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(p, q, r)-Generations and nX-Complementary Generations of the Thompson Group Th

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Abstract. A group G is said to be (l, m, n)-generated if it is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1\rangle$. In 1993 J. Moori posed the question of finding all triples (l, m, n) such that a given non-abelian finite simple group is (l, m, n)-generated. In this paper we partially answer this question for the Thompson group Th. In fact we study (p, q, r)-generation, where p, q and r are distinct primes, and nX-complementary generations of the Thompson group Th.

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§1. Introduction

Let G be a group and nX a conjugacy class of elements of order n in G. Following Woldar [26], the group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that $G =< x, y >$. The element $y = y(x)$ for which $G =< x, y >$ is called complementary. Furthermore, a group G is said to be (lX, mY, nZ)-generated (or (l, m, n)-generated for short) if there exist $x \in lX$, $y \in mY$ and $z \in nZ$ such that $xy = z$ and $G =< x, y >$. If G is (l, m, n)-generated, then we can see that for any permutation π of S_3, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. Therefore we may assume that $l \leq m \leq n$. By [3], if the non-abelian simple group G is (l, m, n)-generated, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Hence for a non-abelian finite simple group G and divisors l, m, n of the order of G such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, it is natural to ask if G is a (l, m, n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [27]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations.

Moori in [20], posed the problem of finding all triples (l, m, n) such that a given non-abelian finite simple group G is (l, m, n)-generated. In a series of papers [13-17] and [20,21], Moori and Ganief established all possible (p, q, r)-generations and nX-complementary generations of the sporadic groups J_1, J_2, J_3, J_4, HS, McL, Co_2, Co_3, and F_{22}, for distinct primes p, q, r and element orders n of $|G|$. Also, the author in [2] and [6-12](joint work) did the same work for the sporadic groups Co_1, ON, Ru and Ly. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

Throughout this paper we use the same notation as in the mentioned papers. In particular, $\Delta(G) = \Delta(lX, mY, nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY, nZ, whose value is the cardinality of the set $\Lambda = \{(x, y)|xy = z\}$, where $x \in lX, y \in mY$ and z is a fixed element of the conjugacy class nZ. In Table IV, we list the values $\Delta(pX, qY, rZ)$, where p, q and r are distinct prime divisors of $|Th|$, using the character table Th. Also, $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ)$ and $\Sigma(H_1 \cup H_2 \cup \cdots \cup H_r)$ denote the number of pairs $(x, y) \in \Lambda$ such that $G = \langle x, y\rangle$ and $\langle x, y\rangle \subseteq H_i$ (for some $1 \le i \le r$), respectively. The number of pairs $(x, y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^\star(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^\star(G) > 0$, then G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G. The number of conjugates of a given subgroup H of G containing a fix element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H(cf. [25]). In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP, [22].

Now we discuss techniques that are useful in resolving generation type questions for finite groups. We begin with a theorem that, in certain situations, is very effective at establishing non-generations.

Theorem 1.1. ([4]) *Let G be a finite centerless group and suppose lX, mY and nZ are G-conjugacy classes for which $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^\star(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

A further useful result that we shall often use is a result from Conder, Wilson and Woldar [4], as follows:

Lemma 1.2. *If G is nX-complementary generated and $(sY)^k = nX$, for some integer k, then G is sY-complementary generated.*

Further useful results that we shall use are:

Lemma 1.3.([15]). *If G is $(2X, sY, tZ)$-generated simple group then G is*

$(sY, sY, (tZ)^2)$-*generated.*

Lemma 1.4. *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

In the present paper we investigate the (p, q, r)-generation and nX-complementary generation for the Thompson group Th, where p, q and r are distinct primes and n is an element order. We prove the following results:

Theorem A. *The Thompson group Th is (p, q, r)-generated if and only if $(p, q, r) \neq (2, 3, 5)$.*

Theorem B. *The Thompson group Th is nX-complementary generated if and only if $nX \notin \{1A, 2A\}$.*

§2. (p, q, r)-Generations of Th

In this section we obtain all the (pX, qY, rZ)-generations of the Thompson group Th, which is a sporadic group of order $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$. Since $31A^{-1} = 31B$, hence, the group Th is $(pX, qY, 31A)$-generated if and only if it is $(pX, qY, 31B)$-generated. Therefore, it is enough to investigate the $(pX, qY, 31A)$-generation of Th.

We will use the maximal subgroups of Th listed in the ATLAS extensively, especially those with order divisible by 13 (for details see [18] and [19]). We listed in Table I, all the maximal subgroups of Th and in Table V, the partial fusion maps of these maximal subgroups into Th (obtained from GAP) that will enable us to evaluate $\Delta^{\star}_{Th}(pX, qY, rZ)$, for prime classes pX, qY and rZ. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z (see Lemma 1.4). For basic properties of the Thompson group Th and information on its maximal subgroups the reader is referred to [5]. It is a well known fact that Th has exactly 16 conjugacy classes of maximal subgroups, as listed in Table I.

If the group Th is $(2, 3, p)$-generated, then by the Conder's result [3], $\frac{1}{2} + \frac{1}{3} + \frac{1}{p} < 1$. Thus we only need to consider the cases $p = 7, 13, 19, 31$.

Table I
The Maximal Subgroups of Th

Group	*Order*	*Group*	*Order*
$^3D_4(2).3$	$2^{12}.3^5.7^2.13$	$2^5.PSL(5,2)$	$2^{15}.3^2.5.7.31$
$2^{1+8}.A_9$	$2^{15}.3^4.5.7$	$U_3(8).6$	$2^{10}.3^5.7.19$
$(3 \times G_2(3)) : 2$	$2^7.3^7.7.13$	$ThN3B$	$2^4.3^{10}$
$ThM7$	$2^4.3^{10}$	$3^5 : 2S_6$	$2^5.3^7.5$
$5^{1+2}.4S_4$	$2^5.3.5^3$	$5^2 : 4S_5$	$2^5.3.5^3$
$7^2 : (3 \times 2S_4)$	$2^4.3^2.7^2$	$L_2(19).2$	$2^3.3^2.5.19$
$L_3(3)$	$2^4.3^3.13$	$A_6.2_3$	$2^4.3^2.5$
$31 : 15$	$3.5.31$	$A_5.2$	$2^3.3.5$

Woldar, in [27] determined which sporadic groups other than F_{22}, F_{23}, F'_{24}, Th, J_4, B and M are Hurwitz groups, i.e. generated by elements x and y with order $o(x) = 2$, $o(y) = 3$ and $o(xy) = 7$. In fact, G is a Hurwitz group if and only if G is $(2,3,7)$-generated. Next, Linton [18], proved that the Thompson group Th is Hurwitz.

For the sake of completeness, in the following lemma, we prove that Th is a Hurwitz group. Therefore, Th is $(2,3,7)$-generated.

Lemma 2.1. *The Thompson group Th is not $(2A,3A,7A)$- and $(2A,3B,7A)$-generated, but it is $(2A,3C,7A)$-generated.*

Proof. From the structure constants, Table iV, we can see that $\Delta_{Th}(2A,3A,7A) < |C_{Th}(7A)|$. So, by Theorem 1.1, $\Delta^\star(G) = 0$ and therefore Th is not $(2A,3A,7A)$-generated. We now consider two cases.

Case $(2A,3B,7A)$. The maximal subgroups of Th that may contain $(2A,3B,7A)$-generated proper subgroups are isomorphic to $^3D_4(2).3$, $2^{1+8}.A_9$, $U_3(8).6$ and $(3\times G_2(3)) : 2$. We calculate that $\Delta(Th) = 1372$ and $\Sigma(^3D_4(2).3) = 343$. Our calculations give:

$$\Delta^\star(Th) \leq \Delta(Th) - 343 = 1029 < 1176 = |C_{Th}(7A)|.$$

Thus, by Theorem 1.1, $\Delta^\star(Th) = 0$, which shows the non-generation of this triple.

Case $(2A,3C,7A)$. From the list of maximal subgroups of Th, Table I, we observe that, up to isomorphisms, $^3D_4(2).3$, $2^5.PSL(5,2)$, $2^{1+8}.A_9$, $U_3(8).6$, $(3 \times G_2(3)) : 2$ and $7^2 : (3 \times 2S_4)$ are the only maximal subgroups of Th that admit $(2A,3C,7A)$-generated subgroups. From the structure constants, Table IV, we calculate $\Delta(Th) = 4704$, $\Sigma(^3D_4(2).3) = \Sigma(2^5.PSL(5,2)) = \Sigma(2^{1+8}.A_9) = \Sigma(U_3(8).6) = \Sigma(7^2 : (3\times 2S_4)) = 0$ and $\Sigma((3\times G_2(3)) : 2) = 42$. Thus, $\Delta^\star(Th) \geq 4704 - 28.42 > 0$. This shows that the Thompson group Th is $(2A,3C,7A)$-generated, proving the lemma. □

By the previous lemma, Th is a Hurwits group. In the following results we not only prove for certain triples (p, q, r) that Th is (p, q, r)-generated, but we also find all generating triples (pX, qY, rZ). We will use some of these generating triples later to find conjugacy classes nX for which Th is nX-complementary generated.

Lemma 2.2. *The Thompson group* Th *is* $(2A, 3X, pY)$*-generated if and only* $p \geq 7$ *and* $(3X, pY) \notin \{(3A, 7A), (3B, 7A), (3A, 13A)\}$.

Proof. As we mentioned above, Th is not $(2, 3, 5)$-generated. Also, by Lemma 2.1, Th is not $(2A, 3A, 7A)$- and $(2A, 3B, 7A)$-generated. We now prove the non-generation of the triple $(2A, 3A, 13A)$. Amongst the maximal subgroups of Th with order divisible by $2 \times 3 \times 13$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to ${}^3D_4(2).3$ and $(3 \times G_2(3)) : 2$. We can see that $\Delta(Th) = 156$, $\Sigma({}^3D_4(2).3) = 39$ and $\Sigma((3 \times G_2(3)) : 2) = 39$. Furthermore, a fixed element of order 13 is contained in three conjugate subgroups of ${}^3D_4(2).3) = 39$ and one conjugate copy of $(3 \times G_2(3)) : 2$ (see Table V).

Table II
Partial Fusion Maps of ${}^3D_4(2)$ into ${}^3D_4(2).3$ and ${}^3D_4(2).3$ into Th

${}^3D_4(2)$-classes	$2a'$	$2b'$	$3a'$	$3b'$	$7a'$	$7b'$	$7c'$	$7d'$	$13a'$	$13b'$	$13c'$
$\to {}^3D_4(2).3$	2a	2b	3a	3b	7a	7a	7a	7b	13a	13a	13a
$\to Th$	2A	2A	3A	3B	7A	7A	7A	7A	13A	13A	13A

Consider the subgroup $H = {}^3D_4(2)$ of Th. In Table I, we obtain the partial fusion map of this subgroup into ${}^3D_4(2).3$ and ${}^3D_4(2).3$ into Th. From the character table of Th [5], we can see that H is a maximal subgroup of ${}^3D_4(2).3$ and ${}^3D_4(2).3$ is a maximal subgroup of Th. Consider the triple $(2b, 3a, 13a)$. Then H is a maximal subgroup of ${}^3D_4(2).3$ with order divisible by 13 and non-empty intersection with the classes 2b, 3a and 13a. We calculate that $\Delta(Th) = 156$, $\Sigma(H) = 117$. Since H does not have a maximal subgroup with order divisible by $2 \times 3 \times 13$, $\Delta^\star({}^3D_4(2).3)(2b, 3a, 13a) = 117$. On the other hand, $\Sigma((3 \times G_2(3)) : 2) = 39$ and $\Sigma((3 \times G_2(3)) : 2)$ does not have a subgroup isomorphic to H. Therefore, there exists at least one pair (x, y) such that $x \in 2A$, $y \in 3A$, $xy \in 13A$ and $< x, y >$ is a subgroup of $(3 \times G_2(3)) : 2$, but it is not a subgroup of ${}^3D_4(2).3$. This shows that

$$\Delta^\star(Th) \leq 156 - 117 - 1 = 38 < 39 = |C_{Th}(13A)|$$

and non-generation of Th by this triple follows from Theorem 1.1. We now prove the $(2A, 3X, pY)-$ generations of other triples. We will treat each triple separately.

Case $(2A, 3A, 13A)$. From the list of maximal subgroups of Th, we observe that, up to isomorphisms, $U_3(8).6$ is the only maximal subgroup of Th that admit $(2A, 3A, 13A)$-generated subgroups. From the structure constants, we calculate $\Delta(Th) = 19$ and $\Sigma(U_3(8).6) = 0$. Thus, $\Delta^\star(Th) = \Delta(Th) = 19 > 0$. This shows that the Thompson group Th is $(2A, 3A, 13A)$-generated.

Case $(2A, 3B, 13A)$. The maximal subgroups of Th that have non-empty intersection with the classes $2A$, $3B$ and $13A$ are, up to isomorphism, ${}^3D_4(2).3$, $(3\times G_2(3)) : 2$ and $L_3(3)$. We calculate that $\Delta(Th) = 1261$, $\Sigma({}^3D_4(2).3) = 91$, $\Sigma((3 \times G_2(3)) : 2) = 13$ and $\Sigma(L_3(3)) = 52$. From Table V it follows that

$$\Delta^\star(Th) \geq 1261 - 3(91) - 13 - 12(52) = 351,$$

and hence Th is $(2A, 3B, 13A)$-generated.

Using similar argument as in above, we can prove the generation of other triples. □

Lemma 2.3. *Let $5 \leq p < q$ are prime divisors of $|Th|$. Then the Thompson group Th is $(2A, pX, qY)$-generated.*

Proof. Set $K = \{(5A, 13A), (13A, 19A), (13A, 31A), (19A, 31A)\}$. From Table V, we can see that for every pairs (pX, qY) in the set K, there is no maximal subgroups that contains $(2A, pX, qY)$-generated proper subgroups. Therefore, $\Delta^\star(Th) = \Delta(Th) > 0$, and so Th is $(2A, pX, qY)$-generated. On the other hand, we can see that $2^5.PSL(5, 2)$ is, up to isomorphism, the only maximal subgroup of Th which intersects the conjugacy classes $2A$, $7A$ and $31A$. Since $\Sigma(2^5.PSL(5, 2)) = 0$, Th is $(2A, 7A, 31A)$-generated. We investigate another triples case by case.

Case $(2A, 5A, 7A)$. The only maximal subgroups that may contain $(2A$, $5A$, $7A)$- generated subgroups are isomorphic to $2^5.PSL(5, 2)$ and $2^{1+8}.A_9$. We calculate that

$$14\Sigma(2^5.PSL(5, 2)) + 21\Sigma(2^{1+8}.A_9) = 14(672) + 21(224) = 14112$$

Since $\Delta(Th) = 362208$, we have $\Delta^\star(Th) > 0$. This proves generation by this triple.

Case $(2A, 5A, 19A)$. From the list of maximal subgroups of Th we observe that, up to isomorphisms, $L_2(19).2$ is the only maximal subgroup of Th that admit $(2A, 5A, 19A)$-generated subgroups. From the structure constants, Table IV, we calculate $\Delta(Th) = 342304$ and $\Sigma(L_2(19).2) = 38$. Thus, $\Delta^\star(Th) \geq \Delta(Th) - 38 > 0$. This shows that the Thompson group Th is $(2A, 5A, 19A)$-generated.

Case $(2A, 5A, 31A)$. In this case, $\Delta(Th) = 320447$ and the only maximal subgroup with non-empty intersection with any conjugacy class in this triple is isomorphic to $2^5.PSL(5,2)$. We calculate, $\Sigma(2^5.PSL(5,2)) = 744$. Our calculations give, $\Delta^\star(Th) \geq \Delta(th) - 3(744) > 0$. Therefore, Th is $(2A, 5A, 31A)$-generated.

Case $(2A, 7A, 13A)$. Amongst the maximal subgroups of Th with order divisible by $2\times7\times13$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to ${}^3D_4(2).3$ and $(3 \times G_2(3)) : 2$. Using Table I, we can see that $\Delta(Th) = 819754$, $\Sigma({}^3D_4(2).3) = 1430$ and $\Sigma((3 \times G_2(3)) : 2) = 1066$. Our calculations give,

$$\Delta^\star(Th) \geq \Delta(Th) - 3(1430) - 1066 > 0,$$

proving the generation of Th by this triple.

Case $(2A, 7A, 19A)$. We have $\Delta(Th) = 753730$. The $(2A, 7A, 19A)$-generated proper subgroups of Th are contained in the maximal subgroups isomorphic to $U_3(8).6$. We calculate further that $\Sigma(U_3(8).6) = 513$. From Table V we conclude that $\Delta^\star(Th) \geq 753730 - 513 > 0$ and the generation of Th by this triple follows. This completes the proof. □

In the following lemma we determine all the generating triples (pX, qY, rZ) for the group Th, where p, q, r are distinct odd primes.

Lemma 2.4. *If p, q and r are odd primes, then the Thompson group Th is (pX, qY, rZ)-generated.*

Proof. The proof is similar to Lemma 2.2 and 2.3 and it omitted. □

We are now ready to state one of main results of this paper.

Theorem A. *The Thompson group Th is (p, q, r)-generated if and only if $(p, q, r) \neq (2, 3, 5)$.*

Proof. The proof follows from the Lemmas 2.1, 2.2, 2.3 and 2.4. □

§3. nX-Complementary Generations of Th

In this section we investigate the nX-complementary generations of the Thompson group Th. Let G be a group and nX be a conjugacy class of elements of order n in G. In [25], Woldar proved that every sporadic simple group is pX-complementary generated, for the greatest prime divisor p of the order of the group. Therefore, Th is $31X$-complementary generated.

As a consequence of a result in [26], a group G is nX-complementary generated if and only if G is (pY, nX, t_pZ)-generated, for all conjugacy classes pY with representatives of prime order and some conjugacy class t_pZ (depending on pY). Using this result, we obtain all of the conjugacy class nX such that Th is nX-complementary generated.

First of all, we show that Th is not $2X$-complementary generated. To see this, we notice that for any positive integer n, $T(2,2,n) \cong D_{2n}$, the dihedral group of order $2n$. Thus if G is a finite group which is not isomorphic to some dihedral group, then G is not $(2X,2X,nY)$-generated, for all classes of involutions and any G-class nY. Thus, Th is not $2X$-complementary generated.

In [26], Woldar proved that every sporadic simple group is pX- complementary generated, for the greatest prime divisor p of the order of the group. So, Th is $31A$- and $31B$-complementary generated.

Lemma 3.1. *The Thompson group Th is $3X$-complementary generated.*

Proof. By Lemmas 1.3, 2.1, 2.2 and 2.3, it is enough to show that there are the conjugacy classes t_1Z, t_2Z and t_3Z such that Th is $(3A,3B,t_1Z)-$, $(3A,3C,t_2Z)-$, and $(3B,3C,t_3Z)$-generated. Suppose $t_1Z = 31A$, $t_2A = t_3Z = 19A$. From Table V, we can see that there is no maximal subgroups contains the triple $(3A,3B,t_1Z)$. Since $\Delta_{Th}(3A,3B,t_1Z) = 14880$, $\Delta^\star(Th) = \Delta(Th) > 0$. This proves the generation by this triple. For other triples, $\Delta_{Th}(3A,3C,19A) = 39990$, $\Delta_{Th}(3B,3C,19A) = 1072848$ and the only maximal subgroups that may contain $(3A,3C,19A)-$ or $(3B,3C,19A)$-generated subgroups is isomorphic to $U_3(8).6$. Next we calculate

$$\begin{aligned}
\Delta^\star_{Th}(3A,3C,19A) &\geq \Delta_{Th}(3A,3C,19A) - \Sigma(U_3(8).6) \\
&= 39990 - 380 > 0 \\
\Delta^\star_{Th}(3B,3C,19A) &\geq \Delta_{Th}(3B,3C,19A) - \Sigma(U_3(8).6) \\
&= 1072848 - 0 > 0
\end{aligned}$$

proving the generation of Th by these triples. □

Lemma 3.2. *The Thompson group Th is pX-complementary generated, for every prime class pX with $p \geq 5$.*

Proof. By a result of Woldar, mentioned above, Th is $31X$-complementary generated. Suppose pX, $5 \leq p \leq 19$, is a conjugacy class with prime order representatives and qY is another conjugacy class with prime order representatives and $q \neq p$. We consider a conjugacy class in the form t_pZ, where t_p is a prime divisor of $|Th|$ different from p and q. Then by Lemmas [2.1-2.4], Th is (qY,pX,t_pZ)-generated. Therefore, it remains to investigate the case $q = p$. Apply Lemma 1.3, we can see that Th is (pX,pX,t_pZ)-generated, for some prime class t_pZ. Therefore, Th is pX-complementary generated, proving the lemma. □

Lemma 3.3. *The Thompson group Th is $4X$-complementary generated.*

Proof. First of all, we assume that $X = A$. For every conjugacy class pY with prime order representatives, we define $t_pY = 19A$. From the list of maximal subgroups of Th we observe that, up to isomorphisms, $U_3(8).6$ is the only maximal subgroup of Th that admit $(pY, 4A, 19A)$-generated subgroups. Then we have,

$$\Delta^\star(Th) = \Delta(Th) - \Sigma(U_3(8).6) > 0.$$

Therefore, Th is $4A$-complementary generated. We next suppose that $X = B$. In this case, for any prime class pY, we define $t_pY = 31A$. The $(pY, 4B, 31A)$-generated proper subgroups of Th are contained in the maximal subgroups isomorphic to $2^5.PSL(5,2)$. Now with the tedious calculations we can see that

$$\Delta^\star(Th) = \Delta(Th) - \Sigma(2^5.PSL(5,2)) > 0.$$

This proves generation by these triples. □

Lemma 3.4. *The Thompson group Th is nX-complementary generated, for every element order $n \geq 5$.*

Proof. In Table III, we compute the power maps of Th. The lemma now follows from Lemmas 3.1-3.3 and Lemma 1.2.

We are now ready to state the second main results of this paper.

Theorem B. *The Thompson group Th is nX-complementary generated if and only if $nX \notin \{1A, 2A\}$.*

Proof. The result follows from Lemmas 3.1-3.4. □

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Table III
The Power Maps of Th

$(6A)^2 = 3C$	$(6B)^2 = 3A$	$(6C)^2 = 3B$	$(8A)^2 = 4A$	$(8B)^2 = 4B$
$(9A)^3 = 3B$	$(9B)^3 = 3B$	$(9C)^3 = 3C$	$(10A)^2 = 5A$	$(12A)^2 = 6B$
$(12B)^2 = 6B$	$(12C)^2 = 6C$	$(12D)^2 = 6A$	$(14A)^2 = 7A$	$(15A)^3 = 5A$
$(15B)^3 = 5A$	$(18A)^3 = 6C$	$(18B)^3 = 6C$	$(20A)^2 = 10A$	$(21A)^3 = 7A$
$(24A)^2 = 12A$	$(24B)^2 = 12B$	$(24C)^2 = 12C$	$(24D)^2 = 12C$	$(27A)^3 = 9B$
$(27B)^3 = 9B$	$(27C)^3 = 9B$	$(28A)^2 = 14A$	$(30A)^2 = 15A$	$(30B)^2 = 15B$
$(36A)^2 = 18A$	$(36B)^2 = 18A$	$(36C)^2 = 18A$	$(39A)^3 = 13A$	$(39B)^3 = 13A$

Table IV
The Structure Constants of Th

pY	$\Delta(2A,3A,pY)$	$\Delta(2A,3B,pY)$	$\Delta(2A,3C,pY)$	$\Delta(2A,5A,pY)$
$7A$	252	1372	4704	362208
$13A$	156	1261	6240	339417
$19A$	19	2166	6194	342304
$31A$	62	1519	5084	320447
pY	$\Delta(2A,7A,pY)$	$\Delta(2A,13A,pY)$	$\Delta(2A,19A,pY)$	$(3A,5A,pY)$
$7A$	-	-	-	1411200
$13A$	819754	-	-	1964898
$19A$	753730	24015278	-	2103528
$31A$	795770	25269867	50957738	2375406
pY	$\Delta(3A,7A,pY)$	$\Delta(3A,13A,pY)$	$\Delta(3A,19A,pY)$	$\Delta(3B,5A,pY)$
$7A$	-	-	-	58788240
$13A$	8386794	-	-	61977591
$19A$	7067620	193359390	-	61643904
$31A$	5965578	182304180	375714234	64174185
pY	$\Delta(3B,7A,pY)$	$\Delta(3B,13A,pY)$	$\Delta(3B,19A,pY)$	$\Delta(3C,5A,pY)$
$7A$	-	-	-	192734640
$13A$	168499786	-	-	178301682
$19A$	173570434	5014848258	-	177905664
$31A$	162605974	4935510837	10083933766	172500678
pY	$\Delta(3C,7A,pY)$	$\Delta(3C,13A,pY)$	$\Delta(3C,19A,pY)$	$\Delta(5A,7A,pY)$
$13A$	421466526	-	-	24924811392
$19A$	419563890	13084126902	-	25096768640
$31A$	440991306	13299654348	27316460898	25723011856
pY	$\Delta(5A,13A,pY)$	$\Delta(5A,19A,pY)$	$\Delta(7A,13A,pY)$	$\Delta(7A,19A,pY)$
$19A$	769350038016	-	2002364205478	-
$31A$	775606154625	1591873859584	1978741938906	4061042386610
pY	$\Delta(13A,19A,pY)$	-	-	-
$19A$	122473293000346	-	-	-

Table V
Partial Fusion Maps of Maximal Subgroups into Th

${}^3D_4(2).3$-classes	2a	2b	3a	3b	3c	3d	3e	3f	4a	4b	4c
$\to Th$	2A	2A	3A	3B	3A	3A	3C	3C	4A	4A	4B
${}^3D_4(2).3$-classes	7a	7b	13a								
$\to Th$	7A	7A	13A								
h	9	9	3								
$2^5.PSL(5,2)$-classes	2a	2b	3a	3b	4a	4b	4c	5a	7a	7b	31a
$\to Th$	2A	2A	3C	3A	4A	4B	4B	5A	7A	7A	31A
h									14	14	3
$2^5.PSL(5,2)$-classes	31b	31c	31d	31e	31f						
$\to Th$	31B	31A	31A	31B	31B						
h	3	3	3	3	3						
$2^{1+8}.A_9$-classes	2a	2b	2c	3a	3b	3c	5a	7a			
$\to Th$	2A	2A	2A	3C	3B	3A	5A	7A			
h								21			
$U_3(8).6$-classes	2a	2b	3a	3b	3c	3d	3e	3f	4a	4b	7a
$\to Th$	2A	2A	3A	3B	3A	3A	3C	3C	4A	4B	7A
h											28
$U_3(8).6$-classes	19a										
$\to Th$	19A										
h	1										
$(3 \times G_3(2)) : 2$-classes	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i
$\to Th$	2A	2A	3A	3B	3A	3A	3B	3A	3C	3B	3A
$(3 \times G_3(2)) : 2$-classes	3j	7a	13a								
$\to Th$	3C	7A	13A								
h		28	1								
$ThN3B$-classes	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i
$\to Th$	2A	2A	3B	3B	3A	3A	3B	3C	3A	3B	3C
$ThN3B$-classes	3j	3k	3l	3m	3n	3o	3p	3q			
$\to Th$	3A	3C	3B	3B	3C	3C	3B	3C			

Table V (Continued)

$ThM7$-classes	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i
$\rightarrow Th$	2A	2A	3B	3C	3B	3A	3A	3B	3C	3B	3C
$ThM7$-classes	3j	3k									
$\rightarrow Th$	3B	3C									
$L_2(19).2$-classes	2a	2b	3a	5a	5b	19a					
$\rightarrow Th$	2A	2A	3B	5A	5A	19A					
h						1					
$3^5.2S_6$-classes	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i
$\rightarrow Th$	2A	2A	3C	3B	3C	3A	3B	3B	3B	3C	3C
$3^5.2S_6$-classes	3j	3k	3l	3m	3n	3o	3p	3q	3r	3s	3t
$\rightarrow Th$	3C	3B	3C	3C	3A	3C	3C	3C	3B	3B	3C
$3^5.2S_6$-classes	5a										
$\rightarrow Th$	5A										
$5^{1+2}.4S_4$-classes	2a	2b	3a	5a	5b						
$\rightarrow Th$	2A	2A	3C	5A	5A						
$5^2.4S_5$-classes	2a	2b	3a	5a	5b	5c					
$\rightarrow Th$	2A	2A	3C	5A	5A	5A					
$7^2:(3\times 2S_4)$-classes	2a	3a	3b	3c	3d	3e	7a				
$\rightarrow Th$	2A	3C	3C	3A	3A	3C	7A				
h							8				
$L_3(3)$-classes	2a	3a	3b	13a	13b	13c	13d				
$\rightarrow Th$	2A	3B	3B	13A	13A	13A	13A				
h				12	12	12	12				
$A_6.2_3$-classes	2a	3a	5a								
$\rightarrow Th$	2A	3B	5A								
$31:15$-classes	3a	3b	5a	5b	5c	5d	31a	31b			
$\rightarrow Th$	3C	3C	5A	5A	5A	5A	31A	31B			
h							1	1			
$A_5.2$-classes	2a	2b	3a	5a							
$\rightarrow Th$	2A	2A	3B	5A							

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