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Propagation of singularities for semilinear wave equation with nonlinearity satisfying null condition

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Abstract. We study the propagation of singularities for nonlinear wave equation $\Box u = F(u, Du)$. Our main result in this paper is Theorem 1.1, which is an extension of Theorem 2.7 in [2]. When the nonlinearity $F(u, Du)$ satisfies the null condition, we improve a condition with respect to regularity of solutions u.

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§1. Introduction

In this paper, we study the propagation of singularities for the following nonlinear wave equation,

$$\Box u = F(u, Du), \tag{1.1}$$

where

$$u = u(x), \quad x = (t, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}, \quad \Box \equiv \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}, \tag{1.2}$$

Du denotes the first partial derivatives of u and $F \in C^\infty$ satisfies the null condition which is defined in Definition 3.1. Typical example of $F(u, Du)$ satisfying the null condition is $f(u)\{(\partial_t u)^2 - |\nabla u|^2\}$. The general case is given in Remark 3.3.

In the case of the linear wave equation

$$\Box u = 0, \tag{1.3}$$

the wave front set of u is locally completely characterized as being invariant under the Hamiltonian flow, and hence it is easily described in terms of the wave front set of the initial data (Hörmander[5]). In particular, the singular support of u is contained in the union of the light cones over the singular support of the initial data.

In the case of nonlinear wave equations in one space dimension, Reed [11] showed that solutions to (1.1) are C^∞ except at the points (t, x) from which the backward characteristics intersect singular points of the initial data at $t = 0$. Therefore the singularities lie on rays issuing from singularities at $t = 0$ as in the linear case.

However this result is specific to the one-dimensional case. The analogous result is false for second order equations when the number of space dimension is greater than one. The counterexamples have been found by Rauch [9] and Rauch-Reed [10], which showed that when the number of space dimensions is greater than one, the solution u to $\Box u = f(u)$ may have other singularities.

Generally, in the case of nonlinear wave equation, its singular support may be larger than that is predicted by the linear case when the number of space dimensions is greater than one. These new singularities are weaker than the original singularities (Beals-Reed [1], Bony [3], Rauch [9]).

However, even in the case of nonlinear wave equations, a phenomenon similar to the linear case is observed when we consider low regularity. In [9], for $\Box u = f(u)$ with a polynomial f, Rauch proved that if $u \in H^s_{\text{loc}}(\mathbb{R}^n)$ for $s > n/2$ and no ray through (t, x) intersects the singular support of the initial data of u then $u \in H^{s+1+\sigma}_{\text{loc}}(t, x)$ for all $\sigma < s - n/2$. These analysis are based on a study of the microlocal regularity of products of distributions. Let $u \in H^s_{\text{loc}}(U) \cap H^r_{ml}(x_0, \xi_0)$ (Definition 2.1) be a solution to (1.1) with singularities on the initial hypersurface or in the past and (x_0, ξ_0) is a point in null bicharacteristic (Definition 2.3) of $\Box$. In [3], Bony showed that u is in H^r_{ml} at all points of null bicharacteristic of $\Box$ as long as $n/2+1 < s \leqq r < 2s-1-n/2$. Beals and Reed [1] gave another proof of this result by using a simple commutator lemma and Rauch's lemma. Beals [2] has shown that for the equation (1.1), u is in H^r_{ml} at all points of null bicharacteristic of $\Box$ as long as $n/2 + 1 < s \leqq r < 3s - n - 2$.

In other words, if r is so small that $s \leqq r < 3s - n - 2$, then microlocal Sobolev H^r_{ml} regularity propagates along null bicharacteristic as in the linear case. If r is sufficiently large, then new singularities are observed. We are interested in the threshold of r. Although numerous attempts have been made to study these analysis, the threshold of r has not been determined exactly. In this paper, we improve lower bound of the threshold in the case that the nonlinear term $F(u, Du)$ satisfies the null condition. The condition for s and r of Theorem 1.1 in this paper is weaker than that of Theorem 2.7 (Beals[2]) in §2, if F satisfies the null condition. We obtain the following theorem.

Theorem 1.1. *Suppose that* U *is a neighborhood of* $x_0 \in \mathbb{R}^n$, $F \in C^\infty$ *satisfies the null condition, and* $u \in H^s_{loc}(U)$, $s > n/2$, *satisfies* (1.1). *Let* Γ *be a null bicharacteristic for* $\Box$ *and suppose that* $u \in H^r_{ml}(x_0, \xi_0)$ *for some point* (x_0, ξ_0) *on* Γ, *then* $u \in H^r_{ml}(\widetilde{\Gamma})$ *for* $n/2 < s \leqq r \leqq 2s - n/2$ *where* $\widetilde{\Gamma}$ *is a connected component of* $\Gamma \cap (U \times \mathbb{R}^n \backslash \{0\})$ *and contains* (x_0, ξ_0).

Remark 1.2. *The definition of null condition, null bicharacteristic and microlocal Sobolev space* $H^r_{ml}(x_0, \xi_0)$ *are given in Definition* 3.1, *Definition* 2.3 *and Definition* 2.1, *respectively.*

Remark 1.3. *If* $F(u, Du)$ *satisfies the null condition, then*

$$F(u, Du) = f(u)\{(\partial_t u)^2 - |\nabla u|^2\} + g(u)\partial_t u + \sum_{i=1}^{n-1} g_i(u)\partial_{x_i} u + h(u).$$

(Proposition 3.2*)*

So we can interpret $F(u, Du)$ *for* $u \in H^s_{loc}(U)$ $(s > n/2)$.

§2. Microlocal analysis

First we give some notation with respect to microlocal analysis. Secondly we introduce the precedence result of microlocal propagation of singularities.

Definition 2.1. *We say that a subset* K *of* $\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \backslash \{0\})$ *is a conic set if* $(x, \xi) \in K$ *implies that* $(x, t\xi) \in K$ *for any* $t > 0$. *Suppose that* U *is a neighborhood of* x_0. $u \in H^s_{loc}(U)$ *means that* $\langle\xi\rangle^s |\widehat{\psi u}(\xi)| \in L^2(\mathbb{R}^n)$ *for all* ψ *in* C_0^∞ *with support in* U. $u \in H^r_{ml}(x_0, \xi_0)$ *means that there exists* $\phi(x) \in C_0^\infty$ *with* $\phi(x_0) = 1$ *and a conic neighborhood* K *of* ξ_0 *in* $\mathbb{R}^n \backslash \{0\}$ *such that*

$$\langle\xi\rangle^r \chi_K(\xi) |\widehat{\phi u}(\xi)| \in L^2(\mathbb{R}^n), \tag{2.1}$$

where χ_K *is the characteristic function of* K *and* $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. *If* Γ *is a closed conic set in* $\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \backslash \{0\})$, *we say that* $u \in H^s_{loc}(U) \cap H^r_{ml}(\Gamma)$ *if* $u \in H^s_{loc}(U)$ *and* $u \in H^r_{ml}(x, \xi)$ *for all* $(x, \xi) \in \Gamma$.

As is easily verified from the definitions and the symbolic calculus, $u \in H^r_{ml}(x_0, \xi_0)$ if and only if there is a classical pseudodifferential operator of order zero with symbol $a(x, \xi)$ microlocally elliptic at (x_0, ξ_0) such that $a(x, D)u(x) \in H^r_{\mathrm{loc}}(\mathbb{R}^n)$. This functional space satisfies the following property. This property is one of the key to solve Theorem 1.1.

Lemma 2.2. *Suppose that* U *is a neighborhood of* x_0. *If* $u \in H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$, $n/2 < s \leqq r \leqq 2s - n/2$, *and* $f \in C^\infty$, *then* $f(x, u) \in H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$.

The first proof of such result was given in Rauch [9]. Rauch proved that $H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$ is an algebra for $n/2 < s \leqq r < 2s - n/2$. Afterward Bony [3] established that $H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$ is preserved for $n/2 < s \leqq r < 2s - n/2$ under the action of smooth functions $f(u)$ by introducing the para product of nonsmooth functions. Moreover Meyer [7] extended this property to $n/2 < s \leqq r \leqq 2s - n/2$.

Next we give a brief explanation for propagation of singularities. In the linear case, it is known that the regularity of microlocal Sobolev space propagates along some integral curve which is called null bicharacteristic, which is defined as follows.

Definition 2.3. *Let $p(x, \xi)$ is a characteristic polynomial of differential operator P. The curves $x(s)$, $\xi(s)$ are bicharacteristics if*

$$\frac{dx_j}{ds} = \frac{\partial p}{\partial \xi_j}(x(s),\ \xi(s)), \quad \frac{d\xi_j}{ds} = -\frac{\partial p}{\partial x_j}(x(s),\ \xi(s)), \quad (j = 1, \ \cdots, \ n). \tag{2.2}$$

Since $\sum_{j=1}^{n}\Big(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big)p = 0,$ *we see that p is constant on each of these curves; one on which p vanishes is called a null bicharacteristic of p.*

Example 2.4. *We consider the null bicharacteristic of $\Box$, with symbol $\tau^2 - |\xi|^2$. Simple calculation shows that the null bicharacteristic through the point $(0, x_0, \tau_0, \xi_0)$ with $|\tau_0| = \pm|\xi_0| \neq 0$ is the straight line*

$$\Gamma = \{(t, x, \tau_0, \xi_0) \ : \ x = x_0 - (\xi_0/\tau_0)t\}. \tag{2.3}$$

In [2] Beals proved that following theorems for propagation of singularities in the sense of microlocal Sobolev spaces.

Theorem 2.5 (Rauch[9], Beals[2])**.** *Suppose that U is a neighborhood of $x_0 \in \mathbb{R}^n$, $f \in C^\infty$, and that $u \in H^s_{loc}(U)$ with $s > n/2$ satisfies*

$$\Box u = f(u). \tag{2.4}$$

Let Γ be a null bicharacteristic for $\Box$ and suppose that $u \in H^r_{ml}(x_0, \xi_0)$ for some point (x_0, ξ_0) on Γ. Then $u \in H^r_{ml}(\widetilde{\Gamma})$ as long as $r < 3s - n + 1$ where $\widetilde{\Gamma}$ is a connected component of $\Gamma \cap (U \times \mathbf{R}^n \backslash \{0\})$ and contains (x_0, ξ_0).

This is proved by a bootstrap argument with Hörmander's propagation of singularities theorem for the linear operator $\Box$ and Lemma 2.2. Moreover in [2] Beals proved the following theorem.

Theorem 2.6 (Beals[2]). *Suppose that U is a neighborhood of $x_0 \in \mathbb{R}^n$ and $f,\ g_\alpha \in C^\infty$, and that $u \in H^s_{loc}(U)$ with $s > n/2$ satisfies*

$$\Box u = f(u) + \sum_{|\alpha|=1} g_\alpha(u) D^\alpha u. \tag{2.5}$$

Let Γ be a null bicharacteristic for $\Box$ and suppose that $u \in H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$ for some point (x_0, ξ_0) on Γ. Then $u \in H^s_{loc}(U) \cap H^r_{ml}(\widetilde{\Gamma})$ for $r < 3s - n$ where $\widetilde{\Gamma}$ is a connected component of $\Gamma \cap (U \times \mathbf{R}^n \backslash \{0\})$ and contains (x_0, ξ_0).

Theorem 2.7 (Beals[2]). *Suppose that U is a neighborhood of $x_0 \in \mathbb{R}^n$, $f \in C^\infty$, and that $u \in H^s_{loc}(U)$ with $s > n/2 + 1$ satisfies*

$$\Box u = f(u, Du). \tag{2.6}$$

Let Γ be a null bicharacteristic for $\Box$ and suppose that $u \in H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$ for some point (x_0, ξ_0) on Γ. Then $u \in H^s_{loc}(U) \cap H^r_{ml}(\widetilde{\Gamma})$ for $r < 3s - n - 2$ where $\widetilde{\Gamma}$ is a connected component of $\Gamma \cap (U \times \mathbf{R}^n \backslash \{0\})$ and contains (x_0, ξ_0).

In Section 3, we give an improvement of Theorem 2.7 with respect to the conditions on s and r for the equation (1.1) under the null condition.

§3. Proof of Theorem 1.1

First we give the following notion of the null condition defined by Klainerman [6]. Klainerman introduced the null condition as a sufficient condition for a global existence of smooth solutions to $\Box u = F(u, u', u'')$.

Definition 3.1. *Let $F(u, v, w)$ a real valued function in the variables*

$$(u, v, w) = (u, v_1, \cdots, v_n, w_{1,1}, \cdots, w_{i,j}, \cdots, w_{n,n})$$

with $i \leqq j$ running from 1 to n, smoothly defined in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\frac{n^2+n}{2}}$. We say that $F(u, Du, D^2u)$ (where $Du,\ D^2u$ denote the first and second partial derivatives of u) satisfies the null condition if, for any $u,\ v,\ w$ and any vector $X = (X_1, \cdots,\ X_n)$ such that $X_1^2 - \sum_{i=2}^n X_i^2 = 0$, the following identities hold

$$\sum_{i,j=1}^n \frac{\partial^2 F}{\partial v_i \partial v_j} X_i X_j = 0 \tag{3.1}$$

$$\sum_{\substack{i,j,k=1 \\ j \leqq k}}^n \frac{\partial^2 F}{\partial v_i \partial w_{j,k}} X_i X_j X_k = 0 \tag{3.2}$$

$$\sum_{\substack{i,j,k,l=1\\ i\leqq j,k\leqq l}}^{n} \frac{\partial^2 F}{\partial w_{i,j}\partial w_{k,l}} X_i X_j X_k X_l = 0. \tag{3.3}$$

As a equivalent condition to null condition, the following important proposition holds for C^∞ function F with no second order derivative terms which satisfies the null condition.

Proposition 3.2. *Suppose that $F(u,v)$ is a C^∞ function with $(u,v) = (u, v_1, \cdots, v_n)$. Then $F(u,v)$ satisfies the null condition if and only if there are some C^∞ functions f, g, g_i and h such that*

$$F(u,v) = f(u)\left(v_1^2 - \sum_{i=2}^{n} v_i^2\right) + \sum_{i=1}^{n} g_i(u)v_i + h(u). \tag{3.4}$$

Proof. By assumption the following identity holds for all u, v and all vector $X = (X_1, \cdots, X_n)$ with $X_1^2 - \sum_{i=2}^{n} X_i^2 = 0$,

$$\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial v_i \partial v_j} X_i X_j = 0. \tag{3.5}$$

If we set for $t \in \mathbb{R}$

$$X_1 = \pm t, \ X_a = t \ \text{ and } \ X_i = 0 \ (i = 2, \ \cdots, \ n \ \text{ and } \ i \neq a),$$

then by (3.5) we have

$$\frac{\partial^2 F}{\partial v_1^2} = -\frac{\partial^2 F}{\partial v_i^2} \quad \text{and} \quad \frac{\partial^2 F}{\partial v_1 \partial v_i} = 0 \ \ (i = 2, \ 3, \ \cdots, \ n). \tag{3.6}$$

Moreover we set for $t, \ s \in \mathbb{R}$

$$X_1 = \pm\sqrt{t^2 + s^2}, \quad X_a = t, \quad X_b = s,$$
$$X_i = 0 \ \ (i = 2, \ \cdots, \ n \ \text{ and } \ i \neq a, b),$$

then by (3.5) and (3.6) we have

$$\frac{\partial^2 F}{\partial v_i \partial v_j} = 0 \ \ (i, \ j = 1, \ 2, \ \cdots, \ n \ \text{ and } \ i \neq j). \tag{3.7}$$

Therefore the result follows from (3.6) and (3.7) immediately. ■

Remark 3.3. *Suppose that F is in C^∞, $u = u(t,x)$ and $(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}$. $F(u, Du) = F(u, \partial_t u, \partial_{x_1} u, \cdots, \partial_{x_{n-1}} u)$ satisfies the null condition if and only if there is some function $f, \ g, \ g_i$ and $h \ \in C^\infty$ such that*

$$F(u, Du) = f(u)\{(\partial_t u)^2 - |\nabla u|^2\} + g(u)\partial_t u + \sum_{i=1}^{n-1} g_i(u)\partial_{x_i} u + h(u). \tag{3.8}$$

Proof of Theorem 1.1. Let

$$\exp\Big[-\int_0^u f(\xi)d\xi\Big] = G'(u) \quad \text{and} \quad v = G(u). \tag{3.9}$$

Then the facts $G''(u) = -f(u)G'(u)$ and $\Box v = G''(u)\{(\partial_t u)^2 - |\nabla u|^2\} + G'(u)\Box u$ with (1.1) and (3.8) imply

$$\Box v = G'(u)\Big(g(u)\partial_t u + \sum_{j=1}^{n-1} g_j(u)\partial_{x_j} u + h(u)\Big), \tag{3.10}$$

where g, g_j and h is C^∞. Since $G \in C^\infty$ and $G'(u(x_0)) \neq 0$, by the inverse mapping theorem, there exists some function $\widetilde{G}$ such that $u = \widetilde{G}(v)$ in the neighborhood of x_0. Therefore we can rewrite the equation (3.10) as the following form

$$\Box v = A(v) + \sum_{|\alpha|=1} B_\alpha(v) D^\alpha v \tag{3.11}$$

where A and B_α are in C^∞. By Lemma 2.2 and (3.9), v is in $H^s_{loc}(U) \cap H^r_{ml}(x_0, \xi_0)$ for $n/2 < s \leqq r \leqq 2s - n/2$. Moreover by Theorem 2.6, v is in $H^s_{loc}(U) \cap H^r_{ml}(\widetilde{\Gamma})$ for $n/2 < s \leqq r \leqq 2s - n/2$. Similarly by Lemma 2.2, u is in $H^s_{loc}(U) \cap H^r_{ml}(\Gamma)$ for $n/2 < s \leqq r \leqq 2s - n/2$. Therefore we have the conclusion of Theorem 1.1. ■

Remark 3.4. *When $n/2 < s \leqq n/2 + 2$, this theorem is better than Theorem* 2.7 *with respect to the conditions on s and r.*

Remark 3.5. *Let $u \in H^s(U) \cap H^r_{ml}(x_0, \xi_0)$ with $n/2 < s \leqq r \leqq 2s - n/2$ satisfies*

$$\Box u = f(u)\{(\partial_t u)^2 - |\nabla u|^2\}, \tag{3.12}$$

where f is C^∞. Let Γ be a null bicharacteristic for $\Box$ through (x_0, ξ_0). In this case, v defined in (3.9) *satisfies $\Box v = 0$ so our problem is reduced to the linear case with respect to v. By using Hörmander's Theorem of propagation of singularities in the linear case, $v \in H^s_{loc} \cap H^r_{ml}(x_0, \xi_0)$ implies $v \in H^s_{loc} \cap H^r_{ml}(\Gamma)$. However we can apply Lemma* 2.2 *only if $n/2 < s \leqq r \leqq 2s - n/2$. Therefore the condition on s and r is needed.*

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