\bar{Y},\bar{Z},\bar{U},\bar{V}) = \alpha(\bar{X})\widetilde{C}(\bar{Y},\bar{Z},\bar{U},\bar{V}). \tag{3.26}$$

Hence we can state the following:

Theorem 3.3. *Let* (M^n, g) *be a Riemannian manifold such that* $M = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$. *If* M *is a* $(WQCS)_n$, *then we get*
(1) *if* $b \neq 0$, *then we find*

$$\beta(Q\cdot) = -\frac{r_1}{bn(n-p)}\beta(\cdot)$$
$$\textit{and} \qquad \delta(Q\cdot) = -\frac{r_1}{bn(n-p)}\delta(\cdot) \quad \textit{on} \quad M_1,$$

(2) *in the case of* $b = 0$,

(i) *if* $r = 0$, *then* M *is a weakly symmetric manifold,*

(ii) *if* $r \neq 0$, *then* $\alpha(\bar{X}) = -\bar{X}\log|r|$ *and* $\nabla_{\bar{X}}\widetilde{C} = \alpha(\bar{X})\widetilde{C}$ *on* M_1 *for* $\bar{X} \in \chi(M_1)$. *Especially, if* r *is a non-zero constant, then the concircular curvature tensor field is parallel on* M_1.

Similarly, from (3.9) and (3.10) we can state the following

Theorem 3.4. *Let* (M^n, g) *be a Riemannian manifold such that* $M = M_1^p \times M_2^{n-p}$ $(2 \leq p \leq n-2)$. *If* M *is a* $(WQCS)_n$, *then we get*
(1) *if* $b \neq 0$, *then we find*

$$\beta(Q\cdot) = -\frac{r_2}{bnp}\beta(\cdot)$$

$$\text{and} \qquad \delta(Q\cdot) = -\frac{r_2}{bnp}\delta(\cdot) \quad \text{on} \quad M_2,$$

where we put

$$r_2 = (n-p-1)\{\frac{pa}{n-1} - (n-2p)b\}\bar{r} + p\{\frac{n-p-1}{n-1}a + (n-2p-2)b\}\overset{*}{r},$$

(2) *in the case of* $b = 0$,

(i) *if* $r = 0$, *then* M *is a weakly symmetric manifold,*

(ii) *if* $r \neq 0$, *then* $\alpha(\overset{*}{X}) = -\overset{*}{X}\log|r|$ *and* $\nabla_{\overset{*}{X}}\widetilde{C} = \alpha(\overset{*}{X})\widetilde{C}$ *on* M_2 *for* $\overset{*}{X} \in \chi(M_2)$. *Especially, if* r *is a non-zero constant, then the concircular curvature tensor field is parallel on* M_2.

§4. $(WQCS)_n$ satisfying certain conditions

Definition 4.1. The Ricci tensor of a Riemannian manifold is said to be *cyclic parallel* if it satisfies the following condition:

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{4.1}$$

for all vector fields X, Y, Z on the manifold i.e., the Ricci tensor S of a Riemannian manifold is cyclic parallel if the cyclic sum of the covariant derivative of S vanishes.

From (4.1) it follows that in such a manifold the scalar curvature r is a constant.
We now consider a $(WQCS)_n$ satisfying the relation (4.1). Taking cyclic sum

with respect to X, Z, U in (2.24) we obtain by virtue of (4.1) and Bianchi identity that

$$\begin{aligned}\{\alpha(X)+\beta(X)+\delta(X)\}[S(Z,U)-\frac{r}{n}g(Z,U)]\\+\{\alpha(Z)+\beta(Z)+\delta(Z)\}[S(X,U)-\frac{r}{n}g(X,U)]\\+\{\alpha(U)+\beta(U)+\delta(U)\}[S(Z,X)-\frac{r}{n}g(Z,X)]=0\end{aligned} \tag{4.2}$$

for $a+(n-2)b\neq 0$ and $\alpha+\beta+\delta\neq 0$ everywhere.

We now choose the vector fields L_1, L_2 and L_3 corresponding to the 1-forms α, β and δ respectively as the unit vector fields such that they are mutually orthogonal to each other. We now suppose that $\alpha(Y)\neq 0$ for all Y. For if, $\alpha(Y)=0$ for all Y then $g(L_1,L_1)=0$, which contradicts to our assumption that L_1 is a unit vector field. Then multiplying both sides of (4.2) by $\alpha(Y)$ we get

$$\begin{aligned}\alpha(Y)\{\alpha(X)+\beta(X)+\delta(X)\}[S(Z,U)-\frac{r}{n}g(Z,U)]\\+\alpha(Y)\{\alpha(Z)+\beta(Z)+\delta(Z)\}[S(X,U)-\frac{r}{n}g(X,U)]\\+\alpha(Y)\{\alpha(U)+\beta(U)+\delta(U)\}[S(Z,X)-\frac{r}{n}g(Z,X)]=0.\end{aligned} \tag{4.3}$$

Setting $X=Y=e_i$ in (4.3) and taking summation over i, $1\leq i\leq n$, we have

$$\begin{aligned}S(Z,U)-\frac{r}{n}g(Z,U)+\{\alpha(Z)+\beta(Z)+\delta(Z)\}[\alpha(QU)-\frac{r}{n}\alpha(U)]\\+\{\alpha(U)+\beta(U)+\delta(U)\}[\alpha(QZ)-\frac{r}{n}\alpha(Z)]=0.\end{aligned} \tag{4.4}$$

Since the manifold under consideration is of constant scalar curvature, using (2.23) in (4.4) we get

$$S(Z,U)=\frac{r}{n}g(Z,U),\quad \text{which means that the manifold is Einstein.}$$

In a similar manner multiplying (4.3) by $\beta(Y)$ and $\delta(Y)$ respectively we obtain that the manifold is Einstein. This leads to the following:

Theorem 4.1. *If in a $(WQCS)_n$, the Ricci tensor is cyclic parallel and $a+(n-2)b\neq 0$ then it is an Einstein manifold unless $\alpha+\beta+\delta$ is non-vanishing everywhere.*

Corollary 4.1. *If a $(WQCS)_n$ is Ricci symmetric then it is an Einstein manifold, provided that $a+(n-2)b\neq 0$ and $\alpha+\beta+\delta\neq 0$ everywhere.*

Again in [7] it is shown that if an Einstein $(WQCS)_n$ is a $(WS)_n$ then the scalar curvature of the manifold vanishes, provided that $a \neq 0$ and $\alpha+\beta+\delta \neq 0$. Hence by virtue of Theorem 4.1 we can state the following:

Theorem 4.2. *If a $(WQCS)_n$ with cyclic parallel Ricci tensor is a $(WS)_n$ then the scalar curvature of the manifold vanishes, provided that $a \neq 0$, $a + (n-2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Next in [7] it is proved that if in an Einstein $(WQCS)_n$ the scalar curvature vanishes then it is a $(WS)_n$, provided that $a \neq 0$. Hence by virtue of Theorem 4.1 we can state the following:

Theorem 4.3. *If in a $(WQCS)_n$ with cyclic parallel Ricci tensor the scalar curvature vanishes, then it is a $(WS)_n$, provided that $a \neq 0$, $a + (n-2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Therefore if a $(WQCS)_n$ satisfying (4.1) is of non-vanishing scalar curvature then in view of Theorem 4.3 we can state the following:

Theorem 4.4. *If in a $(WQCS)_n$ with non-vanishing scalar curvature, the Ricci tensor is cyclic parallel then it cannot be a $(WS)_n$, provided that $a \neq 0$, $a + (n-2)b \neq 0$ and $\alpha + \beta + \delta \neq 0$ everywhere.*

Definition 4.2. A vector field L on a Riemannian manifold is said to be *concurrent* [6] if $\nabla_X L = \rho X$, where ρ is a constant.

In particular, if $\rho = 0$ then L is said to be a parallel vector field.

Let us now consider a $(WQCS)_n$ such that the vector field $L = L_2 + L_3$ defined by $g(X, L) = \beta(X) + \delta(X)$ is a concurrent vector field. Then making use of Ricci identity we have

$$R(X, Y, L, U) = 0 \quad \text{which implies that} \tag{4.5}$$

$$S(Y, L) = 0. \tag{4.6}$$

Now the relation (2.15) can be written as

$$S(X, L) = \frac{r}{n} g(X, L), \quad \text{provided that} \quad a + (n-2)b \neq 0. \tag{4.7}$$

From (4.6) and (4.7) it follows that

$$r = 0, \quad \text{if} \quad ||L||^2 \neq 0.$$

This leads to the following:

Theorem 4.5. *If in a $(WQCS)_n$ the vector field L defined by $g(X, L) = \beta(X) + \delta(X)$ is a concurrent vector field then it is of vanishing scalar curvature, provided that $a + (n-2)b \neq 0$ and $||L||^2 \neq 0$.*

Since $r = 0$, from (2.20) and (2.21) we get

$$\beta(QX) = \delta(QX) = 0 \text{ if } a + (n-2)b \neq 0. \tag{4.8}$$

Now using (4.8) and $r = 0$ in (2.14) we obtain

$$\begin{aligned} &\{a + (n-2)b\}(\nabla_X S)(Z, U) \\ &= \{a + (n-2)b\}[\alpha(X)S(Z, U) + \beta(Z)S(X, U) + \delta(U)S(Z, X)] \\ &+ a[\beta(R(X, Z)U) + \delta(R(X, U)Z)] + b[\beta(X)S(Z, U) \\ &- \beta(Z)S(X, U) + \delta(X)S(Z, U) - \delta(U)S(Z, X)]. \end{aligned} \tag{4.9}$$

Again from $\nabla_X L = \rho X$, we have

$$(\nabla_X S)(Z, L) = -\rho S(Z, X). \tag{4.10}$$

Setting $U = L$ in (4.9) and then using (4.5) and (4.6) we obtain by virtue of (4.10) that

$$[\rho\{a + (n-2)b\} + \{a + (n-3)b\}\delta(L)]S(Z, X) + a\delta(R(X, L)Z) = 0. \tag{4.11}$$

From (4.5) we have

$$\begin{aligned} &R(L, U, X, Y) = 0, \quad \text{which implies that} \\ &R(U, L, Y, X) = 0 \quad \text{for all vector fields} \quad U, X, Y. \end{aligned}$$

The last relation yields (for $X = L_3$) that $\delta(R(U, L)Y) = 0$ for all vector fields $U, Y \in \chi(M)$. Hence $\delta(R(X, L)Z) = 0$ for all $X, Z \in \chi(M)$. Consequently (4.11) reduces to

$$S(Z, X) = 0 \quad \text{for all} \quad X \quad \text{and} \quad Z,$$

provided that $\rho\{a + (n-2)b\} + \{a + (n-3)b\}\delta(L) \neq 0$.
Thus (2.1) takes the form $W(X, Y, Z, U) = aR(X, Y, Z, U)$ and hence (1.5) reduces to

$$\begin{aligned} (\nabla_X R)(Y, Z, U, V) &= \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) \\ &+ \beta(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) \\ &+ \delta(V)R(Y, Z, U, X) \end{aligned}$$

for $a \neq 0$, which implies that the manifold is a $(WS)_n$. Thus we can state the following:

Theorem 4.6. *If in a $(WQCS)_n$ with $a \neq 0$ and $a+(n-2)b \neq 0$ the non-null vector field L defined by $g(X,L) = \beta(X) + \delta(X)$ is a concurrent vector field then it is a $(WS)_n$, provided that $\rho\{a+(n-2)b\} + \{a+(n-3)b\}\delta(L) \neq 0$.*

Corollary 4.2. *If in a $(WQCS)_n$ with $a \neq 0$ and $a+(n-2)b \neq 0$ the non-null vector field L defined by $g(X,L) = \beta(X) + \delta(X)$ is a parallel vector field then it is a $(WS)_n$, provided that $\{a+(n-3)b\}\delta(L) \neq 0$.*

The above corollary certainly improves the Theorem 4.5 of [7].

Definition 4.3. A vector field L on a Riemannian manifold is said to be *recurrent* [6] if $\nabla_X L = \mu(X)L$, where μ is a non-zero 1-form, called the associated 1-form of the recurrent vector field.

In particular, if $\mu(X)$ is a constant then the recurrent vector field reduces to a concurrent vector field.
Now we consider a $(WQCS)_n$ such that the vector field L defined by $g(X,L) = \beta(X) + \delta(X)$ is a recurrent vector field. Then we have

$$\nabla_X \nabla_Y L = (X\mu(Y))L + \mu(X)\mu(Y)L$$

and hence using Ricci identity we get

$$R(X,Y,L,U) = 2d\mu(X,Y)g(L,U) \quad \text{which implies that}$$

$$R(X,Y,L,U) = 0, \quad \text{if the 1-form } \mu \text{ is closed.}$$

Then $S(Y,L) = 0$ and hence $r = 0$. Therefore proceeding similarly as before we obtain that the manifold is a $(WS)_n$. Hence we can state the following:

Theorem 4.7. *If in a $(WQCS)_n$ with $a \neq 0$ and $a+(n-2)b \neq 0$, the vector field L defined by $g(X,L) = \beta(X) + \delta(X)$ is a recurrent vector field such that the associated 1-form of the recurrent vector field is closed then it is a $(WS)_n$, provided that $a+(n-3)b \neq 0$ and $\delta(L) \neq 0$.*

§5. Some examples of $(WQCS)_n$

This section deals with several examples of $(WQCS)_n$. We calculate the components of the curvature tensor, the Ricci tensor, the quasi-conformal curvature tensor and its covariant derivative.

EXAMPLE 1. Let $M^4 = \{(x^1,x^2,x^3,x^4) \in \mathbf{R}^4 | x^1 < 0,\ x^3 > 0\}$ be an open subset of $\mathbf{R}^4$ endowed with the metric

$$ds^2 = x^1(x^3)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2. \tag{5.1}$$

Then the only non-vanishing components of the Christoffel's symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the quasi-conformal curvature tensor and its covariant derivatives are

$$\begin{aligned}
&\Gamma^2_{11} = \frac{1}{2}(x^3)^2, \quad \Gamma^3_{11} = -x^1x^3 = -\Gamma^2_{13},\\
&R_{1313} = x^1, \quad S_{11} = -x^1, \quad r = 0,\\
&W_{1313} = (a+b)x^1, \qquad W_{1414} = bx^1,\\
&W_{1313,1} = (a+b), \qquad W_{1414,1} = b.
\end{aligned}$$

Here ',' denotes the covariant differentiation with respect to the metric tensor g. Therefore our M^4 with the considered metric g in (5.1) is a Riemannian manifold of vanishing scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric. We put

$$\alpha_i(\partial_i) = \alpha_i = \begin{cases} \frac{1}{2x^1} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_i(\partial_i) = \beta_i = \begin{cases} \frac{1}{3x^1} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_i(\partial_i) = \delta_i = \begin{cases} \frac{1}{6x^1} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Then (M^4, g) is a $(WQCS)_4$. Hence we can state the following:

Theorem 5.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in (5.1). Then (M^4, g) is a weakly quasi-conformally symmetric manifold with vanishing scalar curvature which is neither quasi-conformally symmetric nor quasi-conformally recurrent.*

EXAMPLE 2. Let $M^n = \mathbf{R}^n (n \geq 4)$ be endowed with the metric

$$ds^2 = f \cdot (dx^1)^2 + \sum_{i=2}^{n-1}(dx^i)^2 + 2dx^1dx^n, \tag{5.2}$$

where f is a continuously differentiable function of $x^1, x^2, ..., x^{n-1}$ such that

$$f < 0, \quad af_{\cdot mmk} + b\sum_{j=2}^{n-1} f_{\cdot jjk} \neq 0 \quad \text{and} \quad af_{\cdot mm} + b\sum_{j=2}^{n-1} f_{\cdot jj} \neq 0 \tag{5.3}$$

for $2 \leq m \leq n-1$ and $1 \leq k \leq n-1$ and '$\cdot$' denotes the partial differentiation with respect to the coordinates. Then the only non-vanishing components of the Christoffel's symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the quasi-conformal curvature tensors and their covariant derivatives are given by the following:

$$\Gamma^m_{11} = -\Gamma^n_{1m} = -\frac{1}{2}f_{\cdot m}, \quad \Gamma^n_{11} = \frac{1}{2}f_{\cdot 1},$$
$$R_{1m1m} = \frac{1}{2}f_{\cdot mm}, \quad S_{11} = -\frac{1}{2}\sum_{j=2}^{n-1} f_{\cdot jj}, \quad r = 0,$$
$$W_{1m1m} = \frac{1}{2}\left(af_{\cdot mm} + b\sum_{j=2}^{n-1} f_{\cdot jj}\right),$$
$$W_{1m1m,k} = \frac{1}{2}\left(af_{\cdot mmk} + b\sum_{j=2}^{n-1} f_{\cdot jjk}\right).$$

Thus (M^n, g) is neither quasi-conformally flat nor quasi-conformally symmetric. We set

$$\alpha_i(\partial_i) = \alpha_i = \begin{cases} \partial_i \log |af_{\cdot mm} + b\sum_{j=2}^{n-1} f_{\cdot jj}| & \text{for} \quad i = 1, 2, ..., n-1 \\ 0 & \text{for} \quad i = n, \end{cases}$$
$$\beta_i(\partial_i) = \beta_i = \begin{cases} -\frac{1}{2} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$
$$\delta_i(\partial_i) = \delta_i = \begin{cases} \frac{1}{2} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Then (M^n, g) is a $(WQCS)_n$. Hence we can state the following:

Theorem 5.2. *Let (M^n, g) be a Riemannian manifold equipped with the metric given in (5.2). Then (M^n, g) is a weakly quasi-conformally symmetric manifold with vanishing scalar curvature which is neither quasi-conformally symmetric nor quasi-conformally recurrent.*

EXAMPLE 3. Let $M^n = \{(x^1, x^2, ..., x^n) \in \mathbf{R}^n | x^1 < 0, x^3 > 0\}$ be endowed with the metric

$$ds^2 = x^1(x^3)^2(dx^1)^2 + 2dx^1dx^2 + \sum_{i=3}^{n}(dx^i)^2. \tag{5.4}$$

Then the only non-vanishing components of the Christoffel's symbols, the curvature tensor, the Ricci tensor, the scalar curvature, the quasi-conformal curvature tensor and their covariant derivatives are given by the following:

$$\begin{aligned}
&\Gamma^2_{11} = \frac{1}{2}(x^3)^2, \quad \Gamma^3_{11} = -x^1x^3 = -\Gamma^2_{13},\\
&R_{1313} = x^1, \quad S_{11} = -x^1, \quad r = 0,\\
&W_{1313} = (a+b)x^1, \qquad W_{1k1k} = bx^1,\\
&W_{1313,1} = (a+b), \qquad W_{1k1k,1} = b
\end{aligned}$$

for $4 \leq k \leq n$. We put

$$\alpha_i(\partial_i) = \alpha_i = \begin{cases} \frac{1}{2x^1} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_i(\partial_i) = \beta_i = \begin{cases} \frac{1}{3x^1} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_i(\partial_i) = \delta_i = \begin{cases} \frac{1}{6x^1} & \text{for} \quad i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. Then it can be easily shown that (M^n, g) is a $(WQCS)_n$, which is neither quasi-conformally symmetric nor quasi-conformally recurrent. Hence we can state the following:

Theorem 5.3. *Let (M^n, g) $(n \geq 4)$ be a Riemannian manifold equipped with the metric given in* (5.4)*. Then (M^n, g) $(n \geq 4)$ is a weakly quasi-conformally symmetric manifold with vanishing scalar curvature which is neither quasi-conformally symmetric nor quasi-conformally recurrent.*

Let (M^4_1, g_1) be a Riemannian manifold in Example 1 and $(\mathbf{R}^{n-4}, g_0)$ be an $(n-4)$-dimensional Euclidean space with standard metric g_0. Then (M^n, g) in Example 3 is a product manifold of (M^4_1, g_1) and $(\mathbf{R}^{n-4}, g_0)$. Thus we can state the following:

Theorem 5.4. *Let (M^n, g) $(n \geq 5)$ be a Riemannian manifold endowed with the metric given in* (5.4)*. Then (M^n, g) $(n \geq 4)$ is a decomposable weakly quasi-conformally symmetric manifold $(M^4_1, g_1) \times (\mathbf{R}^{n-4}, g_0)$ with vanishing scalar curvature.*

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