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A pseudo-inverse of the fundamental form and its application to affine immersions of non-degenerate surfaces

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Abstract. We choose a canonical transversal bundle of an affine immersion if a pseudo-inverse of the fundamental form exists. In particular, well-known canonical transversal bundles for a non-degenerate surface in $\mathbb{R}^4$ are generalized to those for a non-degenerate surface in 4-dimensional manifold with torsion-free connection.

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§1. Introduction

For affine immersions, some canonical choices of a transversal bundle are known. Among them, the following are relevant to this paper. For a non-degenerate hypersurface in $\mathbb{R}^{n+1}$, there exists a canonical equiaffine transversal bundle which is spanned by the Blaschke normal field. For a non-degenerate surface in $\mathbb{R}^4$, Burstin and Mayer [3], Klingenberg [7], and Nomizu and Vrancken [11] respectively gave canonical transversal bundles, where the last is equiaffine. If the affine metric is positive definite, then Scharlach and Vrancken [13] gave canonical transversal bundles which generalize these three transversal bundles. For an immersion $f : M^n \to \mathbb{R}^{n+r}$ ($r \leq \frac{1}{2}n(n+1)$), Weise [14] gave a canonical transversal bundle under a regularity condition, where he used a "konjugierte Elemente" of the affine fundamental form to construct the transversal bundle. Recently, revising the regularity condition and the "konjugierte Elemente" (pseudo-inverse elements), Wiehe [15] gave a canonical equiaffine transversal bundle for an n-dimensional manifold in $\mathbb{R}^{n+r}$

($r \leq \frac{1}{2}n(n+1)$). Investigating geometry of splittings for a short exact sequence of vector bundles with connection, Abe and Ishii [2] gave a canonical unimodular splitting if a pseudo-inverse of the fundamental form exists. For a regular immersion into $\mathbb{R}^{n+r}$, this splitting gives the equiaffine transversal bundle by Wiehe.

In this paper, for an immersion $f : M \to \tilde{M}$, we first specify the theory in [2] to the following short exact sequence of vector bundles over M:

$$0 \to TM \xrightarrow{\iota} f^{\#}(T\tilde{M}) \xrightarrow{p} f^{\#}(T\tilde{M})/\iota(TM) \to 0$$

with the pull-back $\tilde{\nabla}$ of a torsion-free connection on $\tilde{M}$. We note that a transversal bundle for the immersion f is given by a splitting of this short exact sequence. We set $B := p\tilde{\nabla}\iota$ and call B the fundamental form. We include the proofs for the results shown in [2] in order to make this paper self-contained. Moreover, the regularity condition and the pseudo-inverse of the fundamental form defined by Wiehe in [15] for $\tilde{M} = \mathbb{R}^{n+r}$ are generalized. Our main purpose is to construct a two parameter family of splittings for a non-degenerate surface of codimension two, which gives all known canonical transversal bundles in the special case of $\tilde{M} = \mathbb{R}^4$.

In Section 2, we set up notation and terminology used in this paper. In Section 3, we introduce the notion of a pseudo-inverse of the fundamental form and recall some of the results in [2] to give a canonical choice of a transversal bundle. This section also includes new result about the regularity condition. In Section 4, we study affine surfaces of codimension two.

§2. Preliminaries

Throughout this paper we assume that all manifolds and mappings are smooth and all vector bundles are real. Let M be an n-dimensional manifold. Let V, W be vector bundles over M, $\Gamma(V)$ the space of cross-sections of V, and $\mathfrak{C}(V)$ the set of covariant derivatives of connections on V. Let $\mathrm{Hom}(V, W)$ be the vector bundle of which fiber $\mathrm{Hom}(V, W)_x$ at $x \in M$ is the vector space $\mathrm{Hom}(V_x, W_x)$ of linear mappings from V_x to W_x. The space of vector bundle homomorphisms from V to W is denoted by $\mathrm{HOM}(V, W)$. We note that $\mathrm{HOM}(V, W)$ can be canonically identified with the space $\Gamma(\mathrm{Hom}(V, W))$. For non-negative integer k, we denote the space of V-valued k-forms on M by $A^k(V)$ and $A^k := A^k(M \times \mathbb{R})$.

Let $\tilde{M}$ be an $(n+r)$-dimensional manifold and $f : M \to \tilde{M}$ an immersion. We denote the pull-back bundle through f of $T\tilde{M}$ by $\tilde{T} := f^{\#}(T\tilde{M})$, the bundle mapping by $f_{\#} : \tilde{T} \to T\tilde{M}$, and its restriction to the fiber by $f_{\#x}$ for $x \in M$. We define a linear mapping $\iota_x : T_xM \to \tilde{T}_x$ by $\iota_x := (f_{\#x})^{-1} \circ f_{*x}$ for each $x \in M$, where $f_{*x} : T_xM \to T_{f(x)}\tilde{M}$ is the differential of f at x and the symbol $\circ$ is used to denote the composition of mappings. We often omit the symbol $\circ$ for simplicity. We define a bundle homomorphism $\iota : TM \to \tilde{T}$ by $\iota|_{T_xM} := \iota_x$. The mapping ι will be omitted if there is no ambiguity and set $T := TM = \iota(TM)$. Let $i : T \to \tilde{T}$ be the inclusion and $p : \tilde{T} \to \tilde{T}/T$ the canonical projection. We set $\mathrm{INV}_L(i) := \{\gamma \in \mathrm{HOM}(\tilde{T}, T) | \gamma i = \mathrm{id}_T\}$ and $\mathrm{INV}_R(p) := \{\mu \in \mathrm{HOM}(\tilde{T}/T, \tilde{T}) | p\mu = \mathrm{id}_{\tilde{T}/T}\}$.

Take $D \in \mathfrak{C}(T\tilde{M})$ and let $\tilde{\nabla} \in \mathfrak{C}(\tilde{T})$ be the pull-back connection on $\tilde{T}$ from D defined by $\tilde{\nabla} := f^{\#}D$. We set $B := p\tilde{\nabla}i$, where $p\tilde{\nabla}i$ is defined by $(p\tilde{\nabla}i)_X := p \circ \tilde{\nabla}_X \circ i$ for $X \in \Gamma(T)$. We call B the *fundamental form of the immersion* f. We note that $B \in A^1(\mathrm{Hom}(T, \tilde{T}/T))$ because of $pi = 0$. Let γ be an element of $\mathrm{INV}_L(i)$. Then there exists a unique $\hat{\gamma} \in \mathrm{INV}_R(p)$ such that $i\gamma + \hat{\gamma}p = \mathrm{id}_{\tilde{T}}$. We set ${}^{\gamma}\nabla := \gamma\tilde{\nabla}i$ (resp. $\nabla^{\hat{\gamma}} := p\tilde{\nabla}\hat{\gamma}$) and call ${}^{\gamma}\nabla$ (resp. $\nabla^{\hat{\gamma}}$) the *induced connection* on T (resp. the *transversal connection* on $\tilde{T}/T$). Since $\gamma i = \mathrm{id}_T$ and $p\hat{\gamma} = \mathrm{id}_{\tilde{T}/T}$, we have

Lemma 2.1. *For ${}^{\gamma}\nabla$ and $\nabla^{\hat{\gamma}}$, we have*

$$ {}^{\gamma}\nabla \in \mathfrak{C}(T) \quad \text{and} \quad \nabla^{\hat{\gamma}} \in \mathfrak{C}(\tilde{T}/T). $$

Now we state the relation between $\gamma \in \mathrm{INV}_L(i)$ and an affine immersion. For $\gamma \in \mathrm{INV}_L(i)$, we have a decomposition $\tilde{T} = T \oplus N_\gamma$, where N_γ is defined by $N_\gamma := \mathrm{Im}\hat{\gamma} \cong \tilde{T}/T$. We call N_γ the *transversal bundle* with respect to γ. Conversely we see that a decomposition $T \oplus N = \tilde{T}$ gives $\gamma \in \mathrm{INV}_L(i)$ as follows. We set $p_T : \tilde{T} \to T$ (resp. $p_N : \tilde{T} \to N$) the projection homomorphism. Since $p_T i = \mathrm{id}_T$, the corresponding homomorphism $\gamma := p_T \in \mathrm{INV}_L(i)$ is defined by the decomposition. For $T \oplus N = \tilde{T}$, let ∇^T (resp. B^N) be the induced connection on T (resp. the *affine fundamental form*) defined by $\nabla^T := p_T\tilde{\nabla}i \in \mathfrak{C}(T)$ (resp. $B^N := p_N\tilde{\nabla}i \in A^1(\mathrm{Hom}(T, N))$). We call (f, N) an *affine immersion with the transversal bundle* N from (M, ∇^T) to $(\tilde{M}, D)$. It follows that the correspondence between an affine immersion (f, N) and $p_T \in \mathrm{INV}_L(i)$ is one-to-one, where p_T is defined by the decomposition $T \oplus N = \tilde{T}$. The homomorphism $\hat{p}_T : \tilde{T}/T \to \tilde{T}$ induces the isomorphism $\tilde{p}_T : \tilde{T}/T \to N$. Since $p_N = \tilde{p}_T p$, we see that $B^N = \tilde{p}_T B$ and $\tilde{p}_T^{-1} B^N = B$.

For $\gamma, \bar{\gamma} \in \mathrm{INV}_L(i)$, since $(\bar{\gamma} - \gamma)i = 0$, we see that there exists a unique $\lambda \in \mathrm{HOM}(\tilde{T}/T, T)$ such that $\lambda p = \bar{\gamma} - \gamma$.

Theorem 2.2. *Take $\gamma, \bar{\gamma} \in \mathrm{INV}_L(i)$ and let $\lambda \in \mathrm{HOM}(\tilde{T}/T, T)$ be the homomorphism such that $\lambda p = \bar{\gamma} - \gamma$. For $\bar{\gamma}$, the geometric objects ${}^{\bar{\gamma}}\nabla \in \mathfrak{C}(T)$ and $\nabla^{\hat{\bar{\gamma}}} \in \mathfrak{C}(\tilde{T}/T)$ satisfy the following equations*:

$$ {}^{\bar{\gamma}}\nabla = {}^{\gamma}\nabla + \lambda B, \tag{2.1} $$

$$ \nabla^{\hat{\bar{\gamma}}} = \nabla^{\hat{\gamma}} - B\lambda, \tag{2.2} $$

Proof. We shall prove the last equation. The other equation can be easily obtained. Since $i\gamma + \hat{\gamma}p = \mathrm{id}_{\tilde{T}} = i\bar{\gamma} + \hat{\bar{\gamma}}p$, we have $\hat{\bar{\gamma}} = \hat{\gamma} - i\lambda$. Thus we have

$$\nabla^{\hat{\bar{\gamma}}} = p\tilde{\nabla}\hat{\bar{\gamma}} = p\tilde{\nabla}(\hat{\gamma} - i\lambda) = \nabla^{\hat{\gamma}} - B\lambda.$$

□

§3. A pseudo-inverse of the fundamental form

From now on, we assume that $\tilde{\nabla}$ is torsion-free. Since $[X, Y] \in \Gamma(T)$ for any $X, Y \in \Gamma(T)$, we see that B is symmetric, i.e., $B_X Y = B_Y X$. We first give the definition of pseudo-inverses. For $B \in \mathrm{HOM}(T, \mathrm{Hom}(T, \tilde{T}/T))$, we denote the corresponding element of $\Gamma(T^* \circledcirc T^* \otimes \tilde{T}/T)$ (resp. $\mathrm{HOM}(T \circledcirc T, \tilde{T}/T)$) to B by $\hat{B}$ (resp. $\tilde{B}$), where T^* is the dual bundle of T and $\circledcirc$ is the symmetric tensor product. For a symmetric $\mathfrak{B} \in \mathrm{HOM}(\mathrm{Hom}(T, \tilde{T}/T), T)$, we denote the corresponding element of $\Gamma(T \circledcirc T \otimes (\tilde{T}/T)^*)$ (resp. $\mathrm{HOM}(\tilde{T}/T, T \circledcirc T)$) to $\mathfrak{B}$ by $\hat{\mathfrak{B}}$ (resp. $\tilde{\mathfrak{B}}$).

Definition 3.1. If a symmetric $\mathfrak{B} \in \mathrm{HOM}(\mathrm{Hom}(T, \tilde{T}/T), T)$ satisfies the following equations, we say that $\mathfrak{B}$ is a *pseudo-inverse of B*:

$$\mathfrak{B} \circ B = r\mathrm{id}_T \text{ and } \tilde{B} \circ \tilde{\mathfrak{B}} = n\mathrm{id}_{\tilde{T}/T}. \tag{3.1}$$

This is a generalization of the definitions of those of "konjugierte Elemente" in [14] and pseudo-inverse elements in [15]. Even if there exists a pseudo-inverse $\mathfrak{B}$ of B, then $\mathfrak{B}$ is not unique in general. Considering the ranks of the affine subbundles with respect to $\mathrm{INV}_L(B)$ and $\mathrm{INV}_R(\tilde{B})$, we have

Lemma 3.1. *For $r = 1$ or $r = \frac{1}{2}n(n+1)$, a pseudo-inverse $\mathfrak{B}$ of B is unique if a pseudo-inverse of B exists.*

We secondly recall the theory in [2] to give a canonical transversal bundle. We assume that there exists a pseudo-inverse of B. Let $\mathfrak{B}$ be a pseudo-inverse of B. We denote the corresponding element of $\mathrm{HOM}(\mathrm{Hom}(\tilde{T}/T, T), T^*)$ (resp. $\mathrm{HOM}(T^*, \mathrm{Hom}(\tilde{T}/T, T)))$ to the dual mapping of B (resp. $\mathfrak{B}$) by B^* (resp. $\mathfrak{B}^*$) and the corresponding element of $\mathrm{HOM}(T^* \otimes (\tilde{T}/T), T)$ to $\mathfrak{B}$ by the same symbol $\mathfrak{B}$. From now on, X, Y, Z (resp. ξ) always denote elements of $\Gamma(T)$ (resp. $\Gamma(\tilde{T}/T)$). For $\gamma \in \mathrm{INV}_L(i)$, we denote the dual connection of ${}^\gamma\nabla$ by ${}^\gamma\nabla^*$ and set

$$(\hat{\nabla}^\gamma_X \mathfrak{B})(\eta, \xi) := {}^\gamma\nabla_X(\mathfrak{B}(\eta, \xi)) - \mathfrak{B}({}^\gamma\nabla^*{}_X \eta, \xi) - \mathfrak{B}(\eta, \nabla^{\hat{\gamma}}_X \xi),$$

where $\eta \in A^1$. The symbol $\mathcal{C}_T$ denotes the contraction with respect to X and η. Then we have $\mathcal{C}_T(\hat{\nabla}^\gamma \mathfrak{B}) \in \mathrm{HOM}(\tilde{T}/T, T)$. We define $H^\gamma_{\mathfrak{B}} \in \mathrm{HOM}(\tilde{T}/T, T)$ by

$$H^{\gamma}_{\mathfrak{B}} := -\frac{1}{n+r}(\mathcal{C}_T(\hat{\nabla}^{\gamma}\mathfrak{B}) - \frac{1}{n+2r}\mathfrak{B}^*(B^*(\mathcal{C}_T(\hat{\nabla}^{\gamma}\mathfrak{B})))).$$

Theorem 3.2.([2]) *Let $\mathfrak{B}$ be a pseudo-inverse of B. For $\mathfrak{A} \in \mathrm{HOM}(\tilde{T}/T, T)$, there exists a unique $\gamma_{\mathfrak{A}} \in \mathrm{INV}_L(i)$ which satisfies*

$$H^{\gamma_{\mathfrak{A}}}_{\mathfrak{B}} = \mathfrak{A}.$$

Proof. Take $\gamma, \bar{\gamma} \in \mathrm{INV}_L(i)$ and let $\lambda \in \mathrm{HOM}(\tilde{T}/T, T)$ be the homomorphism such that $\lambda p = \bar{\gamma} - \gamma$. From (2.1)–(2.2), we have

$$(\hat{\nabla}^{\bar{\gamma}}_X\mathfrak{B})(\eta, \xi) = (\hat{\nabla}^{\gamma}_X\mathfrak{B})(\eta, \xi) + \lambda B_X\mathfrak{B}(\eta, \xi) + \mathfrak{B}((\lambda B)^*_X\eta, \xi) + \mathfrak{B}(\eta, B_X\lambda(\xi))$$

for $\eta \in A^1$, where $(\lambda B)^*_X : T^* \to T^*$ is the dual of $\lambda B_X : T \to T$. From (3.1), we have

$$\mathcal{C}_T(\hat{\nabla}^{\bar{\gamma}}\mathfrak{B})\xi = \mathcal{C}_T(\hat{\nabla}^{\gamma}\mathfrak{B})\xi + (n+r)\lambda(\xi) + \mathfrak{B}(\mathcal{C}_T((\lambda B)^*), \xi).$$

Since $\mathcal{C}_T((\lambda B)^*) = \mathrm{tr}(\lambda B) = B^*(\lambda)$, we have

$$\mathfrak{B}(\mathcal{C}_T((\lambda B)^*), \xi) = \mathfrak{B}(B^*(\lambda), \xi) = \mathfrak{B}^*(B^*(\lambda))(\xi).$$

Since $B^* \circ \mathfrak{B}^* = r\mathrm{id}_{T^*}$, we obtain

$$B^*(\mathcal{C}_T(\hat{\nabla}^{\bar{\gamma}}\mathfrak{B})) = B^*(\mathcal{C}_T(\hat{\nabla}^{\gamma}\mathfrak{B})) + (n+2r)B^*(\lambda).$$

Thus we see that

$$H^{\bar{\gamma}}_{\mathfrak{B}} = H^{\gamma}_{\mathfrak{B}} - \lambda.$$

Hence we have $\bar{\gamma} + (H^{\bar{\gamma}}_{\mathfrak{B}} - \mathfrak{A})p = \gamma + (H^{\gamma}_{\mathfrak{B}} - \mathfrak{A})p$. We set $\gamma_{\mathfrak{A}} := \gamma + (H^{\gamma}_{\mathfrak{B}} - \mathfrak{A})p$. Since $\gamma_{\mathfrak{A}} = \gamma_{\mathfrak{A}} + (H^{\gamma_{\mathfrak{A}}}_{\mathfrak{B}} - \mathfrak{A})p$, we have $H^{\gamma_{\mathfrak{A}}}_{\mathfrak{B}} = \mathfrak{A}$. □

Next, we define an equiaffine $\gamma \in \mathrm{INV}_L(i)$. Hereafter we assume that T and $\tilde{T}$ are orientable. We define the line bundle $\mathrm{Det}\tilde{T}$ by $\mathrm{Det}\tilde{T} := \wedge^{n+r}\tilde{T}^*$. We set $\mathfrak{V}(\tilde{T}) := \{\tilde{\omega} \in \Gamma(\mathrm{Det}\tilde{T}) | \tilde{\omega}$ is everywhere non-zero $\}$ and call $\tilde{\omega} \in \mathfrak{V}(\tilde{T})$ the *volume element* on $\tilde{T}$. Let $i^* : \wedge^n\tilde{T} \to \mathrm{Det}(T)$ (resp. $p^* : \mathrm{Det}(\tilde{T}/T) \to \wedge^r\tilde{T}^*$) be the induced homomorphism with respect to $i : T \to \tilde{T}$ (resp. $p : \tilde{T} \to \tilde{T}/T$). For $\tilde{\omega} \in \mathfrak{V}(\tilde{T})$, we define the *induced volume element* $\omega_T \in \mathfrak{V}(T)$ from $\tilde{\omega}$ with respect to a volume element $\omega_Q \in \mathfrak{V}(\tilde{T}/T)$ on the quotient bundle $\tilde{T}/T$ by

$$\omega_T := i^*\hat{\omega}_T,$$

where $\hat{\omega}_T \in \wedge^n\tilde{T}^*$ satisfies $\hat{\omega}_T \wedge p^*\omega_Q = \tilde{\omega}$. For a given $\tilde{\omega}$, the correspondence between $\omega_Q \in \mathfrak{V}(\tilde{T}/T)$ and $\omega_T \in \mathfrak{V}(T)$ is one-to-one.

Definition 3.2. Let γ be an element of $\mathrm{INV}_L(i)$. For $\omega_T \in \mathfrak{V}(T)$, if ${}^{\gamma}\nabla\omega_T = 0$, we say that γ is *equiaffine* with respect to ω_T.

For $\gamma \in \mathrm{INV}_L(i)$ and $\omega_T \in \mathfrak{V}(T)$, let $\nu^{\gamma}_{\omega_T} \in A^1$ be the connection form of ${}^{\gamma}\nabla$ relative to a frame field ω_T, i.e., $\nu^{\gamma}_{\omega_T}(X)\omega_T := {}^{\gamma}\nabla_X\omega_T$. Then we have

Lemma 3.3. *Take $\gamma, \bar{\gamma} \in \mathrm{INV}_L(i)$ and let $\lambda \in \mathrm{HOM}(\tilde{T}/T, T)$ be the homomorphism such that $\lambda p = \bar{\gamma} - \gamma$. Then for $\omega_T \in \mathfrak{V}(T)$, we have*

$$\nu^{\bar{\gamma}}_{\omega_T} = \nu^{\gamma}_{\omega_T} - B^*(\lambda).$$

Proof. From (2.1), we have

$$^{\bar{\gamma}}\nabla\omega_T = {}^{\gamma}\nabla\omega_T - \mathrm{tr}(\lambda B)\omega_T = (\nu^{\gamma}_{\omega_T} - B^*(\lambda))\omega_T.$$

□

We shall use the following lemma to obtain an equiaffine $\gamma \in \mathrm{INV}_L(i)$. From Lemma 3.3 and the equation: $H^{\bar{\gamma}}_{\mathfrak{B}} = H^{\gamma}_{\mathfrak{B}} - \lambda$, where $\lambda p = \bar{\gamma} - \gamma$, we have

Lemma 3.4. *Let $\omega_T \in \mathfrak{V}(T)$ be a volume element on T. For $\gamma, \bar{\gamma} \in \mathrm{INV}_L(i)$, we have*

$$\nu^{\gamma}_{\omega_T} - B^*(H^{\gamma}_{\mathfrak{B}}) = \nu^{\bar{\gamma}}_{\omega_T} - B^*(H^{\bar{\gamma}}_{\mathfrak{B}}).$$

For $\gamma \in \mathrm{INV}_L(i)$ and $\omega_T \in \mathfrak{V}(T)$, we set $\nu_{\omega_T,\mathfrak{B}} := \nu^{\gamma}_{\omega_T} - B^*(H^{\gamma}_{\mathfrak{B}})$. Lemma 3.4 shows that $\nu_{\omega_T,\mathfrak{B}}$ is independent of the choice of $\gamma \in \mathrm{INV}_L(i)$. Then we obtain

Corollary 3.5. *The homomorphism $\gamma_{\mathfrak{A}}$ in Theorem 3.2 is equiaffine with respect to ω_T if the following equation is satisfied*:

$$\mathfrak{A} = -\frac{1}{r}\mathfrak{B}^*(\nu_{\omega_T,\mathfrak{B}}).$$

Proof. Take $\gamma \in \mathrm{INV}_L(i)$ and set $\gamma_{\mathfrak{A}} := \gamma + (H^{\gamma}_{\mathfrak{B}} + \frac{1}{r}\mathfrak{B}^*(\nu_{\omega_T,\mathfrak{B}}))p$. Since $B^* \circ \mathfrak{B}^* = r\mathrm{id}_{T^*}$, we have

$$B^*(\mathfrak{B}^*(\kappa)) = r\kappa$$

for $\kappa \in A^1$. Then we have

$$B^*(H^{\gamma}_{\mathfrak{B}} + \frac{1}{r}\mathfrak{B}^*(\nu^{\gamma}_{\omega_T} - B^*(H^{\gamma}_{\mathfrak{B}}))) = \nu^{\gamma}_{\mathfrak{B}}.$$

From Lemma 3.3, we obtain $\nu^{\gamma_{\mathfrak{A}}}_{\omega_T} = 0$. □

Now we introduce a regularity condition on the immersion and apply Corollary 3.5 to generalize the result of Wiehe [15]. Take $\omega_Q \in \mathfrak{V}(\tilde{T}/T)$ and let $\omega_Q B \in \Gamma((\otimes^{2r}T)^*)$ be defined by

$$(\omega_Q B)(Y_1, Y_2, \ldots, Y_{2r-1}, Y_{2r}) := \omega_Q(B_{Y_1}Y_2, \ldots, B_{Y_{2r-1}}Y_{2r})$$

for $Y_1, \ldots, Y_{2r} \in \Gamma(T)$. Let $Y^1_1, \ldots, Y^{2r}_n$ be elements of $\Gamma(T)$. We define $(\omega_Q B)^n \in \Gamma((\otimes^{2r}(\otimes^n T))^*)$ by

$$\begin{aligned}&(\omega_Q B)^n(Y^1_1, \ldots, Y^1_n, \cdots, Y^{2r}_1, \ldots, Y^{2r}_n)\\ &:= (\omega_Q B)(Y^1_1, \ldots, Y^{2r}_1) \cdots (\omega_Q B)(Y^1_n, \ldots, Y^{2r}_n),\end{aligned}$$

and $\mathfrak{A}_n((\omega_Q B)^n) \in \Gamma((\otimes^{2r}(\wedge^n T))^*)$ by

$$
\begin{aligned}
(3.2) \qquad & \mathfrak{A}_n((\omega_Q B)^n)(Y_1^1, \ldots, Y_n^1, \cdots, Y_1^{2r}, \ldots, Y_n^{2r}) \\
&:= \frac{1}{n!} \sum_{\sigma_1, \ldots, \sigma_{2r}} \mathrm{sgn}\sigma_1 \cdots \mathrm{sgn}\sigma_{2r} (\omega_Q B)(Y^1_{\sigma_1(1)}, \ldots, Y^{2r}_{\sigma_{2r}(1)}) \cdots \\
&\qquad \times (\omega_Q B)(Y^1_{\sigma_1(n)}, \ldots, Y^{2r}_{\sigma_{2r}(n)}),
\end{aligned}
$$

where $\sigma_1, \ldots, \sigma_{2r}$ are permutations on $\{1, \ldots, n\}$. Take $\tilde{\omega} \in \mathfrak{V}(\tilde{T})$ and let $\omega_T \in \mathfrak{V}(T)$ be the induced volume element form $\tilde{\omega}$ with respect to ω_Q. There exists a unique $\det_{\omega_T}(\omega_Q B) \in A^0$ such that

$$\mathfrak{A}_n((\omega_Q B)^n) = \det_{\omega_T}(\omega_Q B)\omega_T \otimes \cdots \otimes \omega_T.$$

We set

$$\omega(B) := |\det_{\omega_T}(\omega_Q B)|^{\frac{1}{n+2r}} \omega_T.$$

Note that $\omega(B)$ depends only on B and the orientation given by ω_T.

Definition 3.3. If the fundamental form B satisfies $\omega(B)_x \neq 0$ at each $x \in M$, we say that the immersion f is *regular*.

The assumption that f is regular is depends only on the immersion $f : M \to \tilde{M}$ and the connection D on $\tilde{M}$. Let N be a subbundle of $\tilde{T}$ such that $T \oplus N = \tilde{T}$ and B^N the affine fundamental form. We set $\omega_N := \tilde{p}_T^{-1*}\omega_Q \in \mathfrak{V}(N)$. Since $B = \tilde{p}_T^{-1} B^N$, we obtain $\omega_Q B = \omega_N B^N$. Then we have

$$|\det_{\omega_T}(\omega_N B^N)|^{\frac{1}{n+2r}} \omega_T = \omega(B),$$

whose left hand side coincides with that of Wiehe [15] in the case of $\tilde{M} = \mathbb{R}^{n+r}$. Let $X_1, \ldots, X_n$ be a unimodular local frame field for T with respect to ω_T, i.e., $\omega_T(X_1, \ldots, X_n) = 1$, and $\xi_1, \ldots, \xi_r$ a unimodular local frame field for $\tilde{T}/T$ with respect to ω_Q. Let $X^1, \ldots, X^n$ (resp. $\xi^1, \ldots, \xi^r$) be the dual of $X_1, \ldots, X_n$ (resp. $\xi_1, \ldots, \xi_r$) and $B^\alpha_{jk} := \xi^\alpha(B_{X_j} X_k)$. We set

$$a_{j_1 \cdots j_{2r}} := (\omega_Q B)(X_{j_1}, \ldots X_{2r}) = \delta_{\alpha_1 \cdots \alpha_r} B^{\alpha_1}_{j_1 j_2} \cdots B^{\alpha_r}_{j_{2r-1} j_{2r}}.$$

Then we have

$$\det_{\omega_T}(\omega_Q B) = \frac{1}{n!} \delta^{j_1^1 \cdots j_1^n} \cdots \delta^{j_{2r}^1 \cdots j_{2r}^n} a_{j_1^1 \cdots j_{2r}^1} \cdots a_{j_1^n \cdots j_{2r}^n}.$$

If f is regular, then we define $\mathfrak{B}' \in \mathrm{HOM}(\mathrm{Hom}(T, \tilde{T}/T), T)$ by requiring

$$
\begin{aligned}
&\det_{\omega_T}(\omega_Q B)\hat{\mathfrak{B}}'(X^j, X^k, \xi_\alpha) = \det_{\omega_T}(\omega_Q B)\mathfrak{B}'^{jk}_\alpha \\
&= r\delta_{\alpha\alpha_2 \cdots \alpha_r} B^{\alpha_2}_{j_3 j_4} \cdots B^{\alpha_r}_{j_{2r-1} j_{2r}} \mathrm{adj}_{\omega_T}(\omega_Q B)^{jk j_3 \cdots j_{2r}},
\end{aligned}
$$

where

$$\mathrm{adj}_{\omega_T}(\omega_Q B)^{j_1\cdots j_{2r}} := \frac{1}{(n-1)!}\delta^{j_1 l_1^2\cdots l_1^n}\cdots\delta^{j_{2r} l_{2r}^2\cdots l_{2r}^n} a_{l_1^2\cdots l_{2r}^2}\cdots a_{l_1^n\cdots l_{2r}^n}$$

is the classical adjoint as in Wiehe [15]. Since

$$n\det_{\omega_T}(\omega_Q B) = a_{j_1\cdots j_{2r}}\mathrm{adj}_{\omega_T}(\omega_Q B)^{j_1\cdots j_{2r}},$$

we see that $\mathfrak{B}'$ satisfies

$$\mathfrak{B}'^{jk}_{\alpha} B^{\alpha}_{jl} = r\delta^k_l \quad \text{and} \quad \mathfrak{B}'^{jk}_{\alpha} B^{\beta}_{jk} = n\delta^{\beta}_{\alpha},$$

that is, $\mathfrak{B}'$ is a pseudo-inverse of B. For the independence of the choice of the transversal bundle N, we use $\mathfrak{B}'$ instead of the pseudo-inverse $\mathfrak{B}^N \in \mathrm{HOM}(\mathrm{Hom}(T,N),T)$ of B^N by Wiehe. Then we have

Lemma 3.6. *If an immersion $f : M \to \tilde{M}$ is regular, then there exists a unique pseudo-inverse $\mathfrak{B}' \in \mathrm{HOM}(\mathrm{Hom}(T,\tilde{T}/T),T)$ which is independent of the choice of the transversal bundle N.*

Corollary 3.7. *If an immersion $f : M \to \tilde{M}$ is regular, then there exists a unique equiaffine $\gamma_{\omega(B)} \in \mathrm{INV}_L(i)$ which satisfies*

$$H^{\gamma_{\omega(B)}}_{\mathfrak{B}'} = -\frac{1}{r}\mathfrak{B}'^*(\nu_{\omega(B),\mathfrak{B}'}).$$

Note that if $\tilde{M} = \mathbb{R}^{n+q}$ and $\tilde{\nabla}\tilde{\omega} = 0$, then the equation of Corollary 3.5 reduces to

$$H^{\gamma_{\omega(B)}}_{\mathfrak{B}'} = -\frac{1}{r}\mathfrak{B}'^*(\nu_{\omega(B),\mathfrak{B}'}) = 0,$$

where the transversal bundle $\mathrm{Im}\hat{\gamma}_{\omega(B)}$ coincides with the transversal bundle given by Wiehe [15].

If $r = 1$, then $f : M \to \tilde{M}$ is regular if and only if f is non-degenerate, that is, the affine fundamental form is non-degenerate. From Lemma 3.1, we have

Corollary 3.8. *If an immersion $f : M \to \tilde{M}$ with $r = 1$ is non-degenerate, then there exists a unique pseudo-inverse $\mathfrak{B}'$ and the transversal bundle $\mathrm{Im}\hat{\gamma}_{\omega(B)}$ gives the Blaschke immersion.*

For an immersion $f : M \to \tilde{M}$ with $r = \frac{1}{2}n(n+1)$, we say that f is *non-degenerate* if $\tilde{B}$ is surjective (see [12]). If $r = \frac{1}{2}n(n+1)$, then $f : M \to \tilde{M}$ is regular if and only if f is non-degenerate. From Lemma 3.1, we have

Corollary 3.9. *If an immersion $f : M \to \tilde{M}$ with $r = \frac{1}{2}n(n+1)$ is non-degenerate, then there exist a unique pseudo-inverse $\mathfrak{B}'$ and a unique equiaffine $\gamma_{\omega(B)} \in \mathrm{INV}_L(i)$ which satisfies*

$$H^{\gamma_{\omega(B)}}_{\mathfrak{B}'} = -\frac{1}{r}\mathfrak{B}'^*(\nu_{\omega(B),\mathfrak{B}'}).$$

In particular, if $\tilde{M} = \mathbb{R}^{n+\frac{1}{2}n(n+1)}$, then $\mathrm{Im}\hat{\gamma}_{\omega(B)}$ coincides with the transversal bundle given by Weise [14], where the regularity condition coincides with that in Wiehe [15].

§4. Non-degenerate surfaces of codimension two

Hereafter we assume that $\dim M = 2$ and $\dim \tilde{M} = 4$. We shall generalize various notions on affine immersions into $\mathbb{R}^4$ to those into $\tilde{M}$. We first introduce the affine metric g given in Nomizu-Vrancken [11], for example. Take $\tilde{\omega} \in \mathfrak{V}(\tilde{T})$, $\omega_Q \in \mathfrak{V}(\tilde{T}/T)$, and let $\omega_T \in \mathfrak{V}(T)$ be the induced volume element from $\tilde{\omega}$ with respect to ω_Q. Let X_1, X_2 be a local unimodular frame field for T with respect to ω_T. Let G be a symmetric bilinear form on M defined by

$$G(Y,Z) := \frac{1}{2}\sum_{\sigma} \mathrm{sgn}\sigma(\omega_Q B)(X_{\sigma(1)}, Y, X_{\sigma(2)}, Z),$$

where σ is a permutation on $\{1,2\}$. We set

$$\det_{\omega_T} G := \det(G(X_j, X_k)).$$

For a local frame field X_1, X_2 for T and ξ_1, ξ_2 for $\tilde{T}/T$, we write $\hat{B}$ as $B^{\alpha}_{jk} X^j \otimes X^k \otimes \xi_\alpha \in \Gamma(T^* \circledcirc T^* \otimes (\tilde{T}/T))$.

Proposition 4.1. *It follows that*

$$\det_{\omega_T}(\omega_Q B) = 4\det_{\omega_T} G.$$

Proof. Let X_1, X_2 be a local unimodular frame field for T with respect to ω_T. We first compute $\det_{\omega_T} G$. By a straightforward calculation, we have

$$\begin{aligned}\det_{\omega_T} G = \omega_Q(\xi_1,\xi_2)^2(&B^1_{11}B^2_{21}B^1_{12}B^2_{22} - B^1_{11}B^2_{21}B^1_{22}B^2_{12} - B^1_{21}B^2_{11}B^1_{12}B^2_{22} \\ &+B^1_{21}B^2_{11}B^1_{22}B^2_{12} - \frac{1}{4}((B^1_{11}B^2_{22})^2 + (B^1_{22}B^2_{11})^2 - 2B^1_{11}B^2_{22}B^1_{22}B^2_{11})).\end{aligned}$$

Next, we compute $\det_{\omega_T}(\omega_Q B)$ by (3.2). Then we have

$$\begin{aligned}\det_{\omega_T}(\omega_Q B) = \frac{1}{2}(\omega_Q(\xi_1,\xi_2))^2(&-(B^1_{11}B^2_{12} - B^1_{12}B^2_{11})(B^1_{22}B^2_{21} - B^1_{21}B^2_{22}) \\ &-(B^1_{11}B^2_{21} - B^1_{21}B^2_{11})(B^1_{22}B^2_{12} - B^1_{12}B^2_{22}) \\ &+(B^1_{11}B^2_{22} - B^1_{22}B^2_{11})(B^1_{22}B^2_{11} - B^1_{11}B^2_{22}) \\ &-(B^1_{12}B^2_{11} - B^1_{11}B^2_{12})(B^1_{21}B^2_{22} - B^1_{22}B^2_{21}) \\ &-(B^1_{12}B^2_{22} - B^1_{22}B^2_{12})(B^1_{21}B^2_{11} - B^1_{11}B^2_{21}) \\ &-(B^1_{21}B^2_{11} - B^1_{11}B^2_{21})(B^1_{12}B^2_{22} - B^1_{22}B^2_{12}) \\ &-(B^1_{21}B^2_{22} - B^1_{22}B^2_{21})(B^1_{12}B^2_{11} - B^1_{11}B^2_{12}) \\ &+(B^1_{22}B^2_{11} - B^1_{11}B^2_{22})(B^1_{11}B^2_{22} - B^1_{22}B^2_{11}) \\ &-(B^1_{22}B^2_{12} - B^1_{12}B^2_{22})(B^1_{11}B^2_{21} - B^1_{21}B^2_{11}) \\ &-(B^1_{22}B^2_{21} - B^1_{21}B^2_{22})(B^1_{11}B^2_{12} - B^1_{12}B^2_{11})).\end{aligned}$$

Comparing $\det_{\omega_T} G$ with $\det_{\omega_T}(\omega_Q B)$, we have the assertion. □

If G is non-degenerate, we say that f is non-degenerate (see [11]). From Lemma 4.1, we have

Corollary 4.2. *The symmetric bilinear form G is non-degenerate if and only if the immersion $f : M \to \tilde{M}$ is regular.*

In the remainder of this section, we assume that the immersion $f : M \to \tilde{M}$ is non-degenerate. We define the *affine metric* g on M by

$$g(Y, Z) := G(Y, Z)/|\det_{\omega_T} G|^{\frac{1}{3}}.$$

It is clear that g is non-degenerate and

$$\omega_g := |\det_{\omega_T} g|^{\frac{1}{2}} \omega_T = |\det_{\omega_T} G|^{\frac{1}{6}} \omega_T = (\tfrac{1}{2})^{\frac{1}{3}} |\det_{\omega_T}(\omega_Q B)|^{\frac{1}{6}} \omega_T = (\tfrac{1}{2})^{\frac{1}{3}} \omega(B).$$

Note that g and ω_g depend only on B and the orientation given by ω_T.

From Lemma 3.6 and Proposition 4.1, there exists a pseudo-inverse $\mathfrak{B}'$ of B. But in this section, we shall show the uniqueness of the pseudo-inverse of the fundamental form and actually construct the pseudo-inverse by B. Let X_1, X_2 be a local orthonormal frame field for T with respect to g, that is,

$$\begin{aligned} g(X_1, X_1) &= \epsilon_1, \\ g(X_1, X_2) &= 0, \\ g(X_2, X_2) &= \epsilon_2, \end{aligned}$$

where $\epsilon_i = 1$ or -1 for $i = 1, 2$. The following lemma can be proved in the same way as in the proof of Theorem 4.1 in [11].

Lemma 4.3. *Let X_1, X_2 be a local orthonormal frame field for T with respect to g. Then there exists a unique local frame field ξ_1, ξ_2 for $\tilde{T}/T$ such that*

$$\begin{array}{ll} \omega_T(X_1, X_2)\omega_Q(\xi_1, \xi_2) = 1, & \\ B^1_{11} = 1, & B^2_{11} = 0, \\ B^1_{12} = 0, & B^2_{12} = \epsilon_2, \\ B^1_{22} = -\epsilon_1\epsilon_2, & B^2_{22} = 0. \end{array}$$

Let X_1, X_2 be a local orthonormal frame field for T with respect to g and ξ_1, ξ_2 a local frame field for $\tilde{T}/T$ that satisfies as in Lemma 4.3.

Lemma 4.4. *A pseudo-inverse of B is unique.*

Proof. Let $\mathfrak{B}', \mathfrak{B}$ be symmetric pseudo-inverses of B and set $\mathfrak{D} := \mathfrak{B} - \mathfrak{B}'$. Then $\mathfrak{D} \in \mathrm{HOM}(\mathrm{Hom}(T, \tilde{T}/T), T)$ satisfies $\mathfrak{D} \circ B = 0$ and $\tilde{B} \circ \tilde{\mathfrak{D}} = 0$. We compute $\mathfrak{D}$ by the frame field as in Lemma 4.3. Then we have $\mathfrak{D} = 0$. □

Let $\mathfrak{B} := \mathfrak{B}'$ be the pseudo-inverse of B. For X_1, X_2, ξ_1, ξ_2, we write $\hat{\mathfrak{B}}$ as $\mathfrak{B}^{jk}_\alpha X_j \otimes X_k \otimes \xi^\alpha \in \Gamma(T \odot T \otimes (\tilde{T}/T)^*)$. From Lemma 4.3, $\mathfrak{B}$ satisfies the following:

Lemma 4.5. *We have*

$$\begin{aligned} \mathfrak{B}_1^{11} &= 1, & \mathfrak{B}_2^{11} &= 0, \\ \mathfrak{B}_1^{12} &= 0, & \mathfrak{B}_2^{12} &= \epsilon_2, \\ \mathfrak{B}_1^{22} &= -\epsilon_1\epsilon_2, & \mathfrak{B}_2^{22} &= 0. \end{aligned}$$

From Theorem 3.2, for $\mathfrak{A} \in \mathrm{HOM}(\tilde{T}/T, T)$, there exists a unique $\gamma_{\mathfrak{A}} \in \mathrm{INV}_L(i)$. In order to give a canonical choice of $\mathfrak{A} \in \mathrm{HOM}(\tilde{T}/T, T)$, we shall construct $\mathfrak{A}$ from the fundamental form B as follows. We set $P(X, Y) := \tilde{\nabla}_X i(Y) - i(\nabla^g_X Y)$, where $\nabla^g \in \mathfrak{C}(T)$ is the Levi-Civita connection for g. Let $\mathrm{tr}_g P \in \Gamma(\tilde{T})$ be defined by

$$\mathrm{tr}_g P := \epsilon_1(\tilde{\nabla}_{X_1} X_1 - \nabla^g_{X_1} X_1) + \epsilon_2(\tilde{\nabla}_{X_2} X_2 - \nabla^g_{X_2} X_2).$$

From Lemma 4.3, we see that $\mathrm{tr}_g P \in \Gamma(T)$. We set

$$\nu_g := \textstyle\sum_i (\epsilon_i X^i(\mathrm{tr}_g P)) X^i \in A^1,$$

i.e., the metric dual of $\mathrm{tr}_g P$ and

$$\mu_g := X^1(\mathrm{tr}_g P) X^2 - X^2(\mathrm{tr}_g P) X^1 \in A^1.$$

Note that μ_g is independent of the choice of a positively oriented orthonormal frame field with respect to g and ω_T. Since the equation: $\nu_g \wedge \mu_g = g(\mathrm{tr}_g P, \mathrm{tr}_g P)\omega_g = \nu_g(\mathrm{tr}_g P)\omega_g$, we see that if $g(\mathrm{tr}_g P, \mathrm{tr}_g P) \neq 0$, then ν_g and μ_g are linearly independent. For $s, t \in \mathbb{R}$, we obtain the element $\mathfrak{B}^*(\nu_{\omega(B),\mathfrak{B}} + s\nu_g + t\mu_g)$ of $\mathrm{HOM}(\tilde{T}/T, T)$, which is given by B. Then we have

Theorem 4.6. *For $s, t \in \mathbb{R}$, there exists a unique $\gamma(s,t) \in \mathrm{INV}_L(i)$ which satisfies*

$$H_{\mathfrak{B}}^{\gamma(s,t)} = -\frac{1}{2}\mathfrak{B}^*(\nu_{\omega(B),\mathfrak{B}} + s\nu_g + t\mu_g).$$

In particular, $\gamma(0,0)$ is the equiaffine with respect to $\omega(B)$.

If $\tilde{M} = \mathbb{R}^4$, $\epsilon_1 = \epsilon_2 = 1$, and $\tilde{\nabla}\tilde{\omega} = 0$, then $\gamma(s,t)$ gives a family of transversal bundle $\mathrm{Im}\hat{\gamma}(s,t)$ which coincides with the family of transversal bundle given by Theorem 5.3 in [13], where the complex number c in [13] satisfies

$$c = -(6s - 1 + 6\sqrt{-1}t).$$

Let $\gamma_{BM} \in \mathrm{INV}_L(i)$ (resp. $\gamma_K, \gamma_{NV} \in \mathrm{INV}_L(i)$) give the transversal bundle $\mathrm{Im}\hat{\gamma}_{BM}$ (resp. $\mathrm{Im}\hat{\gamma}_K, \mathrm{Im}\hat{\gamma}_{NV}$) by Burstin-Mayer (resp. Klingenberg, Nomizu-Vrancken). We set

$$C(X, Y, Z) := (\hat{\nabla}^\gamma_X B)_Y Z = \nabla^{\hat{\gamma}}_X B_Y Z - B_{\gamma\nabla_X Y} Z - B_Y {}^\gamma\nabla_X Z,$$

and call C the *cubic form*. Then we obtain

Corollary 4.7. *If $\tilde{M} = \mathbb{R}^4$ and $\tilde{\nabla}\tilde{\omega} = 0$, then γ_{BM}, γ_K, and γ_{NV} satisfy the following*:

$$\gamma_{BM} = \gamma(\frac{1}{2}, 0),$$
$$\gamma_K \quad = \gamma(\frac{1}{6}, 0),$$
$$\gamma_{NV} = \gamma(0, 0).$$

Proof. From Theorem 4.6, we see that $\gamma_{NV} = \gamma(0,0)$. For $\lambda \in \mathrm{HOM}(\tilde{T}/T, T)$, we set $\lambda_\alpha^j := X^j(\lambda(\xi_\alpha))$. We take $\gamma \in \mathrm{INV}_L(i)$ and set $\lambda_{BM}p := \gamma_{BM} - \gamma, \lambda_K p := \gamma_K - \gamma$, and $\lambda_{NV}p := \gamma_{NV} - \gamma$. To obtain s, t of $\gamma(s,t)$ for γ_{BM} and γ_K, we express λ_{BM}, λ_K, and λ_{NV} by using the cubic form C. Since $\gamma_{NV} = \gamma + H^\gamma_{\mathfrak{B}}$, we obtain $\gamma_{NV} = H^\gamma_{\mathfrak{B}}$. We express γ_{NV} by C as follows. From (3.1), we have

$$(\hat{\nabla}^\gamma_X \mathfrak{B})B + \mathfrak{B} \circ \hat{\nabla}^\gamma_X B = 0 \;\text{ and }\; (\hat{\nabla}^\gamma_X \tilde{B})\tilde{\mathfrak{B}} + \tilde{B} \circ \hat{\nabla}^\gamma_X \tilde{\mathfrak{B}} = 0.$$

Since the ambient space is an affine space, we see that C is symmetric in all three variables. We write $\hat{\nabla}^\gamma_{X_j}\mathfrak{B}$ as $\mathfrak{B}^{kl}_{\alpha;j} X_k \otimes X_l \otimes \xi^\alpha$ and C as $C^\alpha_{jkl} X^j \circledcirc X^k \circledcirc X^l \otimes \xi_\alpha$. Then we have the following:

$$\begin{aligned}
\mathfrak{B}^{11}_{1;1} &= -C^1_{111},\\
\mathfrak{B}^{11}_{1;2} &= -C^1_{112},\\
\mathfrak{B}^{12}_{1;1} &= -\frac{\epsilon_2}{2}C^2_{111} + \frac{\epsilon_1}{2}C^2_{122},\\
\mathfrak{B}^{12}_{1;2} &= -\frac{\epsilon_2}{2}C^2_{112} + \frac{\epsilon_1}{2}C^2_{222},\\
\mathfrak{B}^{22}_{1;1} &= -C^1_{122},\\
\mathfrak{B}^{22}_{1;2} &= -C^1_{222},\\
\mathfrak{B}^{11}_{2;1} &= -\epsilon_2 C^1_{112} - \frac{\epsilon_1\epsilon_2}{2}C^2_{111} - \frac{1}{2}C^2_{122},\\
\mathfrak{B}^{11}_{2;2} &= -\epsilon_2 C^1_{122} - \frac{\epsilon_1\epsilon_2}{2}C^2_{112} - \frac{1}{2}C^2_{222},\\
\mathfrak{B}^{12}_{2;1} &= -C^2_{112},\\
\mathfrak{B}^{12}_{2;2} &= -C^2_{122},\\
\mathfrak{B}^{22}_{2;1} &= \epsilon_1 C^1_{112} - \frac{1}{2}C^2_{111} - \frac{\epsilon_1\epsilon_2}{2}C^2_{122},\\
\mathfrak{B}^{22}_{2;2} &= \epsilon_1 C^1_{122} - \frac{1}{2}C^2_{112} - \frac{\epsilon_1\epsilon_2}{2}C^2_{222}.
\end{aligned}$$

It follows that we have the following equations:

$$\begin{aligned}
\lambda_{NV}{}^1_1 &= \frac{1}{24}(5C^1_{111} + \epsilon_2 C^2_{112} - 3\epsilon_1 C^2_{222} + \epsilon_1\epsilon_2 C^1_{122}),\\
\lambda_{NV}{}^2_1 &= \frac{1}{24}(5C^1_{222} - \epsilon_1 C^2_{122} + 3\epsilon_2 C^2_{111} + \epsilon_1\epsilon_2 C^1_{112}),\\
\lambda_{NV}{}^1_2 &= \frac{1}{24}(5\epsilon_2 C^1_{112} + 3\epsilon_1\epsilon_2 C^2_{111} + 7C^2_{122} + \epsilon_1 C^1_{222}),\\
\lambda_{NV}{}^2_2 &= \frac{1}{24}(7C^2_{112} - 5\epsilon_1 C^1_{122} + 3\epsilon_1\epsilon_2 C^2_{222} - \epsilon_2 C^1_{111}).
\end{aligned}$$

We compute γ_{BM} by virtue of (5.2) in [11]. Then we have

$$\begin{aligned}
\lambda_{BM}{}_1^1 &= \frac{1}{12}(C_{111}^1 - \epsilon_2 C_{112}^2 - \epsilon_1\epsilon_2 C_{122}^1 - 3\epsilon_1 C_{222}^2),\\
\lambda_{BM}{}_1^2 &= \frac{1}{12}(3\epsilon_2 C_{111}^2 - \epsilon_1\epsilon_2 C_{112}^1 + \epsilon_1 C_{122}^2 + C_{222}^1),\\
\lambda_{BM}{}_2^1 &= \frac{1}{6}(2\epsilon_2 C_{112}^1 + C_{112}^2 + \epsilon_1 C_{222}^1),\\
\lambda_{BM}{}_2^2 &= \frac{1}{6}(-\epsilon_2 C_{111}^1 - 2\epsilon_1 C_{122}^1 + C_{112}^2).
\end{aligned}$$

We compute γ_K by virtue of Theorem 6.1 in [11]. Then we have

$$\begin{aligned}
\lambda_K{}_1^1 &= \frac{1}{6}(C_{111}^1 - \epsilon_1 C_{222}^2),\\
\lambda_K{}_1^2 &= \frac{1}{6}(\epsilon_2 C_{111}^2 + C_{222}^1),\\
\lambda_K{}_2^1 &= \frac{1}{12}(3C_{122}^2 + 3\epsilon_2 C_{112}^1 + \epsilon_1 C_{111}^2 + \epsilon_1\epsilon_2 C_{222}^1),\\
\lambda_K{}_2^2 &= \frac{1}{12}(3C_{112}^2 - 3\epsilon_1 C_{122}^1 - \epsilon_2 C_{111}^1 + \epsilon_1\epsilon_2 C_{222}^2).
\end{aligned}$$

Computing $\mathfrak{B}^*(\nu_g)$ and $\mathfrak{B}^*(\mu_g)$, we have the assertion. □

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