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On focal points of submanifolds in symmetric spaces

Naoyuki Koike

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Abstract. The first purpose of this paper is to give a smart proof of the Morse index theorem for squared distance function of submanifolds in s a symmetric space. The second purpose is to classify focal points into strong ones and weak ones and to give a class of submanifolds in symmetric spaces all of whose focal points (other than conjugate points) are strong ones. The third purpose is to construct examples of submanifolds in symmetric spaces admitting weak ones.

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§1. Introduction

Let M be an n-dimensional immersed submanifold in an m-dimensional complete Riemannian manifold N and p be a point of N. Denote by $P(N, M\times p)$ the set of all H^1-paths $\gamma : [0,1] \to N$ with $(\gamma(0), \gamma(1)) \in M \times \{p\}$. For the energy functional $E : P(N, M \times p) \to \mathbf{R}$ ($E(\gamma) = \int_0^1 ||\dot\gamma(t)||^2 dt$), the following facts (i) $\sim$ (iii) hold:

(i) A path $\gamma\,(\in P(N, M \times p))$ is a critical point of E if and only if γ is a geodesic normal to M at $\gamma(0)$ parametrized by an affine parameter,

(ii) If p is not a focal point of M, then E is a Morse function,

(iii) The index of a critical point γ of E is equal to the number (counting the multiplicities) of focal points of $(M, \gamma(0))$ lying in $\gamma((0,1))$ (the Morse index theorem).

See Page 132$\sim$134 of [S] about the proof of the Morse index theorem (iii). In similar to the above facts (i) and (ii), the following facts (i′) and (ii′) hold for the squared distance function $d_p^2 : M \to \mathbf{R}$ ($d_p^2(x) = d(p,x)^2$ (d : the distance function of N)):

(i′) Let $x \in M \setminus C_p$, where C_p is the cut locus of p. The point x is a critical point of d_p^2 if and only if $\overrightarrow{xp}$ is normal to M, where $\overrightarrow{xp}$ is the initial velocity vector of the minimal geodesic γ_{xp} with $\gamma_{xp}(0) = x$ and $\gamma_{xp}(1) = p$,

(ii′) If p is not a focal point of M, then each critical point of d_p^2 which does not belong to C_p is non-degenerate.

The first purpose of this paper is to prove smartly the following fact similar to the Morse index theorem (iii) in the case where the ambient space is a symmetric space.

Theorem A. *Let M be a submanifold in a symmetric space N and p be a point of N. The index of a critical point x of $d_p^2 : M \to \mathbf{R}$ with $x \notin C_p$ is equal to the number (counting multiplicities) of focal points of (M, x) lying in $\overline{xp} \setminus \{p\}$, where $\overline{xp} := \gamma_{xp}([0,1])$.*

Remark 1. This fact has already shown by K. Nomizu and L. Rodriguez [NR] in the case where the ambient space is the Euclidean space.

We consider a family of normal geodesics of M whose initial velocity vectors are parallel with respect to the normal connection of M and define a notion of a strong focal point as a point where such a family of normal geodesics focus (at 1-jet level). Also, we call non-strong focal points (other than conjugate points) weak focal points. In the case where the ambient manifold N is of constant curvature, all focal points (other than conjugate points) become strong focal points. However, this fact does not hold for a general symmetric space. The second purpose of this paper is to show the following fact.

Theorem B. *If M is a submanifold with root decomposable normal bundle in a symmetric space N, then all focal points (other than conjugate points) of M are strong ones.*

Remark 2. The following submanifolds have root decomposable normal bundle:

(i) (General) submanifolds in real space forms,

(ii) Complex submanifolds in complex space forms,

(iii) Generic submanifolds in complex space forms, where a generic submanifold implies a submanifold M satisfying $J(T^{\perp}M) \subset TM$ (J : the complex structure of the complex space form),

(iv) Submanifolds with abelian normal bundle of the sense of [TT] in an arbitrary symmetric space, where we note that all hypersurfaces in an arbitrary symmetric space have abelian normal bundle.

The third purpose of this paper is to construct examples of submanifolds in symmetric spaces admitting weak focal points (see §4).

In §2, we prepare the basic notions and facts. In §3, we prove Theorems A and B. In §4, we give examples of submanifolds admitting weak focal points.

Throughout this paper, unless otherwise mentioned, we assume that all geometric objects are of class C^∞ and that all manifolds are connected ones without boundary.

§2. Basic notions and facts

In this secton, we recall the basic notions and facts. Let $N = G/K$ be a symmetric space of compact type or non-compact type, $(\mathfrak{g}, \sigma)$ be its orthogonal symmetric Lie algebra and $\mathfrak{p}$ be the eigenspace of σ for -1. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$ of N at eK, where e is the identity element of G. Let $\mathfrak{h}$ be a maximal abelian subspace of $\mathfrak{p}$. For each linear function α on $\mathfrak{h}$, we set $\mathfrak{p}_\alpha := \{X \in \mathfrak{p} \,|\, \mathrm{ad}(H)^2 X = \varepsilon\alpha(H)^2 X \text{ for all } H \in \mathfrak{h}\}$, where ad is the adjoint representation of $\mathfrak{g}$ and $\varepsilon = -1$ (resp. $\varepsilon = 1$) in the case where N is of compact type (resp. of non-compact type). If $\mathfrak{p}_\alpha \neq \{0\}$, then the function α is called a (*restricted*) *root for* $\mathfrak{h}$ and $\mathfrak{p}_\alpha$ is called the *root space for* α. Also, we call each element of $\mathfrak{p}_\alpha$ a *root vector for* α. For $w \in T_{gK}N$, we define linear transformations D^{co}_w and D^{si}_w of $T_{gK}N$ by

$$D^{co}_w = g_* \circ \cos(\sqrt{-1}\mathrm{ad}(g_*^{-1}w)) \circ g_*^{-1},$$
$$D^{si}_w = g_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}(g_*^{-1}w))}{\sqrt{-1}\mathrm{ad}(g_*^{-1}w)} \circ g_*^{-1},$$

respectively, where g_* is the differential of g. Also, we define a linear transformation D^{ct}_w by $D^{ct}_w := (D^{si}_w)^{-1} \circ D^{co}_w$ when $(D^{si}_w)^{-1}$ exists. A Jacobi field J along a geodesic γ in N is described as

$$J(s) = P_{\gamma|_{[0,s]}}(D^{co}_{s\dot\gamma(0)}J(0) + s \cdot D^{si}_{s\dot\gamma(0)}J'(0)), \tag{2.1}$$

where $P_{\gamma|_{[0,s]}}$ is the parallel translation along $\gamma|_{[0,s]}$. Let M be an immersed submanifold in N and A be its shape tensor. We omit the notation of the immersion. Let $p \in N$ and x be a critical point of d_p^2 with $x \neq C_p$. By imitating the proof of Lemma 3.1 in [K2], we can show that the Hessian $(\mathrm{Hess}\, d_p^2)_x$ of the squared distance function d_p^2 at x is given by

$$(\mathrm{Hess}\, d_p^2)_x(X, Y) = 2\langle (D^{ct}_{\overrightarrow{xp}} - A_{\overrightarrow{xp}})X, Y\rangle \quad (X, Y \in T_xM). \tag{2.2}$$

If, for each $\xi(\neq 0) \in T^\perp M$, there exists a maximal abelian subspace $\mathfrak{h}$ in $\mathfrak{p}$ containing $g_*^{-1}\xi$ such that $g_*^{-1}(T^\perp_x M) = \mathfrak{h} \cap g_*^{-1}(T^\perp_x M) + \sum_{\alpha \in \triangle_+}(\mathfrak{p}_\alpha \cap g_*^{-1}(T^\perp_x M))$ (x : the base point of ξ, $x = gK$), then M is said to *have root decomposable normal bundle*, where $\triangle_+$ is the positive root system with respect to $\mathfrak{h}$ (under some lexicographical ordering of $\mathfrak{h}$). Note that M has root decomposable normal bundle if and only if, for each normal vector ξ of M, the operator $R(\cdot, \xi)\xi$ leaves T_xM invariant (x : the base point of ξ), where R is the curvature tensor of N.

At the end of this section, we define a new notion of a strong focal point. First we recall the notion of a focal point. Let M be an immersed submanifold in a general Riemannian manifold N. Denote by $\exp^\perp$ the normal exponential map of M. Let $\xi \in U^\perp M$ and γ_ξ be the non-extendable geodesic in N with $\dot\gamma_\xi(0) = \xi$. If $\mathrm{Ker}\,(\exp^\perp_*)_\xi \neq \{0\}$, then the point $\exp^\perp(\xi)$ (resp. $||\xi||$) is called a *focal point* (resp. a *focal radius*) *of* $(M, \pi(\xi))$ *along* γ_ξ, where π is the bundle projection of the normal bundle $T^\perp M$. Also, $\dim \mathrm{Ker}(\exp^\perp_*)_\xi$ is called the *multiplicity* of the focal point. Now we shall define the notion of a strong focal point. Let $H_\xi\,(\subset T_\xi(T^\perp M))$ be the horizontal space at ξ with respect to the normal connection. If $\mathrm{Ker}(\exp^\perp_*)_\xi \cap H_\xi \neq \{0\}$, then we call the point $\exp^\perp(\xi)$ (resp. $||\xi||$) a *strong focal point* (resp. a *strong focal radius*) *of* $(M, \pi(\xi))$ *along* γ_ξ. Also, we call $\dim(\mathrm{Ker}(\exp^\perp_*) \cap H_\xi)$ the *multiplicity* of the strong focal point. If p is a non-strong focal point (other than conjugate points) along γ_ξ, then we call p a *weak focal point along* γ_ξ. A strong focal point is catched as a point where the normal geodesics whose initial vectors are parallel with respect to the normal connection focus (at 1-jet level). This is a geometrical meaning of a strong focal point. We think that the parallelism condition of the initial vectors is a geometrically essential condition. Hence we think that it is important to investigate the strongness of a focal point.

§3. Proofs of Theorems A and B

In this section, we prove Theorem A smartly.

Proof of Theorem A. Let x be a critical point of d_p^2 with $x \notin C_p$ and k_1 be the index of the critical point x. Also, let k_2 be the number (counting the multiplicities) of focal points of (M, x) lying in $\overline{xp} \setminus \{p\}$. We must show $k_1 = k_2$. Set $Q(s) := \mathrm{pr}_{T_x} \circ D^{ct}_{s\overrightarrow{xp}} - sA_{\overrightarrow{xp}} \circ \mathrm{pr}_{T_x}$ $(0 \le s \le 1)$, where $D^{ct}_{s\overrightarrow{xp}}$ is as in §2 and pr_{T_x} is the orthogonal projection of T_xN onto T_xM. According to (2.1), a Jacobi field J along $\gamma_{\overrightarrow{xp}}$ with $J(0) = X (\neq 0) (\in T_xM)$ is described as

$$
\begin{aligned}
J(s) &= P_{\gamma_{\overrightarrow{xp}}|_{[0,s]}}(D^{co}_{s\overrightarrow{xp}}X + sD^{si}_{s\overrightarrow{xp}}J'(0)) \\
&= P_{\gamma_{\overrightarrow{xp}}|_{[0,s]}}D^{si}_{s\overrightarrow{xp}}(D^{ct}_{s\overrightarrow{xp}}X - A_{s\overrightarrow{xp}}X + sJ'(0)_\perp) \\
&= P_{\gamma_{\overrightarrow{xp}}|_{[0,s]}}D^{si}_{s\overrightarrow{xp}}(Q(s)X + (D^{ct}_{s\overrightarrow{xp}}X)_\perp + sJ'(0)_\perp),
\end{aligned}
$$

where $(\cdot)_\perp$ is the normal component of $\cdot$. Hence $J(s_0) = 0$ if and only if $Q(s_0)X = 0$ and $(D^{ct}_{s_0\overrightarrow{xp}}X)_\perp + s_0J'(0)_\perp = 0$, where $0 < s_0 < 1$. Also, for each $X (\neq 0) (\in T_xM)$ and each $s_0 \in (0,1)$, there exists a unique Jacobi field J along $\gamma_{\overrightarrow{xp}}$ with $J(0) = X$ and $(D^{ct}_{s_0\overrightarrow{xp}}X)_\perp + s_0J'(0)_\perp = 0$. After all we see that $\gamma_{\overrightarrow{xp}}(s_0)$ is a focal point with multiplicity ν along $\gamma_{\overrightarrow{xp}}$ if and only if $\dim \mathrm{Ker}Q(s_0) = \nu$. This fact deduces $k_2 = \sum_{0<s<1} \dim \mathrm{Ker}Q(s)$. Next we shall show $k_1 = \sum_{0<s<1} \dim \mathrm{Ker}Q(s)$. Set $g_s := \langle \frac{1}{s}Q(s)\cdot, \cdot\rangle$ $(s \in (0,1])$ and

$F_X(s) := g_s(X, X)$. Since $F_X(s) = \langle(\frac{1}{s}D^{ct}_{s\overrightarrow{xp}} - A_{\overrightarrow{xp}})X, X\rangle$ $(s \in (0, 1])$, we have

$$\frac{dF_X}{ds} = -\langle\frac{1}{s^2}(D^{si}_{s\overrightarrow{xp}})^{-2}X, X\rangle < 0.$$

Thus the function F_X is decreasing over $(0, 1]$ for each $X(\neq 0) \in T_xM$. Since $Q(0)$ is the identity transformation of T_xM, there exists a positive number ε such that g_s is positive definite for every $s \in (0, \varepsilon)$. On the other hand, since $\mathrm{Hess}_x d_p^2 = 2g_1$ by (2.2), the index of g_1 is equal to k_1. From these facts, we see that $\sum\limits_{0<s<1} \dim \mathrm{Ker} Q(s) = k_1$. After all we can obtain $k_1 = k_2$. □

Next we prove Theorem B.

Proof of Theorem B. Let M be a submanifold with root decomposable normal bundle in N. Let p be a focal point of (M, x) other than a conjugate point. That is, there exists a Jacobi field J along $\gamma_{\overrightarrow{xp}}$ with $J(0)(\neq 0) \in T_xM$ and $J(1) = 0$. According to (2.1), a Jacobi field J along $\gamma_{\overrightarrow{xp}}$ is described as

$$J(s) = P_{\gamma_{\overrightarrow{xp}}|_{[0,s]}} D^{si}_{s\overrightarrow{xp}}(D^{ct}_{s\overrightarrow{xp}}J(0) - A_{s\overrightarrow{xp}}J(0) + sJ'(0)_\perp).$$

Since M has root decomposable normal bundle, we have $D^{ct}_{s\overrightarrow{xp}}J(0) \in T_xM$. Hence it follows from $J(1) = 0$ that $J'(0)_\perp = 0$. Let $\delta : [0, 1] \times (-\varepsilon, \varepsilon) \to M$ be a normal geodesic variation of $\gamma_{\overrightarrow{xp}}$ having J as the variation vector field, where ε is a positive number. Then we have $\nabla_{J(0)}\frac{\partial}{\partial s} = \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}|_{(0,0)} = \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}|_{(0,0)} = J'(0)$, where t is the second parameter of δ and ∇ is the Levi-Civita connection of N. It follows from $J'(0)_\perp = 0$ that $\nabla^\perp_{J(0)}\frac{\partial}{\partial s} = 0$. Define a curve $\alpha : (-\varepsilon, \varepsilon) \to T^\perp M$ by $\alpha(t) = \frac{\partial}{\partial s}|_{(0,t)}$ $(t \in (-\varepsilon, \varepsilon))$. The relation $\nabla^\perp_{J(0)}\frac{\partial}{\partial s} = 0$ implies $\dot\alpha(0) \in H_{\overrightarrow{xp}}$, where $\dot\alpha(0)$ is the velocity vector of α at $t = 0$. Note that $\dot\alpha(0) \neq 0$ because of $\pi_*\dot\alpha(0) = J(0) \neq 0$, where π is the bundle projection of $T^\perp M$. Also, we have $(\exp^\perp_*)_{\overrightarrow{xp}}(\dot\alpha(0)) = J(1) = 0$. Hence we have $\dot\alpha(0) \in \mathrm{Ker}\,(\exp^\perp_*)_{\overrightarrow{xp}} \cap H_{\overrightarrow{xp}}$. This implies that p is a strong focal point of (M, x). □

§4. Examples of submanifolds admitting weak focal points

In this section, we give some examples of submanifolds in a symmetric space admitting weak focal points. Those examples show that the assumption that the submanifold has root decomposable normal bundle is indispensable in Theorem B. Let S be a geodesic sphere in a symmetric space G/K of compact type or non-compact type such that its radius is smaller than the injective radius of G/K. Denote by p_0 its center. Take $x_0 = g_0K \in S$ such that $\alpha(g_{0*}^{-1}\overrightarrow{x_0p_0})$'s ($\alpha \in \triangle_+ \cup \{0\}$) are mutually distinct, where $\triangle_+$ is the positive (restricted) root system with respect to a maximal abelian subspace $\mathfrak{h}$ containing $g_{0*}^{-1}\overrightarrow{x_0p_0}$ (under some lexicographic ordering of $\mathfrak{h}$). It is clear that such a point x_0 exists. Let $\{e_i^\alpha \,|\, i = 1, \cdots, m_\alpha\}$ be a base of the root space $\mathfrak{p}_\alpha$, where $\alpha \in \triangle_+ \cup \{0\}$. Note that $\mathfrak{p}_0 = \mathfrak{h}$. Take a linearly independent system $\{X_k := e_{i_k}^{\alpha_k} + e_{j_k}^{\beta_k} \,|\, k = 1, \cdots, n\}$ of $\mathfrak{p} = T_{eK}(G/K)$ satisfying

($\natural$) $X_k (k = 1, \cdots, n)$ are orthogonal to $g_{0*}^{-1}\overrightarrow{x_0p_0}$, $\alpha_k \neq \beta_k (k = 1, \cdots, n)$ and $\{e_{i_k}^{\alpha_k}, e_{j_k}^{\beta_k}\}$'s $(k = 1, \cdots, n)$ are pairwise disjoint.

Proposition 1. *Let M be a submanifold in S through x_0 satisfying $T_{x_0}M = \mathrm{Span}\{g_{0*}X_1, \cdots, g_{0*}X_n\}$. Then the point p_0 is a weak focal point of M along the normal geodesic $\gamma_{\overrightarrow{x_0p_0}}$ and there does not exist a strong focal point of M along the normal geodesic $\gamma_{\overrightarrow{x_0p_0}}$.*

Proof. It is clear that p_0 is a focal point of M along $\gamma_{\overrightarrow{x_0p_0}}$. We show that there does not exist a strong focal point of M along $\gamma_{\overrightarrow{x_0p_0}}$. Suppose that $\gamma_{\overrightarrow{x_0p_0}}(s_0)$ is a strong focal point along $\gamma_{\overrightarrow{x_0p_0}}$. Then there exists a Jacobi field J along $\gamma_{\overrightarrow{x_0p_0}}$ satisfying $J(0) \neq 0$ ($\in T_{x_0}M$), $J'(0) = -A_{\overrightarrow{x_0p_0}}J(0)$ and $J(s_0) = 0$, where A is the shape tensor of M. For simplicity, set $X := J(0)$. From (2.1), the Jacobi field J is described as

$$J(s) = P_{\gamma_{\overrightarrow{x_0p_0}}|_{[0,s]}}(D^{co}_{s\overrightarrow{x_0p_0}}X - sD^{si}_{s\overrightarrow{x_0p_0}}(A_{\overrightarrow{x_0p_0}}X)).$$

From $J(s_0) = 0$, we have $D^{co}_{s_0\overrightarrow{x_0p_0}}X - s_0D^{si}_{s_0\overrightarrow{x_0p_0}}(A_{\overrightarrow{x_0p_0}}X) = 0$, which is equivalent to

$$\cos(\sqrt{-\varepsilon}\alpha(s_0g_{0*}^{-1}\overrightarrow{x_0p_0}))(g_{0*}^{-1}X)_\alpha - \frac{\sin(\sqrt{-\varepsilon}\alpha(s_0g_{0*}^{-1}\overrightarrow{x_0p_0}))}{\sqrt{-\varepsilon}\alpha(g_{0*}^{-1}\overrightarrow{x_0p_0})}(g_{0*}^{-1}A_{\overrightarrow{x_0p_0}}X)_\alpha = 0 \quad (\alpha \in \triangle_+ \cup \{0\}),$$

where $\varepsilon = -1$ (resp. 1) when G/K is of compact type (resp. of non-compact type) and $(\cdot)_\alpha$ is the $\mathfrak{p}_\alpha$-component of $\cdot$. Hence we have

$$g_{0*}^{-1}A_{\overrightarrow{x_0p_0}}X = \sum_{\alpha \in \triangle_+ \cup \{0\}} \frac{\sqrt{-\varepsilon}\alpha(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\alpha(g_{0*}^{-1}\overrightarrow{x_0p_0}))}(g_{0*}^{-1}X)_\alpha. \tag{4.1}$$

Here we note that $\tan(\sqrt{-\varepsilon}\alpha(s_0g_{0*}^{-1}\overrightarrow{x_0p_0})) \neq 0$ because $\gamma_{\overrightarrow{x_0p_0}}(s_0)$ is not a conjugate point along $\gamma_{\overrightarrow{x_0p_0}}$. Since $T_{x_0}M = \mathrm{Span}\{g_{0*}X_1, \cdots, g_{0*}X_n\}$, we can express as $g_{0*}^{-1}X=\sum\limits_{k=1}^n b_kX_k$ $(b_k\in\mathbf{R})$. From (4.1), we have

(4.2)
$$
\begin{aligned}
&g_{0*}^{-1}A_{\overrightarrow{x_0p_0}}X\\
&\quad=\sum_{\alpha\in\triangle_+\cup\{0\}}\sum_{k=1}^n\frac{\sqrt{-\varepsilon}\alpha(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\alpha(g_{0*}^{-1}\overrightarrow{x_0p_0}))}\times b_k(e_{i_k}^{\alpha_k}+e_{j_k}^{\beta_k})_\alpha\\
&\quad=\sum_{k=1}^n b_k\{\frac{\sqrt{-\varepsilon}\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0}))}e_{i_k}^{\alpha_k}+\frac{\sqrt{-\varepsilon}\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0}))}e_{j_k}^{\beta_k}\}.
\end{aligned}
$$

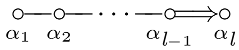
Since $\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0})\neq\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0})$, we have $\frac{\sqrt{-\varepsilon}\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0}))}$ $\neq\frac{\sqrt{-\varepsilon}\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0}))}$. Hence the vector $\frac{\sqrt{-\varepsilon}\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\alpha_k(g_{0*}^{-1}\overrightarrow{x_0p_0}))}e_{i_k}^{\alpha_k}$ $+\frac{\sqrt{-\varepsilon}\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0})}{\tan(\sqrt{-\varepsilon}s_0\beta_k(g_{0*}^{-1}\overrightarrow{x_0p_0}))}e_{j_k}^{\beta_k}$ is linearly independent of X_k. This fact implies that the right-hand side of (4.2) does not belong to $g_{0*}^{-1}T_{x_0}M\setminus\{0\}$ because $\{e_{i_k}^{\alpha_k},e_{j_k}^{\beta_k}\}$'s $(k=1,\cdots,n)$ are pairwise disjoint. Therefore, we have $A_{\overrightarrow{x_0p_0}}X=0$. On the other hand, since M is a submanifold in S, we have $\mathrm{Ker}A_{\overrightarrow{x_0p_0}}=\{0\}$. After all we obtain $X=0$. This contradicts $X\neq0$. Therefore, we see that there does not exist a strong focal point along $\gamma_{\overrightarrow{x_0p_0}}$. $\square$

By using this proposition, we give some examples of submanifolds in G/K admitting weak focal points.

Example 1. We consider the case where G/K is the simply connected rank one symmetric space $\mathbf{F}P^m(c)$ of compact type, where $\mathbf{F}=\mathbf{C}$, $\mathbf{Q}$ or $\mathbf{Cay}$ and $m\geq2$ when $\mathbf{F}=\mathbf{C}$ or $\mathbf{Q}$ and $m=2$ when $\mathbf{F}=\mathbf{Cay}$. Set $q:=\dim_{\mathbf{R}}\mathbf{F}$ and denote by $\{\phi_1,\cdots,\phi_{q-1}\}$ the $\mathbf{F}$-structure of $\mathbf{F}P^m(c)$. The positive root system $\triangle_+$ for a maximal abelian subspace $\mathfrak{h}$ of $\mathfrak{p}=T_{eK}(G/K)$ (under the lexicographical ordering determined by a unit vector v of $\mathfrak{h}$) is given by $\triangle_+=\{\sqrt{c}\langle v,\cdot\rangle,\ \frac{\sqrt{c}}{2}\langle v,\cdot\rangle\}$ and the root spaces $\mathfrak{p}_{\sqrt{c}\langle v,\cdot\rangle}$ and $\mathfrak{p}_{\frac{\sqrt{c}}{2}\langle v,\cdot\rangle}$ are given by $\mathfrak{p}_{\sqrt{c}\langle v,\cdot\rangle}=\mathrm{Span}\{\phi_1v,\cdots,\phi_{q-1}v\}$ and $\mathfrak{p}_{\frac{\sqrt{c}}{2}\langle v,\cdot\rangle}=\mathrm{Span}\{v,\phi_1v,\cdots,\phi_{q-1}v\}^\perp$. According to these facts, for each $n\leq q-1$, we can find a linearly independent system of $\mathfrak{p}$ consisting of n pieces of vectors satisfying the above condition $(\natural)$. According to Proposition 1, for each $n\leq q-1$, we can construct an n-dimensional submanifold in $\mathbf{F}P^m(c)$ admitting weak focal points. Similarly, for each $n\leq q-1$, we can construct such an n-dimensional submanifold in the simply connected rank one symmetric space $\mathbf{F}H^m(c)$ of non-compact type, where $\mathbf{F}=\mathbf{C}$, $\mathbf{Q}$ or $\mathbf{Cay}$ and $m\geq2$ when $\mathbf{F}=\mathbf{C}$ or $\mathbf{Q}$ and $m=2$ when $\mathbf{F}=\mathbf{Cay}$.

Example 2. We consider the case where G/K is the Grassmannian manifold $SO(m)/(SO(l)\times SO(m-l))$, where $2\le l\le \frac{m}{2}$. The positive root system $\triangle_+$ for a maximal abelian subspace $\mathfrak{h}$ of $\mathfrak{p}=T_{eK}(G/K)$ is given as follows:

$$\begin{aligned}\triangle_+ =&\{\alpha_i+\cdots+\alpha_j \,|\, 1\le i\le j\le l\}\\ &\cup\{\alpha_i+\cdots+\alpha_j+2(\alpha_{j+1}+\cdots+\alpha_l)\,|\,1\le i\le j\le l-1\},\end{aligned}$$

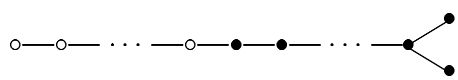
where $\{\alpha_1,\alpha_2,\cdots,\alpha_l\}$ is the fundamental root system ().

The Satake diagram of the orthogonal symmetric Lie algebra associated with $SO(m)/(SO(l)\times SO(m-l))$ is as in Diagrams 1 and 2.

$\left(\begin{array}{c}\text{the number of white circles} = l\\ \text{the number of black circles} = [\frac{m}{2}]-l\end{array}\right)$

$(\,m\ :\ \text{odd}\,)$

Diagram 1.



$\left(\begin{array}{c}\text{the number of white circles} = l\\ \text{the number of black circles} = \frac{m}{2}-l\end{array}\right)$

$(\,m\ :\ \text{even}\,)$

Diagram 2.

According to these Satake diagrams, the multiplicities of positive roots are as in Table 1.

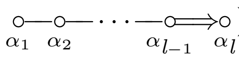
positive root	multiplicity
$\alpha_i+\cdots+\alpha_j\quad(1\le i\le j\le l-1)$	1
$\alpha_i+\cdots+\alpha_l\quad(1\le i\le l)$	$m-2l$
$\alpha_i+\cdots+\alpha_j+2(\alpha_{j+1}+\cdots+\alpha_l)\quad(1\le i\le j\le l-1)$	1

Table 1.

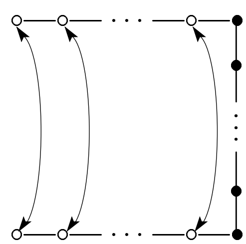
According to Table 1, for each $n \leq [\frac{l(m-l)-1}{2}] = [\frac{1}{2}(\dim SO(m)/(SO(l) \times SO(m-l)) -1)]$, we can find a linearly independent system of $\mathfrak{p}$ consisting of n-pieces of vectors satisfying the condition ($\natural$), where $[\cdot]$ is the Gauss's symbol of $\cdot$. Hence, according to Proposition 1, for each $n \leq [\frac{l(m-l)-1}{2}]$, we can construct an n-dimensional submanifold in $SO(m)/(SO(l) \times SO(m-l))$ admitting weak focal points. Similarly, we can construct such an $n\,(\leq [\frac{l(m-l)-1}{2}])$-dimensional submanifold in the dual $SO_0(l, m-l)/(SO(l) \times SO(m-l))$ of $SO(m)/(SO(l) \times SO(m-l))$.

Example 3. We consider the case where G/K is the complex Grassmannian manifold $SU(m)/S(U(l) \times U(m-l))$, where we assume that $2 \leq l \leq \frac{m}{2}$. First we consider the case of $l < \frac{m}{2}$. Then the positive root system $\triangle_+$ for a maximal abelian subspace $\mathfrak{h}$ of $\mathfrak{p} = T_{eK}(G/K)$ is given as follows:

$$
\begin{aligned}
\triangle_+ = &\{\alpha_i + \cdots + \alpha_j \,|\, 1 \leq i \leq j \leq l\} \\
&\cup \{\alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_l) \,|\, 1 \leq i \leq j \leq l-1\} \\
&\cup \{2\alpha_l\},
\end{aligned}
$$

where $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ is the fundamental root system (

).
The Satake diagram of the orthogonal symmetric Lie algebra associated with $SU(m)/S(U(l) \times U(m-l))$ $(l < \frac{m}{2})$ is as in Diagram 3.



$$\left(\begin{array}{c}\text{the number of white circles} = 2l \\ \text{the number of black circles} = m-2l-1\end{array}\right)$$

Diagram 3.

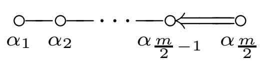
According to this Satake diagram, the multiplicities of positive roots are as in Table 2.

positive root	multiplicity
$\alpha_i + \cdots + \alpha_j \quad (1 \le i \le j \le l-1)$	2
$\alpha_i + \cdots + \alpha_l \quad (1 \le i \le l)$	$2(m-2l)$
$\alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_l) \quad (1 \le i \le j \le l-1)$	2
$2(\alpha_i + \cdots + \alpha_l) \quad (1 \le i \le l)$	1

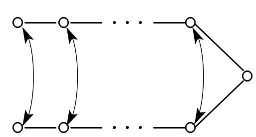
Table 2.

According to Table 2, for each $n \le l(m-l) - 1 = \frac{1}{2}\dim SU(m)/S(U(l) \times U(m-l)) - 1$, we can find a linearly independent system of $\mathfrak{p}$ consisting of n-pieces of vectors satisfying the condition ($\natural$). Hence, according to Proposition 1, for each $n \le \frac{1}{2}\dim SU(m)/S(U(l) \times U(m-l)) - 1$, we can construct an n-dimensional submanifold in $SU(m)/S(U(l) \times U(m-l))$ admitting weak focal points. Next we consider the case of $l = \frac{m}{2}$. Then the positive root system $\triangle_+$ for a maximal abelian subspace $\mathfrak{h}$ of $\mathfrak{p}$ is given as follows:

$$\begin{aligned}\triangle_+ = & \{\alpha_i + \cdots + \alpha_j \,|\, 1 \le i \le j \le \frac{m}{2}\} \\ & \cup \{\alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_{\frac{m}{2}-1}) + \alpha_{\frac{m}{2}} \mid 1 \le i \le j \le \frac{m}{2} - 2\} \\ & \cup \{2(\alpha_i + \cdots + \alpha_{\frac{m}{2}-1}) + \alpha_{\frac{m}{2}} \mid 1 \le i \le \frac{m}{2} - 1\},\end{aligned}$$

where $\{\alpha_1, \alpha_2, \cdots, \alpha_{\frac{m}{2}}\}$ is the fundamental root system (

). The Satake diagram of the orthogonal symmetric Lie algebra associated with $SU(m)/S(U(\frac{m}{2}) \times U(\frac{m}{2}))$ is as in Diagram 4.



(the number of white circles $= m - 1$)

Diagram 4.

According to this Satake diagram, the multiplicities of positive roots are as in Table 3.

positive root	multiplicity
$\alpha_i \quad (1 \le i \le l-1)$	2
α_l	1
$\alpha_i + \cdots + \alpha_j \quad (1 \le i < j \le l)$	2
$\alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_{l-1}) + \alpha_l$ $(1 \le i \le j \le l-2)$	2
$2(\alpha_i + \cdots + \alpha_{l-1}) + \alpha_l \quad (1 \le i \le l-1)$	1

Table 3.

According to Table 3, for each $n \le \frac{m^2}{4} - 1 = \frac{1}{2}\dim\, SU(m)/S(U(\frac{m}{2}) \times U(\frac{m}{2})) - 1$, we can find a linearly independent system of $\mathfrak{p}$ consisting of n-pieces of vectors satisfying the condition ($\natural$). Hence, according to Proposition 1, for each $n \le \frac{1}{2}\dim SU(m)/S(U(\frac{m}{2}) \times U(\frac{m}{2})) - 1$, we can construct an n-dimensional submanifold in $SU(m)/S(U(\frac{m}{2}) \times U(\frac{m}{2}))$ admitting weak focal points. Similarly, we can construct such an $n\,(\le l(m-l)-1)$-dimensional submanifold in the dual $SU(l, m-l)/S(U(l) \times U(m-l))$ of $SU(m)/S(U(l) \times U(m-l))$, where $2 \le l \le \frac{m}{2}$.

Similarly, we can construct examples of submanifolds admitting weak focal points in other symmertric spaces.

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Naoyuki Koike
Department of Mathematics, Faculty of Science, Tokyo University of Science
26 Wakamiya Shinjuku-ku, Tokyo 162-8601 Japan

E-mail: `koike@ma.kagu.sut.ac.jp`