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A note on perturbed three point inequalities

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Abstract. The main purpose of this paper is to use a variant of Grüss inequality to obtain a sharp perturbed generalized three point inequality for functions whose nth derivative is bounded both above and below. Thus we provide improvement of a previous result.

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§1. Introduction

In 1935, G. Grüss (see for example [4, p.296]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

Theorem 1. *Let h, $g : [a, b] \to \mathbf{R}$ be two integrable functions such that $\phi \leq h(t) \leq \Phi$ and $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a, b]$, where ϕ, Φ, γ and Γ are real numbers. Then we have*

$$|T(h, g)| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{1}$$

where

$$T(h, g) = \frac{1}{b-a}\int_a^b h(t)g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \tag{2}$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

In 2000, M. Matić, J. Pečarić and N. Ujević [3] established the following premature Grüss inequality.

Theorem 2. *Let $h, g : [a,b] \to \mathbf{R}$ be two integrable functions such that $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, where γ, $\Gamma \in \mathbf{R}$. Then we have*

$$|T(h,g)| \leq \frac{\Gamma - \gamma}{2}[T(h,h)]^{\frac{1}{2}}, \tag{3}$$

where $T(h,g)$ is as defined in (2).

In 2002, X. L. Cheng and J.Sun [2] have got the following variant of the Grüss inequality.

Theorem 3. *Let $h, g : [a,b] \to \mathbf{R}$ be two integrable functions such that $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, where γ, $\Gamma \in \mathbf{R}$. Then*

$$|T(h,g)| \leq \frac{\Gamma - \gamma}{2(b-a)} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s)\, ds \right| dt, \tag{4}$$

where $T(h,g)$ is as defined in (2).

It is not difficult to find that the premature Grüss inequality (3) provides a sharper bound than the Grüss inequality (1) and the variant of Grüss inequality (4) provides a sharper bound than the premature Grüss inequality (3).

In [1], Theorem 2 has been used to provide a perturbed generalized three point inequality. But unfortunately, there exist some careless mistakes which led to wrong results (see Theorem 5, Corollary 8 and Corollary 9 in [1]). In this paper, we will give the correct results, and, a sharp perturbed generalized three point inequality is obtained by using Theorem 3.

§2. Perturbed rules from premature inequalities

In this section, we will provide a perturbed generalized three point inequality and its two special cases to replace Theorem 5, Corollary 8 and Corollary 9 in [1]. We need the following integral identity whose proof can be found in [1].

Lemma. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$, $n \geq 1$ is absolutely continuous on $[a,b]$. Further, let $\alpha : [a,b] \to [a,b]$ and $\beta : [a,b] \to [a,b]$ such that $\alpha(x) \leq x$ and $\beta(x) \geq x$. Then, for all $x \in [a,b]$ the following identity holds:*

(5)

$$(-1)^n \int_a^b K_n(x,t) f^{(n)}(t)\, dt = \int_a^b f(t)\, dt - \sum_{k=1}^{n} \frac{1}{k!} \left[R_k(x) f^{(k-1)}(x) + S_k(x) \right],$$

where the kernel $K_n : [a,b]^2 \to \mathbf{R}$ *is given by*

$$K_n(x,t) := \begin{cases} \frac{(t-\alpha(x))^n}{n!}, & t \in [a,x], \\ \frac{(t-\beta(x))^n}{n!}, & t \in (x,b], \end{cases} \tag{6}$$

and

$$\begin{cases} R_k(x) = (\beta(x)-x)^k + (-1)^{k-1}(x-\alpha(x))^k, \\ S_k(x) = (\alpha(x)-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-\beta(x))^k f^{(k-1)}(b). \end{cases} \tag{7}$$

Theorem 4. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$, $n \geq 1$ *is absolutely continuous on* $[a,b]$. *Further, let* $\alpha : [a,b] \to [a,b]$ *and* $\beta : [a,b] \to [a,b]$ *such that* $\alpha(x) \leq x$ *and* $\beta(x) \geq x$. *Assume that there exist constants* $\gamma, \Gamma \in \mathbf{R}$ *with* $\gamma \leq f^{(n)}(t) \leq \Gamma$ *a.e. on* $[a,b]$. *Then for all* $x \in [a,b]$ *we have*

$$\begin{aligned} \Bigg| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\big[R_k(x) f^{(k-1)}(x) + S_k(x)\big] \\ - (-1)^n \frac{\theta_n(x)}{(n+1)!} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \Bigg| \\ \leq \frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} I(x,n), \end{aligned} \tag{8}$$

where

$$\begin{aligned} I(x,n) = \frac{1}{(n+1)\sqrt{2n+1}} \Bigg\{ n^2(b-a)Q_n(x) \\ + (2n+1) \sum_{i=1,\, j>i}^{4} z_i z_j \Big[z_i^n - \big((-1)^{i+j} z_j\big)^n \Big]^2 \Bigg\}^{\frac{1}{2}}, \end{aligned} \tag{9}$$

$$z_1 = \alpha(x) - a,\ z_2 = x - \alpha(x),\ z_3 = \beta(x) - x,\ z_4 = b - \beta(x),$$
$$Q_n(x) = z_1^{2n+1} + z_2^{2n+1} + z_3^{2n+1} + z_4^{2n+1},$$
$$\theta_n(x) = (-1)^n z_1^{n+1} + z_2^{n+1} + (-1)^n z_3^{n+1} + z_4^{n+1}$$

and $R_k(x)$, $S_k(x)$ *are as given by* (7).

Proof. Applying the premature Grüss inequality (3) by associating $f^{(n)}(t)$ with $g(t)$ and $K_n(x,t)$ with $h(t)$, gives

$$\begin{aligned} \left| \int_a^b K_n(x,t) f^{(n)}(t)\,dt - \left(\int_a^b K_n(x,t)\,dt \right) \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \\ \leq (b-a) \frac{\Gamma-\gamma}{2} \big[T(K_n,K_n)\big]^{\frac{1}{2}}, \end{aligned} \tag{10}$$

where from (2),

$$T(K_n,K_n)=\frac{1}{b-a}\int_a^b K_n(x,t)^2\,dt-\left(\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\right)^2.$$

Now, from (6),

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K_n(x,t)\,dt\\
&=\frac{1}{b-a}\left[\int_a^x\frac{\big(t-\alpha(x)\big)^n}{n!}\,dt+\int_x^b\frac{\big(t-\beta(x)\big)^n}{n!}\,dt\right]\\
&=\frac{1}{(b-a)(n+1)!}\Big[\big(x-\alpha(x)\big)^{n+1}+(-1)^n\big(\alpha(x)-a\big)^{n+1}\\
&\qquad\qquad+\big(b-\beta(x)\big)^{n+1}+(-1)^n\big(\beta(x)-x\big)^{n+1}\Big]\\
&=\frac{1}{(b-a)(n+1)!}\theta_n(x)
\end{aligned}\tag{11}$$

and

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K_n(x,t)^2\,dt\\
&=\frac{1}{(b-a)(n!)^2}\left[\int_a^x\big(t-\alpha(x)\big)^{2n}\,dt+\int_x^b\big(t-\beta(x)\big)^{2n}\,dt\right]\\
&=\frac{1}{(b-a)(n!)^2(2n+1)}\Big[\big(x-\alpha(x)\big)^{2n+1}+\big(\alpha(x)-a\big)^{2n+1}\\
&\qquad\qquad+\big(b-\beta(x)\big)^{2n+1}+\big(\beta(x)-x\big)^{2n+1}\Big]\\
&=\frac{1}{(b-a)(n!)^2(2n+1)}Q_n(x).
\end{aligned}\tag{12}$$

Hence, substitution of (11) and (12) into (10) gives

$$\begin{aligned}
&\left|\int_a^b K_n(x,t)f^{(n)}(t)\,dt-\frac{\theta_n(x)}{(n+1)!}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right|\\
&\le\frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}I(x,n),
\end{aligned}\tag{13}$$

where

$$\begin{aligned}
&(2n+1)(n+1)^2I(x,n)^2\\
&=(n+1)^2(b-a)Q_n(x)-(2n+1)\theta_n^2(x)\\
&=n^2(b-a)Q_n(x)+(2n+1)\big[(z_1+z_2+z_3+z_4)Q_n(x)-\theta_n^2(x)\big]\\
&=n^2(b-a)Q_n(x)+(2n+1)\sum_{i=1,j>i}^{4}z_iz_j\big[z_i^n-((-1)^{i+j}z_j)^n\big]^2.
\end{aligned}\tag{14}$$

Thus the inequality (8) with (9) follows from (5), (13) and (14).

The proof is completed. □

Corollary 1. *Let the conditions of Theorem 4 hold. Then for all $x \in [a,b]$ we have*

$$\begin{aligned}\Bigg|\int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{A^k f^{(k-1)}(a) + \left[(-1)^{k-1}A^k + B^k\right] f^{(k-1)}(x) + (-1)^{k-1} B^k f^{(k-1)}(b)}{k!} \\ - \frac{\left[1+(-1)^n\right]\left(A^{n+1}+B^{n+1}\right)}{(n+1)!} \cdot \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\Bigg| \\ \le \frac{\Gamma-\gamma}{(n+1)!}\left[\frac{(n+1)^2}{2n+1}(A+B)\left(A^{2n+1}+B^{2n+1}\right) - \frac{1+(-1)^n}{2}\left(A^{n+1}+B^{n+1}\right)^2\right]^{\frac{1}{2}},\end{aligned} \tag{15}$$

where $A = \frac{x-a}{2}$ and $B = \frac{b-x}{2}$.

Proof. Let $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{x+b}{2}$. Taking $A = \frac{x-a}{2}$, $B = \frac{b-x}{2}$, we have

$$R_k(x) = B^k + (-1)^{(k-1)}A^k, \quad S_k(x) = A^k f^{(k-1)}(a) + (-1)^{(k-1)} B^k f^{(k-1)}(b),$$
$$Q_n(x) = 2\left(A^{2n+1} + B^{2n+1}\right),$$
$$\theta_n(x) = \left[1+(-1)^n\right]\left(A^{n+1}+B^{n+1}\right)$$

and

$$\begin{aligned}&\sum_{i=1,\, j>i}^{4} z_i z_j \left[z_i^n - ((-1)^{i+j} z_j)^n\right]^2 \\ &= 2\left[1+(-1)^{n+1}\right]\left(A^{2n+2}+B^{2n+2}\right) + 4AB\left[A^{2n}+B^{2n}-(1+(-1)^n)A^nB^n\right].\end{aligned}$$

Thus readily giving the left hand side of (15), and for the right hand side of (15), we see from (9),

$$\begin{aligned}I(x,n) &= \frac{1}{n+1}\Bigg\{\frac{4n^2(A+B)\left(A^{2n+1}+B^{2n+1}\right)}{2n+1} \\ &\quad + 2\left[1+(-1)^{n+1}\right]\left(A^{2n+2}+B^{2n+2}\right) \\ &\quad + 4AB\left[A^{2n}+B^{2n}-\left(1+(-1)^n\right)A^nB^n\right]\Bigg\}^{\frac{1}{2}} \\ &= \frac{1}{n+1}\Bigg\{\frac{4(n+1)^2(A+B)\left(A^{2n+1}+B^{2n+1}\right)}{2n+1} \\ &\quad - 2\left[1+(-1)^n\right]\left(A^{n+1}+B^{n+1}\right)^2\Bigg\}^{\frac{1}{2}}.\end{aligned} \tag{16}$$

A simple substitution in (8) of (16) completes the proof. □

Corollary 2. *Let the conditions of Theorem 4 hold. Then*

(17)

$$\begin{aligned}&\Bigg|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{f^{(k-1)}(a)+\big[1+(-1)^{k-1}\big]f^{(k-1)}\big(\frac{a+b}{2}\big)+(-1)^{(k-1)}f^{(k-1)}(b)}{k!}\Big(\frac{b-a}{4}\Big)^k\\&\qquad\qquad-\frac{2\big[1+(-1)^n\big]}{(n+1)!}\Big(\frac{b-a}{4}\Big)^{n+1}\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\Bigg|\\&\le\frac{\Gamma-\gamma}{(n+1)!}\left[\frac{4(n+1)^2}{2n+1}-2(1+(-1)^n)\right]^{\frac{1}{2}}\Big(\frac{b-a}{4}\Big)^{n+1}.\end{aligned}$$

Proof. The proof of (17) follows directly from (15) with $x=\frac{a+b}{2}$. □

§3. A sharp perturbed generalized three point inequality

We now use Theorem 3 to derive a sharp perturbed generalized three point inequality for n-time differentiable functions with evaluations at an interior point and at the end points.

Theorem 5. *Let the conditions of Theorem 4 hold. Then for all $x\in[a,b]$ we have*

(18)

$$\begin{aligned}&\Bigg|\int_a^b f(t)\,dt-\sum_{k=1}^{n}\frac{A^kf^{(k-1)}(a)+\big[(-1)^{k-1}A^k+B^k\big]f^{(k-1)}(x)+(-1)^{k-1}B^kf^{(k-1)}(b)}{k!}\\&\qquad\qquad-\frac{\big[1+(-1)^n\big]\big(A^{n+1}+B^{n+1}\big)}{(n+1)!}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\Bigg|\\&\le\frac{\Gamma-\gamma}{(n+1)!}G(a,b,x;n),\end{aligned}$$

where for an odd n,

$$G(a,b,x;n)=A^{n+1}+B^{n+1}, \tag{19}$$

and for an even n,

(20)

$$G(a,b,x;n)=\begin{cases}\frac{2(AB^{n+1}-BA^{n+1})}{A+B}+\frac{2n(A^{n+1}+B^{n+1})}{(n+1)(A+B)}\sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}, & a\le x\le\xi,\\ \frac{4n(A^{n+1}+B^{n+1})}{(n+1)(A+B)}\sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}, & \xi<x<\eta,\\ \frac{2(BA^{n+1}-AB^{n+1})}{A+B}+\frac{2n(A^{n+1}+B^{n+1})}{(n+1)(A+B)}\sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}, & \eta\le x\le b,\end{cases}$$

with $A = \frac{x-a}{2}$, $B = \frac{b-x}{2}$, ξ *and* η *are the real roots of the equations*

$$\frac{(\xi - a)^{n+1} + (b - \xi)^{n+1}}{(n+1)(b-a)} - (\xi - a)^n = 0 \tag{21}$$

and

$$\frac{(\eta - a)^{n+1} + (b - \eta)^{n+1}}{(n+1)(b-a)} - (b - \eta)^n = 0 \tag{22}$$

respectively, and $a < \xi < \eta < b$.

Proof. Applying the variant of Grüss inequality (4) by associating $f^{(n)}(t)$ with $g(t)$ and $K_n(x,t)$ with $h(t)$, gives

$$\begin{aligned} &\left| \int_a^b K_n(x,t) f^{(n)}(t)\,dt - \left(\int_a^b K_n(x,t)\,dt \right) \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \\ &\leq \frac{\Gamma - \gamma}{2} \int_a^b \left| K_n(x,t) - \frac{1}{b-a} \int_a^b K_n(x,s)\,ds \right| dt, \end{aligned} \tag{23}$$

where the kernel $K_n : [a,b]^2 \to \mathbf{R}$ is given by

$$K_n(x,t) = \begin{cases} \frac{\left(t - \frac{a+x}{2}\right)^n}{n!}, & t \in [a,x], \\ \frac{\left(t - \frac{x+b}{2}\right)^n}{n!}, & t \in (x,b]. \end{cases} \tag{24}$$

For any fixed $x \in [a,b]$, taking $A = \frac{x-a}{2}$ and $B = \frac{b-x}{2}$, we can derive from (23) and (24) that

$$\begin{aligned} &\Bigg| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{A^k f^{(k-1)}(a) + \left[(-1)^{k-1} A^k + B^k\right] f^{(k-1)}(x) + (-1)^{k-1} B^k f^{(k-1)}(b)}{k!} \\ &\qquad - \frac{\left[1+(-1)^n\right]\left(A^{n+1}+B^{n+1}\right)}{(n+1)!} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \Bigg| \\ &\leq \frac{\Gamma - \gamma}{2n!} \Bigg[\int_a^x \left| \left(t - \frac{a+x}{2}\right)^n - \frac{\left[1+(-1)^n\right]\left(A^{n+1}+B^{n+1}\right)}{2(n+1)\left(A+B\right)} \right| dt \\ &\qquad + \int_x^b \left| \left(t - \frac{x+b}{2}\right)^n - \frac{\left[1+(-1)^n\right]\left(A^{n+1}+B^{n+1}\right)}{2(n+1)\left(A+B\right)} \right| dt \Bigg]. \end{aligned} \tag{25}$$

For an odd n, it is easy to find that the last two integrals in (25) is equal to

$$\int_a^x \left| \left(t - \frac{a+x}{2}\right)^n \right| dt + \int_x^b \left| \left(t - \frac{x+b}{2}\right)^n \right| dt = \frac{2\left(A^{n+1}+B^{n+1}\right)}{n+1}. \tag{26}$$

For an even n, the last two integrals in (25) is

$$\int_a^x \left|\left(t-\frac{a+x}{2}\right)^n-\frac{A^{n+1}+B^{n+1}}{(n+1)\big(A+B\big)}\right| dt+\int_x^b \left|\left(t-\frac{x+b}{2}\right)^n-\frac{A^{n+1}+B^{n+1}}{(n+1)\big(A+B\big)}\right| dt$$

which can be calculated as follows:

For brevity, we first put

$$\begin{aligned} p_1(t) &:= \left(t-\frac{a+x}{2}\right)^n-\frac{A^{n+1}+B^{n+1}}{(n+1)\big(A+B\big)},\ \ t\in[a,x], \\ p_2(t) &:= \left(t-\frac{x+b}{2}\right)^n-\frac{A^{n+1}+B^{n+1}}{(n+1)\big(A+B\big)},\ \ t\in[x,b]. \end{aligned}$$

It is easy to find by elementary calculus that the function $p_1(t)$ is strictly decreasing in $(a, \frac{a+x}{2})$ and strictly increasing in $(\frac{a+x}{2}, x)$ as well as the function $p_2(t)$ is strictly decreasing in $(x, \frac{x+b}{2})$ and strictly increasing in $(\frac{x+b}{2}, b)$. Moreover, we have

$$\begin{aligned} p_1(a) = p_1(x) &= \left(\frac{x-a}{2}\right)^n-\frac{(x-a)^{n+1}+(b-x)^{n+1}}{2^n(n+1)(b-a)}, \\ p_2(x) = p_2(b) &= \left(\frac{b-x}{2}\right)^n-\frac{(x-a)^{n+1}+(b-x)^{n+1}}{2^n(n+1)(b-a)}, \\ p_1\left(\frac{a+x}{2}\right) = p_2\left(\frac{x+b}{2}\right) &= -\frac{(x-a)^{n+1}+(b-x)^{n+1}}{2^n(n+1)(b-a)} < 0. \end{aligned}$$

Now we consider another two functions as

$$\begin{aligned} q_1(u) &= \frac{(u-a)^{n+1}+(b-u)^{n+1}}{(n+1)(b-a)}-(u-a)^n,\ \ u\in[a,b], \\ q_2(u) &= \frac{(u-a)^{n+1}+(b-u)^{n+1}}{(n+1)(b-a)}-(b-u)^n,\ \ u\in[a,b]. \end{aligned}$$

It is not difficult to find that $q_1(u)$ is strictly decreasing on $[a,b]$ and $q_2(u)$ is strictly increasing on $[a,b]$ and there exist unique $\xi\in(a,\frac{a+b}{2})$ and unique $\eta\in(\frac{a+b}{2},b)$ such that $q_1(\xi)=0$ and $q_2(\eta)=0$. i.e., ξ and η are the real roots of equations (21) and (22) respectively and $a<\xi<\eta<b$. If $x\in[a,\xi]$, then $q_1(x)\ge 0$ and $q_2(x)<0$ which imply that $p_1(a)=p_1(x)\le 0$ and $p_2(x)=p_2(b)>0$. If $x\in(\xi,\eta)$, then $q_1(x)<0$ and $q_2(x)<0$ which imply that $p_1(a)=p_1(x)>0$ and $p_2(x)=p_2(b)>0$. If $x\in[\eta,b]$, then $q_1(x)<0$ and $q_2(x)\ge 0$ which imply that $p_1(a)=p_1(x)>0$ and $p_2(x)=p_2(b)\le 0$. So there are three possible cases to be determined.

(i) In case $x \in [a,\xi]$, $p_1(t) \le 0$ for $t \in [a,x]$ and $p_2(t)$ has two zeros in (x,b) as $t_3 = \frac{x+b}{2} - \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$ and $t_4 = \frac{x+b}{2} + \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$. We have

$$\begin{aligned}
&\int_a^x \left| \left(t - \frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right| dt \\
&\qquad + \int_x^b \left| \left(t - \frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right| dt \\
&= \int_a^x \left[\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} - \left(t - \frac{a+x}{2}\right)^n \right] dt \\
&\qquad + \int_x^{t_3} \left[\left(t - \frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt \\
&\qquad + \int_{t_3}^{t_4} \left[\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} - \left(t - \frac{x+b}{2}\right)^n \right] dt \\
&\qquad + \int_{t_4}^{b} \left[\left(t - \frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt \\
&= \frac{4\left(AB^{n+1} - BA^{n+1}\right)}{(n+1)(A+B)} + \frac{4n\left(A^{n+1}+B^{n+1}\right)}{(n+1)^2(A+B)} \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}.
\end{aligned} \tag{27}$$

(ii) In case $x \in (\xi,\eta)$, $p_1(t)$ has two zeros in (a,x) as $t_1 = \frac{a+x}{2} - \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$ and $t_2 = \frac{a+x}{2} + \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$, $p_2(t)$ has two zeros in (x,b) as $t_3 = \frac{x+b}{2} - \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$ and $t_4 = \frac{x+b}{2} + \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$. We have

$$\begin{aligned}
&\int_a^x \left| \left(t-\frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right| dt + \int_x^b \left| \left(t-\frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right| dt \\
&= \int_a^{t_1} \left[\left(t-\frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt + \int_{t_1}^{t_2} \left[\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} - \left(t-\frac{a+x}{2}\right)^n \right] dt \\
&\quad + \int_{t_2}^{x} \left[\left(t-\frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt + \int_x^{t_3} \left[\left(t-\frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt \\
&\quad + \int_{t_3}^{t_4} \left[\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} - \left(t-\frac{x+b}{2}\right)^n \right] dt + \int_{t_4}^{b} \left[\left(t-\frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt \\
&= \frac{8n\left(A^{n+1}+B^{n+1}\right)}{(n+1)^2(A+B)} \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}.
\end{aligned} \tag{28}$$

(iii) In case $x \in [\eta, b]$, $p_1(t)$ has two zeros in (a,x) as $t_1 = \frac{a+x}{2} - \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$ and $t_2 = \frac{a+x}{2} + \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}$, $p_2(t) \le 0$ for $t \in [x,b]$. We have

$$\begin{aligned}
&\int_a^x \left| \left(t - \frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right| dt \\
&\qquad + \int_x^b \left| \left(t - \frac{x+b}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right| dt \\
&= \int_a^{t_1} \left[\left(t - \frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt \\
&\qquad + \int_{t_1}^{t_2} \left[\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} - \left(t - \frac{a+x}{2}\right)^n \right] dt \\
&\qquad + \int_{t_2}^{x} \left[\left(t - \frac{a+x}{2}\right)^n - \frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} \right] dt \\
&\qquad + \int_x^b \left[\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)} - \left(t - \frac{x+b}{2}\right)^n \right] dt \\
&= \frac{4\left(BA^{n+1} - AB^{n+1}\right)}{(n+1)(A+B)} + \frac{4n\left(A^{n+1}+B^{n+1}\right)}{(n+1)^2(A+B)} \sqrt[n]{\frac{A^{n+1}+B^{n+1}}{(n+1)(A+B)}}.
\end{aligned} \tag{29}$$

Consequently, the inequality (18) with (19) and (20) follows from (25), (26), (27), (28) and (29).

The proof is completed. □

Remark 1. It is not difficult to prove that the inequality (18) with (19) and (20) is sharp in the sense that we can construct the function f to attain the equality in (18) with (19) and (20). Indeed, for an odd n we may choose f such that

$$f^{(n-1)}(t) = \begin{cases} \gamma(t-a), & a \le t < \frac{a+x}{2}, \\ \Gamma(t - \frac{a+x}{2}) + \frac{x-a}{2}\gamma, & \frac{a+x}{2} \le t < x, \\ \gamma(t - \frac{a+x}{2}) + \frac{x-a}{2}\Gamma, & x \le t < \frac{x+b}{2}, \\ \Gamma(t - \frac{a+b}{2}) + \frac{b-a}{2}\gamma, & \frac{x+b}{2} \le t \le b \end{cases}$$

for any $x \in [a,b]$, and for an even n we may choose f such that

$$f^{(n-1)}(t) = \begin{cases} \gamma(t-a), & a \le t < x, \\ \Gamma(t-x) + (x-a)\gamma, & x \le t < t_3, \\ \gamma(t - t_3 + x - a) + (t_3 - x)\Gamma, & t_3 \le t < t_4, \\ \Gamma(t - t_4 + t_3 - x) + (t_4 - t_3 + x - a)\gamma, & t_4 \le t \le b \end{cases}$$

for any $x \in [a, \xi]$, and

$$f^{(n-1)}(t) = \begin{cases} \Gamma(t-a), & a \le t < t_1, \\ \gamma(t-t_1) + (t_1-a)\Gamma, & t_1 \le t < t_2, \\ \Gamma(t-t_2+t_1-a) + (t_2-t_1)\gamma, & t_2 \le t < t_3, \\ \gamma(t-t_3+t_2-t_1) + (t_3-t_2+t_1-a)\Gamma, & t_3 \le t < t_4, \\ \Gamma(t-t_4+t_3-t_2+t_1-a) + (t_4-t_3+t_2-t_1)\gamma, & t_4 \le t \le b \end{cases}$$

for any $x \in (\xi, \eta)$, and

$$f^{(n-1)}(t) = \begin{cases} \Gamma(t-a), & a \le t < t_1, \\ \gamma(t-t_1) + (t_1-a)\Gamma, & t_1 \le t < t_2, \\ \Gamma(t-t_2+t_1-a) + (t_2-t_1)\gamma, & t_2 \le t < x, \\ \gamma(t-x+t_2-t_1) + (x-t_2+t_1-a)\Gamma, & x \le t \le b \end{cases}$$

for any $x \in [\eta, b]$.

It is clear that the above all $f^{(n-1)}(t)$ are absolutely continuous on $[a, b]$.

Remark 2. If we take $x = \frac{a+b}{2}$, we can obtain the following sharp perturbed three point inequality:

$$\left| \int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{f^{(k-1)}(a) + \left[1+(-1)^{k-1}\right] f^{(k-1)}(\frac{a+b}{2}) + (-1)^{k-1} f^{(k-1)}(b)}{k!} \left(\frac{b-a}{4}\right)^k - \frac{2\left[1+(-1)^n\right]}{(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right|$$

$$\le \begin{cases} \frac{2(\Gamma-\gamma)}{(n+1)!} (\frac{b-a}{4})^{n+1}, & n : \text{odd}, \\ \frac{4n(\Gamma-\gamma)}{(n+1)!(n+1)\sqrt[n]{n+1}} (\frac{b-a}{4})^{n+1}, & n : \text{even}. \end{cases}$$

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