THE BOUNDEDNESS OF MULTILINEAR CALDERÓN–ZYGMUND OPERATORS ON WEIGHTED AND VARIABLE HARDY SPACES

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Abstract: We establish the boundedness of the multilinear Calderón–Zygmund operators from a product of weighted Hardy spaces into a weighted Hardy or Lebesgue space. Our results generalize to the weighted setting results obtained by Grafakos and Kalton [**18**] and recent work by the third author, Grafakos, Nakamura, and Sawano [**20**]. As part of our proof we provide a finite atomic decomposition theorem for weighted Hardy spaces, which is interesting in its own right. As a consequence of our weighted results, we prove the corresponding estimates on variable Hardy spaces. Our main tool is a multilinear extrapolation theorem that generalizes a result of the first author and Naibo [**10**].

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1. Introduction

In this paper we study the boundedness of multilinear Calderón–Zygmund operators (m-CZOs) on products of weighted and variable Hardy spaces. More precisely, we are interested in the following operators. Let $K(y_0, y_1, \ldots, y_m)$ be a kernel that is defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and satisfies the smoothness condition

$$(1.1) \quad |\partial_{y_0}^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K(y_0, y_1, \ldots, y_m)|$$
$$\leq A_{\alpha_0, \ldots, \alpha_m} \left(\sum_{k,l=0}^{m} |y_k - y_l| \right)^{-(mn + |\alpha_0| + \cdots + |\alpha_m|)}$$

for all $\alpha = (\alpha_0, \ldots, \alpha_m)$ such that $|\alpha| = |\alpha_0| + \cdots + |\alpha_m| \leq N$, where N is a sufficiently large integer. An m-CZO is a multilinear operator T that satisfies

$$T \colon L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

for some $1 < q_1, \ldots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and T has the integral representation

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f(y_1) \cdots f(y_m) \, dy_1 \cdots dy_m$$

whenever $f_i \in L_c^\infty(\mathbb{R}^n)$ and $x \notin \cap_i \operatorname{supp}(f_i)$.

Multilinear CZOs were introduced by Coifman and Meyer [2, 3] in the 1970s and were systematically studied by Grafakos and Torres [22]. They showed that m-CZOs are bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, for any $1 < p_1, \ldots, p_m < \infty$ and p defined by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Further, m-CZOs satisfy weak endpoint bounds when $p_i = 1$ for some i. For Lebesgue space bounds, it is sufficient to take $N = 1$ in (1.1) and in fact weaker regularity conditions are sufficient. Bounds for m-CZOs from products of Hardy spaces into Lebesgue spaces were proved by Grafakos and Kalton [18] (see also Grafakos and He [17]). As in the linear case, more regularity is required on the operators: in this case, $N \geq s = \lfloor n(\frac{1}{p} - 1) \rfloor_+$ where $x_+ = \max(0, x)$. Very recently, bounds into Hardy spaces were proved by the third author, Grafakos, Nakamura, and Sawano [20]. To map into Hardy spaces the kernel K must satisfy (1.1) for

$$N > s + \max\left\{ \left\lfloor mn\left(\frac{1}{p_k} - 1\right) \right\rfloor_+ : 1 \leq k \leq m \right\} + mn.$$

Moreover, in the multilinear case the operator T must satisfy an additional cancelation condition:

$$(1.2) \qquad \int x^\alpha T(a_1, \ldots, a_m)(x) \, dx = 0,$$

for all $|\alpha| \leq s$ and all (p_k, ∞, N) atoms a_k. For linear CZOs of convolution type, this condition holds automatically: see [21, Lemma 2.1]. An example of a bilinear CZO that satisfies this cancelation condition is $T = R_1 + R_2$, where R_i is the bilinear Riesz transform

$$R_i(f, g)(x) = \text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{x - y_i}{|(x - y_1, x - y_2)|^3} f(y_1) g(y_2) \, dy_1 \, dy_2.$$

Somewhat surprisingly, neither Riesz transform itself has sufficient cancellation. For more examples of convolution-type multilinear operators that do and do not satisfy this cancelation condition, see [**20, 21**].[1]

Weighted norm inequalities for multilinear operators were first considered by Grafakos and Torres [**23**]. Later, Lerner *et al.* [**25**] characterized the weighted inequalities for m-CZOs using a multilinear generalization of the Muckenhoupt A_p condition. Weighted Hardy spaces were introduced by García-Cuerva [**14**]. A complete treatment of weighted Hardy spaces is due to Strömberg and Torchinsky [**31**]; they proved that (linear) Calderón–Zygmund operators whose kernels have enough regularity map $H^p(w)$ into $L^p(w)$ or $H^p(w)$, for $0 < p < \infty$ and for weights $w \in A_\infty$.

Our goal is to generalize the results of Strömberg and Torchinsky to m-CZOs. To state them, we first define some notation. To do so we rely on some (hopefully) well-known concepts; complete definitions will be given below. Given $w \in A_\infty$, we define

$$r_w = \inf\{r \in (1, \infty) : w \in A_r\}$$

and for $0 < p < \infty$ we define the critical index s_w of w by

$$s_w = \left\lfloor n\left(\frac{r_w}{p} - 1\right)\right\rfloor_+.$$

Our first result gives the boundedness of m-CZOs into weighted Lebesgue spaces.

Theorem 1.1. *Given an integer* $m \geq 1$, $0 < p_1, \ldots, p_m < \infty$, *and* $w_k \in A_\infty$, $1 \leq k \leq m$, *let* T *be an* m-CZO *associated to a kernel* K *that satisfies* (1.1) *for* N *such that*

$$(1.3) \qquad N \geq \max\left\{\left\lfloor mn\left(\frac{r_{w_k}}{p_k} - 1\right)\right\rfloor_+, 1 \leq k \leq m\right\} + (m-1)n.$$

Then

$$T: H^{p_1}(w_1) \times \cdots \times H^{p_m}(w_m) \to L^p(\overline{w}),$$

where $\overline{w} = \prod_{k=1}^m w_k^{\frac{p}{p_k}}$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Our second result gives boundedness of m-CZOs into weighted Hardy spaces.

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[1]We note in passing that the results for m-CZOs in [**20**] are stated for convolution type operators, but as the authors note (see Remark 3.4), their results extend to non-convolution type m-CZOs.

Theorem 1.2. *Given $p, p_1, \ldots, p_m, \overline{w}, w_1, \ldots, w_m$ and T as in Theorem 1.1, suppose the kernel K satisfies (1.1) for N such that*

$$
(1.4) \qquad N > s_{\overline{w}} + \max \left\{ \left\lfloor mn \left(\frac{r_{w_k}}{p_k} - 1 \right) \right\rfloor_+, \, 1 \le k \le m \right\} + mn.
$$

Suppose further that T satisfies the cancellation condition (1.2) for all $|\alpha| \le s_{\overline{w}}$, where for $1 \le k \le m$, a_k is an (N, ∞) atom: i.e., a_k is supported on a cube Q_k, $\|a_k\|_\infty \le 1$, and

$$
(1.5) \qquad \int_{\mathbb{R}^n} x^\beta a_k(x)\, dx = 0
$$

for all $|\beta| \le N$. Then

$$
T \colon H^{p_1}(w_1) \times \cdots \times H^{p_m}(w_m) \to H^p(\overline{w}).
$$

Remark 1.3. In Theorems 1.1 and 1.2, if all the weights $w_k = 1$, then $r_{w_k} = 1$, so we recapture the unweighted results in [**18, 20**].

Remark 1.4. If $p > 1$ and $w \in A_p$, then $H^p(w) = L^p(w)$ (see [**31**]). Therefore, in Theorems 1.1 and 1.2, if $w_k \in A_{p_k}$, then we can replace $H^{p_k}(w_k)$ by $L^{p_k}(w_k)$ in the conclusion.

Remark 1.5. Implicit in the statement of Theorem 1.2 is the assumption that $\overline{w} \in A_\infty$. However, this is always the case: see Lemma 2.1 below.

Remark 1.6. Earlier, Xue and Yan [**33**] proved a version of Theorem 1.1 with the additional restriction that $0 < p_k \le 1$ for all $1 \le k \le m$. We want to thank the authors for calling our attention to their paper, which we had overlooked.

Our next pair of results are the analogs of Theorems 1.1 and 1.2 for the variable Lebesgue spaces. The variable Lebesgue spaces are a generalization of the classical L^p spaces with the exponent p replaced by a measurable exponent function $p(\cdot) \colon \mathbb{R}^n \to (0, \infty)$. It consists of all measurable functions f such that for some $\lambda > 0$

$$
\rho(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.
$$

This becomes a quasi-Banach space with quasi-norm

$$
\|f\|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho(f/\lambda) \le 1 \}.
$$

If $p(x) \ge 1$ a.e., then this is a norm and $L^{p(\cdot)}$ is a Banach space. These spaces were introduced by Orlicz [**29**] in 1931, and have been extensively studied by a number of authors in the past 25 years. For complete details and references, see [**6**]. Variable Hardy spaces were introduced by the first author and Wang [**12**] and independently by Nakai and Sawano [**28**].

In variable Lebesgue exponent spaces, harmonic analysis requires some assumption of regularity on the exponent function $p(\cdot)$. A common assumption that is sufficient for almost all applications is that the exponent function is log-Hölder continuous both locally and at infinity. More precisely, there exist constants C_0, C_∞, and p_∞ such that

$$(1.6) \qquad |p(x) - p(y)| \leq \frac{C_0}{-\log(|x-y|)}, \quad 0 < |x-y| < \frac{1}{2},$$

and

$$(1.7) \qquad |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

Finally, given an exponent function $p(\cdot)$, we define

$$p_- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

As an immediate application of Theorems 1.1 and 1.2, and multilinear Rubio de Francia extrapolation in the scale of variable Lebesgue spaces, we get the following two results.

Theorem 1.7. *Given an integer $m \geq 1$, let $p_1, \ldots, p_m$ be real numbers, and let $q_1(\cdot), \ldots, q_m(\cdot)$ be log-Hölder continuous exponent functions such that $0 < p_k < (q_k)_- \leq (q_k)_+ < \infty$. Define*

$$\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)}.$$

Let T be an m-CZO as in Theorem 1.1 satisfying (1.1) for all $|\alpha| \leq N$ with

$$N \geq \max\left\{ \left\lfloor mn\left(\frac{1}{p_k} - 1\right) \right\rfloor_+, \, 1 \leq k \leq m \right\} + (m-1)n.$$

Then

$$T \colon H^{q_1(\cdot)} \times \cdots \times H^{q_m(\cdot)} \to L^{q(\cdot)}.$$

Theorem 1.8. *Given $q(\cdot), q_1(\cdot), \ldots, q_m(\cdot), p_1, \ldots, p_m$ as in Theorem 1.7, define p by*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Let T be an m-CZO as in Theorem 1.1 satisfying (1.1) for all $|\alpha| \leq N$ with

$$N > \left\lfloor n\left(\frac{1}{p} - 1\right) \right\rfloor_+ + \max\left\{ \left\lfloor mn\left(\frac{1}{p_k} - 1\right) \right\rfloor_+, \, 1 \leq k \leq m \right\} + mn.$$

Suppose further that T satisfies (1.2) for all $|\alpha| \leq \lfloor n(1/p - 1)\rfloor_+$. Then

$$T \colon H^{q_1(\cdot)} \times \cdots \times H^{q_m(\cdot)} \to H^{q(\cdot)}.$$

Remark 1.9. We chose to prove Theorems 1.7 and 1.8 using Rubio de Francia extrapolation because of the simplicity of this approach. However, it should be possible to prove these results directly, adapting the unweighted proof in [**20**] to the variable exponent setting. Many of the technical, weighted results for the proof contained in Sections 2 and 3 also hold in the variable exponent case: see [**12, 28, 30**].

As we were completing this paper we learned that this approach was taken by Tan [**32**]. He proved bilinear versions of Theorems 1.7 and 1.8, with the additional hypothesis that $(q_k)_+ \leq 1$ for all $1 \leq k \leq m$. We want to thank him for sharing with us a preprint of his work.

Remark 1.10. Our techniques should apply to a wider class of operators, namely, we believe that Theorems 1.1, 1.2, 1.7, and 1.8 will hold for the classes of product type and mixed type operators defined on pages 2 and 3 of [**20**].

The remainder of this paper is organized as follows. In Section 2 we give some basic definitions and theorems about weights that we will use in subsequent sections. In particular, we prove a finite atomic decomposition for weighted Hardy spaces that extends the results in [**12**]. In Section 3 we gather together a number of technical lemmas that we need for the proofs of Theorems 1.1 and 1.2. Then in Sections 4 and 5 we prove these results. Finally, in Section 6 we give some basic facts about variable exponent spaces and prove Theorems 1.7 and 1.8. In fact, we prove more general results which include these theorems as special cases. Their statements, however, require additional facts about variable exponent spaces, and so we delay their statement until the final section.

Throughout this paper, we will use n to denote the dimension of the underlying space, $\mathbb{R}^n$, and will use m to denote the "dimension" of our multilinear operators. By a cube Q we will always mean a cube whose sides are parallel to the coordinate axes, and for $\tau > 1$ let τQ denote the cube with same center such that $\ell(\tau Q) = \tau \ell(Q)$. We define the average of a function f on a cube Q by $f_Q = \fint_Q f \, dx = |Q|^{-1} \int_Q f \, dx$. By C, c, etc. we will mean constants that may depend on the underlying parameters in the problem. Sometimes, to emphasize that they (only) depend on certain parameters, we will write $C(X, Y, Z, \dots)$. The values of these constants may change from line to line. If we write $A \lesssim B$, we mean that $A \leq cB$ for some constant c.

2. Weights and weighted Hardy spaces

Weights and weighted norm inequalities. In this section we give some basic definitions and results about A_p weights. For complete information, we refer the reader to [**13, 15, 16**]. By a weight w we always

mean a non-negative, locally integrable function such that $0 < w(x) < \infty$ a.e. For $1 < p < \infty$, we say that w is in the Muckenhoupt class A_p, denoted by $w \in A_p$, if

$$[w]_{A_p} = \sup_Q \left(\fint_Q w \, dx \right) \left(\fint_Q w^{1-p'} \, dx \right)^{p-1} < \infty.$$

When $p = 1$ we say that $w \in A_1$ if there is a constant C such that, for every cube Q and a.e. $x \in Q$,

$$\fint_Q w \, dx \leq C w(x).$$

The infimum over all such constants will be denoted by $[w]_{A_1}$. The A_p classes are nested: for $1 < p < q < \infty$, $A_1 \subsetneq A_p \subsetneq A_q$. Let A_∞ denote the union of all the A_p classes, $p \geq 1$.

Given $w \in A_\infty$, then w is a doubling measure. More precisely, if $w \in A_p$ for some $p \geq 1$, then it follows from the definition that given any cube Q and $\tau > 1$

$$w(\tau Q) \leq C \tau^{np} w(Q).$$

In the study of multilinear weighted norm inequalities, we often need the fact that the convex hull of A_∞ weights is again in A_∞. The following result can be found, for instance, in [33] or in [19, Lemma 5]. For completeness we sketch a short proof, using a multilinear reverse Hölder inequality: if $w_1, \ldots, w_m \in A_\infty$, $1 < p_1, \ldots, p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then for every cube Q

$$\prod_{k=1}^m \left(\fint_Q w_k \, dx \right)^{\frac{p}{p_k}} \lesssim \fint_Q \prod_{k=1}^m w_k^{\frac{p}{p_k}} \, dx.$$

This was originally proved in the bilinear case by the first author and Neugebauer [11]; for simpler proofs in the multilinear case, see [9, 33].

Lemma 2.1. *Given $m \geq 1$, $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, if $w_1, \ldots, w_m \in A_\infty$, then $\overline{w} = \prod_{k=1}^m w_k^{\frac{p}{p_k}} \in A_\infty$.*

Proof: Since each $w_k \in A_\infty$, by choosing C sufficiently large and $\delta < 1$ sufficiently close to 1, we have that, for every cube Q and $E \subset Q$,

$$\frac{w_k(E)}{w_k(Q)} \leq C \left(\frac{|E|}{|Q|} \right)^\delta.$$

But then, if we apply Hölder's inequality and the multilinear reverse Hölder's inequality, we have that

$$\frac{\overline{w}(E)}{\overline{w}(Q)} \lesssim \frac{\prod_{k=1}^{m} \left(\int_E w_k \, dx \right)^{\frac{p}{p_k}}}{\prod_{k=1}^{m} \left(\int_Q w_k \, dx \right)^{\frac{p}{p_k}}} \leq C \left(\frac{|E|}{|Q|} \right)^{\delta}. \qquad \square$$

There is a close connection between Muckenhoupt weights and the Hardy–Littlewood maximal operator, defined by

$$Mf(x) = \sup_Q \fint_Q |f(y)| \, dy \cdot \chi_Q(x),$$

where the supremum is taken over all cubes Q. We have that if $1 < p < \infty$, then the maximal operator is bounded $L^p(w)$ if and only if $w \in A_p$. Moreover, we have a weighted vector-valued inequality that generalizes the Fefferman–Stein inequality. This was first proved by Andersen and John [1]; for an elementary proof via extrapolation, see [7].

Lemma 2.2. *Given* $1 < p, q < \infty$ *and* $w \in A_p$, *then, for any sequence* $\{f_k\}$ *in* $L^p(w)$,

$$\left\| \left(\sum_k (Mf_k)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_k |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Remark 2.3. Below we will repeatedly apply Lemma 2.2 in the following way. Fix $0 < p < \infty$ and $w \in A_\infty$. Then $w \in A_q$ and without loss of generality we may assume $p < q$. Let $r = \frac{q}{p} > 1$. Given a sequence of cubes Q_k, let $Q_k^* = \tau Q_k$, $\tau > 1$. Then $\chi_{Q_k^*} \lesssim M(\chi_{Q_k})$, and the implicit constant depends only on n and τ. But then by Lemma 2.2 we have that, for any non-negative λ_k,

$$\left\| \sum_k \lambda_k \chi_{Q_k^*} \right\|_{L^p(w)} \lesssim \left\| \sum_k M\big(\lambda_k^{\frac{1}{r}} \chi_{Q_k}^{\frac{1}{r}}\big)^r \right\|_{L^p(w)} = \left\| \left(\sum_k M\big(\lambda_k^{\frac{1}{r}} \chi_{Q_k}^{\frac{1}{r}}\big)^r \right)^{\frac{1}{r}} \right\|_{L^q(w)}^r$$

$$\lesssim \left\| \left(\sum_k \lambda_k \chi_{Q_k} \right)^{\frac{1}{r}} \right\|_{L^q(w)}^r = \left\| \sum_k \lambda_k \chi_{Q_k} \right\|_{L^p(w)}.$$

Below we will need to prove a weighted norm inequality for an m-CZO. To do so, we will make use of some recent developments in the theory of harmonic analysis on the domination of singular integrals by sparse operators. Here we sketch the basic definitions; for further information, see, for instance, [5].

A collection of cubes $\mathcal{S}$ is called a *sparse family* if each cube $Q \in \mathcal{S}$ contains measurable subset $E_Q \subset Q$ such that $|E_Q| \geq \frac{1}{2}|Q|$ and the family $\{E_Q\}_{Q \in \mathcal{S}}$ is pairwise disjoint. Given a sparse family $\mathcal{S}$ we define a linear sparse operator

$$T^{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \fint_Q f(y) \, dy \cdot \chi_Q(x).$$

The following estimate is proved in [**8, 27**].

Proposition 2.4. *If* $1 < q < \infty$ *and* $w \in A_q$, *then, given any sparse linear operator* $T^{\mathcal{S}}$,

$$\|T^{\mathcal{S}} f\|_{L^q(w)} = \left\| \sum_{Q \in \mathcal{S}} \fint_Q f \, dy \cdot \chi_Q \right\|_{L^q(w)} \leq C[w]_{A_q}^{\max(1, \frac{1}{q-1})} \|f\|_{L^q(w)}.$$

In a similar way, given a sparse family $\mathcal{S}$ we define the multilinear sparse operator

$$T^{\mathcal{S}}(f_1, \ldots, f_m)(x) = \sum_{Q \in \mathcal{S}} \prod_{k=1}^m \fint_Q f_k(y_k) \, dy_k \cdot \chi_Q(x).$$

The following pointwise domination theorem was proved in [**24**, Theorem 13.2] (see also [**4**]).

Proposition 2.5. *Let* T *be an* m-*CZO whose kernel satisfies* (1.1) *for any* $N \geq 1$. *Then given any collection* $f_1, \ldots, f_m$ *with bounded functions of compact support, there exists* 3^n *sparse families* S_j *such that*

$$|T(f_1, \ldots, f_m)(x)| \leq C \sum_{j=1}^{3^n} T^{\mathcal{S}_j}(|f_1|, \ldots, |f_m|)(x).$$

Weighted Hardy spaces. In this section we define the weighted Hardy spaces and prove a finite atomic decomposition theorem. In defining them we follow Strömberg and Torchinsky [**31**] and we refer the reader there for more information.

Let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz class of smooth functions. For $N_0 \in \mathbb{N}$ to be a large value determined later, define

$$\mathfrak{F}_{N_0} = \left\{ \varphi \in \mathscr{S}(\mathbb{R}^n) : \int (1 + |x|)^{N_0} \left(\sum_{|\alpha| \leq N_0} \left| \frac{\partial^\alpha}{\partial x^\alpha} \varphi(x) \right|^2 \right) dx \leq 1 \right\}.$$

Fix $0 < p < \infty$ and $w \in A_\infty$; we define the weighted Hardy space $H^p(w)$ to be the set of distributions

$$H^p(w) = \{ f \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{M}_{N_0}(f) \in L^p(w) \}$$

with the quasi-norm
$$\|f\|_{H^p(w)} = \|\mathcal{M}_{N_0}(f)\|_{L^p(w)},$$
where the grand maximal function $\mathcal{M}_{N_0}(f)$ is defined by
$$\mathcal{M}_{N_0}(f)(x) = \sup_{\varphi \in \mathfrak{F}_{N_0}} \sup_{t>0} |\varphi_t * f(x)|.$$
Note that in this definition, N_0 is taken to be a large positive integer, depending on n, p, and w, whose value is chosen so that the usual definitions of unweighted Hardy spaces remain equivalent in the weighted setting. Its exact value does not matter for us.

Given an integer $N > 0$, an (N, ∞) atom is a function a such that there exists a cube Q with $\mathrm{supp}(a) \subset Q$, $\|a\|_\infty \le 1$, and for all $|\beta| \le N$
$$\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0.$$
In [**31**, Chapter VIII] it was shown that every $f \in H^p(w)$ has an atomic decomposition: for every $N \ge s_w$ there exist a sequence of non-negative numbers $\{\lambda_k\}$ and a sequence of smooth (N, ∞) atoms $\{a_k\}$ with $\mathrm{supp}(a_k) \subset Q_k$, such that
$$f = \sum_k \lambda_k a_k,$$
and the sum converges in the sense of distributions and in the $H^p(w)$ quasi-norm. Moreover, we have that
$$\left\| \sum_k \lambda_k \chi_{Q_k} \right\|_{L^p(w)} \lesssim \|f\|_{H^p(w)}.$$

Below, we want to use the atomic decomposition to estimate the norm of an m-CZO. One technical obstacle, however, is that this atomic decomposition may be an infinite sum, and therefore it is not immediate that we can exchange sum and integral in the definition of an m-CZO. For the argument to overcome this problem in the unweighted setting, see [**17**]. Our approach here is different: we show that for a dense subset of $H^p(w)$, we can form the atomic decomposition using a finite sequence of atoms. Our result generalizes a result in the unweighted case from [**26**]; in the weighted case it generalizes results proved in [**12, 28**].

To state our result, note that for $N \ge s_w$, if we define
$$\mathcal{O}_N = \left\{ f \in \mathcal{C}_0^\infty : \int_{\mathbb{R}^n} x^\alpha f(x)\,dx = 0, \quad 0 \le |\alpha| \le N \right\},$$
then $\mathcal{O}_N \cap H^p(w)$ is dense in $H^p(w)$.

Theorem 2.6. *Fix $w \in A_\infty$ and $0 < p < \infty$, and let $N \geq s_w$. For each $f \in \mathcal{O}_N \cap H^p(w)$, there exists a finite sequence of non-negative numbers $\{\lambda_k\}_k$ and a sequence $\{a_k\}$ of (N, ∞) atoms, $\mathrm{supp}(a_k) \subset Q_k$, such that $f = \sum_k \lambda_k a_k$ and*

$$(2.1) \qquad \left\| \sum_k \lambda_k \chi_{Q_k} \right\|_{L^p(w)} \leq C \|f\|_{H^p(w)}.$$

The proof of Theorem 2.6 is gotten by a close analysis of the atomic decomposition given above. To prove it, we use the following technical result. It is adapted from the corresponding result from [**31**, Chapter VIII] (in the weighted case) and from the proof of the unweighted version of Theorem 2.6 in [**26**]. (See also the construction of the atomic decomposition in [**12**].) Indeed weights play almost no role in the result except in (4).

Lemma 2.7. *Fix $w \in A_\infty$, $0 < p < \infty$, and $N \geq s_w$, and let $f \in \mathcal{O}_N \cap H^p(w)$. For each $k \in \mathbb{Z}$, let*

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}_{N_0} f(x) > 2^k\}.$$

Then there exists a sequence $\{\beta_{k,i}\}$ of smooth functions with compact support and a family of cubes $\{Q_{k,i}\}$ with finite overlap such that the following hold:

(1) *For each k and all i, $Q_{k,i} \subset Q_{k,i}^* \subset \Omega_k$, where $Q_{k,i}^* = \tau Q_{k,i}$ for a fix constant $\tau > 1$ and the $Q_{k,i}^*$ also have finite overlap.*

(2) *The $\beta_{k,i}$ are (N, ∞) atoms with $\mathrm{supp}(\beta_{k,i}) \subset Q_{k,i}^*$. In particular, $\sum_i |\beta_{k,i}| \lesssim C$ uniformly for all $k \in \mathbb{Z}$.*

(3) *$f = \sum_{k,i} \lambda_{k,i} \beta_{k,i}$, where the convergence is unconditional both pointwise and in the sense of distributions.*

(4) *$\lambda_{k,i} \lesssim 2^k$ for all k, i and $\sum_{k,i} \lambda_{k,i} \chi_{Q_{k,i}} \lesssim \mathcal{M}_{N_0}(f)$. In particular, $\sum_{k,i} \lambda_{k,i} \beta_{k,i}$ also converges absolutely to f in $L^q(w)$, whenever $q > 1$ is such that $w \in A_q$.*

Proof of Theorem 2.6: Fix $f \in \mathcal{O}_N \cap H^p(w)$; by homogeneity we may assume without loss of generality that $\|f\|_{H^p(w)} = 1$. Then there exists $R > 1$ such that $\mathrm{supp}(f) \subset B(0, R) = B$. Let $B^* = B(0, 4R)$. We claim that, for all $x \notin B^*$,

$$(2.2) \qquad \mathcal{M}_{N_0} f(x) \lesssim w(B)^{\frac{-1}{p}} \|f\|_{H^p(w)} \lesssim \frac{1}{w(B^*)^{\frac{1}{p}}}.$$

To prove this, we argue as in [**12**, Lemma 7.11] (cf. inequality (7.7)). There they showed a pointwise inequality: given any $\varphi \in \mathfrak{F}_{N_0}$ and $t > 0$,

$$|f * \varphi_t(x)| \lesssim \inf_{z \in B_*} \mathcal{M}_{N_0} f(z),$$

where $B_* = B\left(0, \frac{1}{2}R\right)$. Therefore, we have that

$$|f * \varphi_t(x)|^p \lesssim \frac{1}{w(B_*)} \int_{B_*} \mathcal{M}_{N_0} f(z)^p w(z) \, dz \leq \frac{1}{w(B_*)};$$

inequality (2.2) follows if we take the supremum over all $\varphi \in \mathfrak{F}_{N_0}$ and $t > 0$, and note that since $w \in A_\infty$, $w(B^*) \lesssim w(B_*)$.

Now let k_0 be the smallest integer such that for all $k > k_0$, $\Omega_k \subset B^*$. More precisely, by (2.2) we can take k_0 to be the largest integer such that $2^{k_0} \leq C w(B^*)^{\frac{-1}{p}}$.

By Lemma 2.7 we can decompose f as

$$f = \sum_{k,i} \lambda_{k,i} \beta_{k,i},$$

where the $\beta_{k,i}$ are (N, ∞) atoms. We will show that this sum can be rewritten as a finite sum of atoms. Set

$$F_1 = \sum_{k \leq k_0} \sum_i \lambda_{k,i} \beta_{k,i} = f - \sum_{k > k_0} \sum_i \lambda_{k,i} \beta_{k,i}.$$

Since the $\beta_{k,i}$ are supported in $\Omega_k \subset B^*$ for all $k > k_0$, the function F_1 is also supported in B^*. Moreover

$$\|F_1\|_\infty \leq \sum_{k \leq k_0} \left\| \sum_i \lambda_{k,i} |\beta_{k,i}| \right\|_{L^\infty} \lesssim \sum_{k \leq k_0} 2^k = C_1 2^{k_0}.$$

Further, F_1 has vanishing moments up to order N. To see this, fix $|\alpha| \leq N$ and $q > 1$ such that $w \in A_q$. Then, since $\mathrm{supp}(\beta_{k,i}) \subset B^*$,

$$\left\| \sum_{k \leq k_0} \sum_i |x^\alpha| |\lambda_{k,i} \beta_{k,i}| \right\|_{L^1} \leq (4R)^{|\alpha|} \left\| \sum_{k \leq k_0} \sum_i |\lambda_{k,i} \beta_{k,i}| \right\|_{L^q(w)} w^{1-q'}(B^*)^{\frac{1}{q'}}$$

$$\lesssim (4R)^{|\alpha|} \|\mathcal{M}_{N_0} f\|_L^q(w) w^{1-q'}(B^*)^{\frac{1}{q'}}$$

$$\lesssim (4R)^{|\alpha|} \|f\|_L^q(w) w^{1-q'}(B^*)^{\frac{1}{q'}} < \infty.$$

Therefore, the series on the left-hand side converges absolutely, so you can exchange the sum and integral; since each $\beta_{k,i}$ has vanishing moments, so does F_1. Therefore, if we set $a_0 = C_1^{-1} 2^{-k_0} F_1$ then a_0 is an (N, ∞) atom supported in B^*.

To estimate the remaining terms, note that f is a bounded function and so there exists an integer $k_\infty > k_0$ such that $\Omega_k = \emptyset$ for all $k \geq k_\infty$. Thus the sum

$$\sum_{k>k_0} \sum_i \lambda_{k,i} \beta_{k,i} = \sum_{k_0 < k < k_\infty} \sum_i \lambda_{k,i} \beta_{k,i}$$

has finite many terms under the summation of k indices. Further, since the sum $\sum_{k,i} \lambda_{k,i} \chi_{Q_{k,i}} \lesssim \mathcal{M}_{N_0} f$ it converges everywhere. Therefore, for each $k_0 < k < k_\infty$ there exists an integer ρ_k such that

$$\sum_{i>\rho_k} \lambda_{k,i} |\beta_{k,i}| \leq 2^{-k_\infty} w(B^*)^{-\frac{1}{p}}.$$

If we define

$$F_2 = \sum_{k_0 < k < k_\infty} \sum_{i>\rho_k} \lambda_{k,i} \beta_{k,i},$$

then F_2 is supported in B^* and

$$\|F_2\|_\infty \leq \sum_{k_0 < k < k_\infty} 2^{-k_\infty} w(B^*)^{-\frac{1}{p}} \leq C_2 w(B^*)^{-\frac{1}{p}}.$$

Moreover, arguing as we did above for F_1, we have that F_2 has vanishing moments for $|\alpha| \leq N$. Thus if we set $a_\infty = C_2^{-1} w(B^*)^{\frac{1}{p}} F_2$, then a_∞ is an (N, ∞) atom.

Therefore, we have shown that we can decompose f as a finite sum of (N, ∞) atoms:

$$f = (C_1 2^{k_0}) a_0 + \sum_{k_0 < k < k_\infty} \sum_{1 \leq i \leq \rho_k} \lambda_{k,i} \beta_{k,i} + C_2 w(B^*)^{-\frac{1}{p}} a_\infty.$$

It remains to prove that (2.1) holds. But by our choice of k_0, we have that $\|C_1 2^{k_0} \chi_{B^*}\|_{L^p(w)} \leq C$, and clearly $\|w(B^*)^{-\frac{1}{p}} \chi_{B^*}\|_{L^p(w)} \leq C$. Finally, by the weighted Fefferman–Stein inequality (see Remark 2.3), we have that

$$\left\| \sum_{k_0 < k < k_\infty} \sum_{1 \leq i \leq \rho_k} \lambda_{k,i} \chi_{Q^*_{k,i}} \right\|_{L^p(w)} \lesssim \left\| \sum_{k_0 < k < k_\infty} \sum_{1 \leq i \leq \rho_k} \lambda_{k,i} \chi_{Q_{k,i}} \right\|_{L^p(w)}$$

$$\lesssim \|\mathcal{M}_{N_0} f\|_{L^p(w)} \lesssim 1.$$

Since $\|f\|_{H^p(w)} = 1$, we get the desired inequality, and this completes the proof of Theorem 2.6. $\qquad\square$

3. Auxiliary results

In this section we state and prove several lemmas on averaging operators and m-CZOs needed for the proofs of Theorems 1.1 and 1.2.

Averaging operators. We begin with a well-known result on the maximal operator M_μ defined with respect to a measure μ:

$$M_\mu f(x) = \sup_Q \frac{1}{\mu(Q)} \int_Q |f| \, d\mu \cdot \chi_Q(x).$$

For a proof, see [**15**, Chapter II].

Proposition 3.1. *Let μ be a doubling measure on $\mathbb{R}^n$. Then the maximal operator M_μ satisfies the weak $(1,1)$ inequality*

$$(3.1) \qquad \sup_{t>0} t \, \mu(\{x \in \mathbb{R}^n : M_\mu f(x) > t\}) \leq C(\mu) \int_{\mathbb{R}}^n |f| \, d\mu,$$

and for $1 < p < \infty$ the strong (p,p) inequality

$$(3.2) \qquad \int_{\mathbb{R}}^n (M_\mu f)^p \, d\mu \leq C(\mu, p) \int_{\mathbb{R}}^n |f|^p \, d\mu.$$

The next three lemmas on averaging operators are weighted extensions of results from [**18**]. Our proofs, however, are different and are motivated by ideas from [**31**].

Lemma 3.2. *Let μ be a doubling measure on $\mathbb{R}^n$ and fix $0 < p < 1$. Then, given any finite collection $\mathcal{J}$ of cubes and any set $\{f_Q : Q \in \mathcal{J}\}$ of non-negative integrable functions with $\mathrm{supp}(f_Q) \subset Q$,*

$$\left\| \sum_{Q \in \mathcal{J}} f_Q \right\|_{L^p(\mu)} \leq C(\mu, p, n) \left\| \sum_{Q \in \mathcal{J}} a_1^\mu(Q) \chi_Q \right\|_{L^p(\mu)},$$

where

$$a_1^\mu(Q) = \mu(Q)^{-1} \int_Q f_Q(x) \, d\mu(x).$$

Proof: Let $F = \sum_{Q \in \mathcal{J}} f_Q$ and $G = \sum_{Q \in \mathcal{J}} a_1^\mu(Q) \chi_Q$ and for each $t > 0$ let

$$L_t = \{x \in \mathbb{R}^n : G(x) > t\}, \quad U_t = \left\{ y \in \mathbb{R}^n : M_\mu \chi_{L_t}(y) > \frac{1}{4} \right\}.$$

By (3.1) we have that $\mu(U_t) \leq C(\mu)\mu(L_t)$. We can now estimate as follows:

$$\mu(\{x \in \mathbb{R}^n : F(x) > t\}) \leq \mu(U_t) + \mu(U_t^c \cap \{x \in \mathbb{R}^n : F(x) > t\})$$

$$\lesssim \mu(L_t) + \frac{1}{t}\int_{U_t^c} F(x)\,d\mu(x)$$

$$\lesssim \mu(L_t) + \frac{1}{t}\sum_{Q \in \mathcal{J}:Q\cap U_t^c \neq \emptyset}\int_Q f_Q(x)\,d\mu(x)$$

$$\lesssim \mu(L_t) + \frac{1}{t}\sum_{Q \in \mathcal{J}:Q\cap U_t^c \neq \emptyset} a_1^\mu(Q)\mu(Q).$$

If $Q \in \mathcal{J}$ is such that $Q \cap U_t^c \neq \emptyset$, then $M_\mu \chi_{L_t}(z) \leq \frac{1}{4}$ for all $z \in Q \cap U_t^c$. In particular, we have

$$\frac{\mu(L_t \cap Q)}{\mu(Q)} \leq \frac{1}{4},$$

and so $\mu(Q) \leq \frac{4}{3}\mu(L_t^c \cap Q)$ for all $Q \in \mathcal{J}$. Thus we have that

$$\mu(\{x \in \mathbb{R}^n : F(x) > t\}) \lesssim \mu(L_t) + \frac{1}{t}\sum_{Q \in \mathcal{J}} a_1^\mu(Q)\mu(Q \cap L_t^c)$$

$$\lesssim \mu(L_t) + \frac{1}{t}\sum_{Q \in \mathcal{J}} a_1^\mu(Q)\int_{L_t^c} \chi_Q(x)\,d\mu(x)$$

$$\lesssim \mu(L_t) + \frac{1}{t}\int_{L_t^c} G(x)\,d\mu(x).$$

Given this estimate, if we multiply by pt^{p-1} and integrate, by Fubini's theorem we get

$$\|F\|_{L^p(\mu)}^p = \int_0^\infty pt^{p-1}\mu(\{x \in \mathbb{R}^n : F(x) > t\})\,dt$$

$$\lesssim \int_0^\infty pt^{p-1}\mu(\{x \in \mathbb{R}^n : G(x) > t\})\,dt$$

$$+ \int_0^\infty pt^{p-2}\int_{\{x\in\mathbb{R}^n:G(x)\leq t\}} G(x)\,d\mu(x)$$

$$= \int_{\mathbb{R}}^n G(x)^p\,d\mu(x) + \int_{\mathbb{R}}^n G(x)\int_{G(x)}^\infty pt^{p-2}\,dt\,d\mu(x)$$

$$\lesssim \|G\|_{L^p(\mu)}^p. \qquad \square$$

Lemma 3.3. *Let μ be a doubling measure on $\mathbb{R}^n$ and fix $1 \leq p < q < \infty$. Then, given any finite collection of cubes $\mathcal{J}$ and any set $\{f_Q : Q \in \mathcal{J}\}$ of non-negative integrable functions with $\mathrm{supp}(f_Q) \subset Q$,*

$$\left\| \sum_{Q \in \mathcal{J}} f_Q \right\|_{L^p(\mu)} \leq C(\mu, p, q, n) \left\| \sum_{Q \in \mathcal{J}} a_q^\mu(Q) \chi_Q \right\|_{L^p(\mu)},$$

where

$$a_q^\mu(Q) = \left(\frac{1}{\mu(Q)} \int_Q |f_Q(x)|^q \, d\mu(x) \right)^{\frac{1}{q}}.$$

Proof: First suppose that $p > 1$; we estimate by duality. Then there exists non-negative $g \in L^{p'}(\mu)$, $\|g\|_{L^{p'}(d\mu)} = 1$, such that

$$\left\| \sum_{Q \in \mathcal{J}} f_Q \right\|_{L^p(\mu)} = \sum_{Q \in \mathcal{J}} \int_Q f_Q(x) g(x) \, d\mu(x)$$

$$\leq \sum_{Q \in \mathcal{J}} \left(\int_Q f_Q(x)^q \, d\mu(x) \right)^{\frac{1}{q}} \left(\int_Q g(x)^{q'} \, d\mu(x) \right)^{\frac{1}{q'}}$$

$$= \sum_{Q \in \mathcal{J}} a_q^\mu(Q) \mu(Q) \left[\frac{1}{\mu(Q)} \int_Q g^{q'} \, d\mu \right]^{\frac{1}{q'}}$$

$$\leq \sum_{Q \in \mathcal{J}} a_q^\mu(Q) \int_Q M_\mu(g^{q'})(x)^{\frac{1}{q'}} \, d\mu(x)$$

$$\leq \int_{\mathbb{R}^n} \left[\sum_{Q \in \mathcal{J}} a_q^\mu(Q) \chi_Q \right] M_\mu(g^{q'})(x)^{\frac{1}{q'}} \, d\mu(x)$$

$$\leq \left\| \sum_{Q \in \mathcal{J}} a_q^\mu(Q) \chi_Q \right\|_{L^p(\mu)} \| M_\mu(g^{q'})^{\frac{1}{q'}} \|_{L^{p'}(\mu)}$$

$$= \left\| \sum_{Q \in \mathcal{J}} a_q^\mu(Q) \chi_Q \right\|_{L^p(d\mu)} \| M_\mu(g^{q'}) \|_{L^{\frac{p'}{q'}}(\mu)}^{\frac{1}{q'}}$$

$$\lesssim \left\| \sum_{Q \in \mathcal{J}} a_q^\mu(Q) \chi_Q \right\|_{L^p(\mu)};$$

the first and third inequalities follow from Hölder's inequality, and the last from (3.2) (since $p' > q'$) and the fact that $\|g\|_{L^{p'}(\mu)} = 1$.

Finally, when $p = 1$ the proof is essentially the same except that we use use the fact that M_μ is bounded on L^∞. This completes the proof.   □

Lemma 3.4. *Let $w \in A_\infty$, and fix $0 < p < \infty$ and $\max(1, p) < q < \infty$. Then, given any collection of cubes $\{Q_k\}_{k=1}^\infty$ and nonnegative integrable functions $\{g_k\}$ with $\mathrm{supp}(g_k) \subset Q_k$,*

$$\left\| \sum_{k=1}^\infty g_k \right\|_{L^p(w)} \leq C(w, p, q, n) \left\| \sum_{k=1}^\infty \left(\frac{1}{w(Q_k)} \int_{Q_k} g_k(x)^q w(x)\, dx \right)^{\frac{1}{q}} \chi_{Q_k} \right\|_{L^p(w)}.$$

Proof: Since $w \in A_\infty$, the measure $\mu = w(x)\, dx$ is doubling. If $p \geq 1$, then if we fix an arbitrary integer K and apply Lemma 3.3 to the functions $\{g_k\}_{k=1}^K$, we immediately get

$$\left\| \sum_{k=1}^K g_k \right\|_{L^p(w)} \leq C(w, p, q, n) \left\| \sum_{k=1}^K \left(\frac{1}{w(Q_k)} \int_{Q_k} g_k(x)^q w(x)\, dx \right)^{\frac{1}{q}} \chi_{Q_k} \right\|_{L^p(w)}.$$

The desired inequality now follows from Fatou's lemma.

When $0 < p < 1$, we can apply Lemma 3.2 to get the same conclusion, using the fact that

$$\frac{1}{w(Q_k)} \int_{Q_k} g_k(x) w(x)\, dx \leq \left(\frac{1}{w(Q_k)} \int_{Q_k} g_k(x)^q w(x)\, dx \right)^{\frac{1}{q}}. \qquad \square$$

Estimates for m-CZOs. In this section we prove three estimates on m-CZOs.

Lemma 3.5. *Let T be the operator as in Theorem 1.1 and fix $w \in A_q$, $q > 1$. Then, given any collection $f_1, \ldots, f_m$ of bounded functions with compact support,*

$$\|T(f_1, f_2, \ldots, f_m)\|_{L^q(w)} \leq C \|f_1\|_{L^q(w)} \|f_2\|_{L^\infty} \cdots \|f_m\|_{L^\infty}.$$

Proof: By the domination estimate in Proposition 2.5 it will suffice to prove this estimate for any multilinear sparse operator $T^{\mathcal{S}}$ and nonnegative functions $f_1, \ldots, f_m$. By the definition of the sparse operator we have

$$T^{\mathcal{S}}(f_1, \ldots, f_m) \leq \|f_2\|_\infty \cdots \|f_m\|_\infty \sum_{Q \in \mathcal{S}} \fint_Q f_1\, dy \cdot \chi_Q$$

$$= \|f_2\|_\infty \cdots \|f_m\|_\infty T^{\mathcal{S}} f_1,$$

where on the right-hand side we now have a linear sparse operator. But then by Proposition 2.4 we have that

$$\|T^{\mathcal{S}}(f_1, \ldots, f_m)\|_{L^q(w)} \lesssim \|T^{\mathcal{S}} f_1\|_{L^q(w)} \|f_2\|_\infty \cdots \|f_m\|_\infty$$

$$\lesssim \|f_1\|_{L^q(w)} \|f_2\|_\infty \cdots \|f_m\|_\infty. \ \square$$

The following lemma was first prove in [**20**]. For completeness we include its short proof.

Lemma 3.6. *For $1 \leq k \leq m$ let a_k be an (N, ∞) atom supported in Q_k and let c_k be the center of Q_k. Then, given any non-empty subset $\Lambda \subset \{1, \ldots, m\}$, we have that, for all $y \notin \cup_{k \in \Lambda} Q_k^*$,*

$$(3.3) \qquad |T(a_1, \ldots, a_m)(y)| \lesssim \frac{\min\{\ell(Q_k) : k \in \Lambda\}^{n+N+1}}{\left(\sum_{k \in \Lambda} |y - c_k|\right)^{n+N+1}}.$$

In particular, we always have that

$$(3.4) \qquad |T(a_1, \ldots, a_m)|\chi_{(Q_1^* \cap \cdots \cap Q_m^*)^c} \lesssim \prod_{k=1}^{m} \left(M(\chi_{Q_k})\right)^{\frac{n+N+1}{mn}}.$$

Proof: Without loss of generality we may assume that $\Lambda = \{1, \ldots, r\}$ for some $1 \leq r \leq m$ and that

$$\ell(Q_1) = \min\{\ell(Q_k) : k \in \Lambda\}.$$

Fix $y \notin \cup_{k \in \Lambda} Q_k^*$; because a_1 has vanishing moments up to order N, we can rewrite

$$T(a_1, \ldots, a_m)(y) = \int_{\mathbb{R}^{mn}} K(y, y_1, \ldots, y_m) a_1(y_1) \cdots a_m(y_m) \, d\vec{y}$$

$$(3.5) \qquad = \int_{\mathbb{R}^{mn}} \left[K(y, y_1, \ldots, y_m) - P_N(y, y_1, y_2, \ldots, y_m)\right]$$

$$\times \, a_1(y_1) \cdots a_m(y_m) \, d\vec{y}$$

$$= \int_{\mathbb{R}^{mn}} K^1(y, y_1, y_2, \ldots, y_m) a_1(y_1) \cdots a_m(y_m) \, d\vec{y},$$

where

$$P_N(y, y_1, y_2, \ldots, y_m) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial_2^\alpha K(y, c_1, y_2, \ldots, y_m)(y_1 - c_1)^\alpha$$

is the Taylor polynomial of degree N of $K(y, \cdot, y_2, \ldots, y_m)$ at c_1 and

$$(3.6) \quad K^1(y, y_1, \ldots, y_m) = K(y, y_1, \ldots, y_m) - P_N(y, y_1, y_2, \ldots, y_m).$$

By the smoothness condition of the kernel and the fact that $|y - y_k| \approx |y - c_k|$ for all $k \in \Lambda$ and $y_k \in Q_k$ we have that

$$|K(y, y_1, \ldots, y_m) - P_N(y, c_1, y_2, \ldots, y_m)|$$

$$\lesssim |y_1 - c_1|^{N+1} \left(\sum_{k \in \Lambda} |y - c_k| + \sum_{j=2}^{m} |y - y_j|\right)^{-mn-N-1}.$$

Thus

$$|T(a_1,\ldots,a_m)(y)| \lesssim \int_{\mathbb{R}^{mn}} \frac{|y_1 - c_1|^{N+1}|a_1(y_1)| \cdots |a_m(y_m)|}{\left(\sum_{k\in\Lambda} |y - c_k| + \sum_{j=2}^m |y - y_j|\right)^{mn+N+1}} \, d\vec{y}$$

$$\lesssim \int_{\mathbb{R}^{(m-1)n}} \frac{\ell(Q_1)^{n+N+1}}{\left(\sum_{k\in\Lambda}|y-c_k|+\sum_{j=2}^m|y_j|\right)^{mn+N+1}} \, dy_2 \cdots dy_m$$

$$\lesssim \frac{\ell(Q_1)^{n+N+1}}{\left(\sum_{k\in\Lambda} |y - c_k|\right)^{n+N+1}},$$

which implies (3.3).

To prove (3.4), fix $y \in (Q_1^* \cap \cdots \cap Q_m^*)^c$; then there exists a non-empty subset Λ of $\{1,\ldots,m\}$ such that $y \notin Q_k^*$ for all $k \in \Lambda$ and $y \in Q_l^*$ for $l \notin \Lambda$. Then by (3.3) we have that

$$|T(a_1,\ldots,a_m)(y)| \lesssim \frac{\min\{\ell(Q_k) : k \in \Lambda\}^{n+N+1}}{\left(\sum_{k\in\Lambda} |y - c_k|\right)^{n+N+1}}$$

$$\lesssim \prod_{k\in\Lambda} \left(\frac{\ell(Q_k)}{\ell(Q_k) + |y - c_k|}\right)^{\frac{n+N+1}{mn}}$$

$$\lesssim \prod_{k=1}^m \left(\frac{\ell(Q_k)}{\ell(Q_k) + |y - c_k|}\right)^{\frac{n+N+1}{mn}}.$$

Inequality (3.4) follows from the definition of the maximal operator. □

Lemma 3.7. *Given $w \in A_q$, $1 \leq q < \infty$, for $1 \leq k \leq m$ let a_k be an (N,∞) atom supported in Q_k and let c_k be the center of Q_k. Suppose Q_1 is the cube such that $\ell(Q_1) = \min\{\ell(Q_k) : 1 \leq k \leq m\}$. Then*

$$(3.7) \quad \|T(a_1,\ldots,a_m)\chi_{Q_1^*}\|_{L^q(w)} \lesssim w(Q_1)^{\frac{1}{q}} \prod_{l=1}^m \inf_{z\in Q_1} M(\chi_{Q_l})(z)^{\frac{n+N+1}{mn}}.$$

Proof: Since the A_p classes are nested, we may assume without loss of generality that $q > 1$. To prove (3.7) we consider two cases: $Q_1^* \cap Q_k^* \neq \emptyset$ for all $2 \leq k \leq m$ or this intersection is empty for at least one value of k. In the first case, since $\ell(Q_1) = \min\{\ell(Q_k) : 1 \leq k \leq m\}$ we have $Q_1^* \subset 3Q_k^*$ for all $1 \leq k \leq m$. This implies

$$\inf_{z\in Q_1} M(\chi_{Q_k})(z) \gtrsim 1,$$

for all $1 \leq k \leq m$, and so Lemma 3.5 yields

$$
\begin{aligned}
\|T(a_1, \ldots, a_m)\chi_{Q_1^*}\|_{L^q(w)} &\leq \|T(a_1, \ldots, a_m)\|_{L^q(w)} \\
&\lesssim \|a_1\|_{L^q(w)}\|a_2\|_{L^\infty}\cdots\|a_m\|_{L^\infty} \\
&\lesssim w(Q_1)^{\frac{1}{q}}\prod_{k=1}^{m}\inf_{z\in Q_1}M(\chi_{Q_k})(z)^{\frac{n+N+1}{mn}}.
\end{aligned}
$$

In the second case, since $Q_1^* \cap Q_k^* = \emptyset$ for some k, the set

$$
\Lambda = \{2 \leq k \leq m : Q_1^* \cap Q_k^* = \emptyset\}
$$

is non-empty. Fix any point $y \in \mathbb{R}^n$. Then arguing as in the previous proof we have that

$$
(3.8)\quad T(a_1, \ldots, a_m)(y) = \int_{\mathbb{R}^{mn}} K^1(y, y_1, y_2, \ldots, y_m)a_1(y_1)\cdots a_m(y_m)\,d\vec{y},
$$

where $K^1(y, y_1, \ldots, y_m)$ is defined by (3.6). For $y_1 \in Q_1$ we have that, for some $\xi_1 \in Q_1$ and for all $y_l \in Q_l$, $1 \leq l \leq m$,

$$
(3.9)\quad |K^1(y, y_1, \ldots, y_m)| \leq C\ell(Q_1)^{N+1}\left(|y-\xi_1| + \sum_{j=2}^{m}|y-y_j|\right)^{-mn-N-1}.
$$

For all $k \in \Lambda$, since $Q_1^* \cap Q_k^* = \emptyset$, $|y-\xi_1| + |y-y_k| \geq |\xi_1 - y_k| \gtrsim |c_1 - c_k|$. Therefore, for all $y_1 \in Q_1^*$ and $y_k \in Q_k$, $k \in \Lambda$,

$$
|K^1(y, y_1, \ldots, y_m)| \lesssim \ell(Q_1)^{N+1}\left(\sum_{k\in\Lambda}|c_1 - c_k| + \sum_{j=2}^{m}|y-y_j|\right)^{-mn-N-1}.
$$

If we combine this inequality with (3.8), we get

$$
\begin{aligned}
|T(a_1, \ldots, a_m)(y)| &\lesssim \frac{\ell(Q_1)^{n+N+1}}{\left(\sum_{k\in\Lambda}|c_1 - c_k|\right)^{n+N+1}} \\
&\lesssim \frac{\ell(Q_1)^{n+N+1}}{\left(\sum_{k\in\Lambda}[\ell(Q_1) + |c_1 - c_k| + \ell(Q_k)]\right)^{n+N+1}}.
\end{aligned}
$$

Since $Q_1^* \subset 3Q_l^*$ for all $l \notin \Lambda$, the last inequality gives us

$$
\|T(a_1, \ldots, a_m)\|_{L^\infty} \lesssim \prod_{k=1}^{m}\inf_{z\in Q_1}M(\chi_{Q_k})(z)^{\frac{n+N+1}{mn}};
$$

since $w \in A_q$ is doubling, this implies that

$$
\|T(a_1, \ldots, a_m)\chi_{Q_1^*}\|_{L^q(w)} \lesssim w(Q_1)^{\frac{1}{q}}\prod_{k=1}^{m}\inf_{z\in Q_1}M(\chi_{Q_k})(z)^{\frac{n+N+1}{mn}}.
$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Proof of Theorem 1.1

For $1 \leq k \leq m$, let $w_k \in A_\infty$ and fix arbitrary functions $f_k \in H^{p_k}(w_k) \cap \mathcal{O}_N(\mathbb{R}^n)$. By Theorem 2.6, we have the finite atomic decompositions

(4.1) $$f_k = \sum_{j_k=1}^{N_0} \lambda_{k,j_k} a_{k,j_k},$$

where $\lambda_{k,j_k} \geq 0$ and a_{k,j_k} are (N, ∞)-atoms that satisfy

$$\operatorname{supp}(a_{k,j_k}) \subset Q_{k,j_k}, \quad |a_{k,j_k}| \leq \chi_{Q_{k,j_k}}, \quad \int_{Q_{k,j_k}} x^\alpha a_{k,j_k}(x)\, dx = 0$$

for all $|\alpha| \leq N$, and

(4.2) $$\left\| \sum_{j_k} \lambda_{j_k} \chi_{Q_{j_k}} \right\|_{L^{p_k}(w_k)} \leq C \|f_k\|_{H^{p_k}(w_k)}.$$

Set $\overline{w} = \prod_{k=1}^m w_k^{\frac{p}{p_k}}$. Again by Theorem 2.6, it will suffice to prove that

$$\|T(f_1, \ldots, f_m)\|_{L^p(\overline{w})} \lesssim \prod_{k=1}^m \left\| \sum_{j_k} \lambda_{k,j_k} \chi_{Q_{k,j_k}} \right\|_{L^{p_k}(w_k)}.$$

Since T is m-linear, we have that, for a.e. $x \in \mathbb{R}^n$,

$$T(f_1, \ldots, f_m)(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} T(a_{1,j_1}, \ldots, a_{m,j_m})(x).$$

Given a cube Q, let $Q^* = 2\sqrt{n}Q$. For each m-tuple, $(j_1, \ldots, j_m)$, define $R_{j_1,\ldots,j_m}$ to be the smallest cube among $Q^*_{1,j_1}, \ldots, Q^*_{m,j_m}$. To estimate $\|T(f_1, \ldots, f_m)\|_{L^p(\overline{w})}$ we will split $T(f_1, \ldots, f_m)$ into two parts:

$$|T(f_1, \ldots, f_m)(x)| \leq G_1(x) + G_2(x),$$

where

$$G_1(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} |T(a_{1,j_1}, \ldots, a_{m,j_m})| \chi_{R_{j_1,\ldots,j_m}}(x)$$

and

$$G_2(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} |T(a_{1,j_1}, \ldots, a_{m,j_m})| \chi_{(R_{j_1,\ldots,j_m})^c}(x).$$

We first estimate $\|G_2\|_{L^p(\overline{w})}$. By (3.4) we have that

$$|T(a_{1,j_1},\ldots,a_{m,j_m})(x)|\chi_{(R_{j_1,\ldots,j_m})^c}(x) \lesssim \prod_{k=1}^m M(\chi_{Q_{k,j_k}})(x)^{\frac{n+N+1}{mn}};$$

thus

$$G_2 \lesssim \sum_{j_1}\cdots\sum_{j_m}\prod_{k=1}^m \lambda_{k,j_k} M(\chi_{Q_{k,j_k}})^{\frac{n+N+1}{mn}} = \prod_{k=1}^m\left[\sum_{j_k}\lambda_{k,j_k} M(\chi_{Q_{k,j_k}})^{\frac{n+N+1}{mn}}\right].$$

By condition (1.3), Hölder's inequality and the weighted Fefferman–Stein vector-valued inequality (see Remark 2.3), we get

$$\|G_2\|_{L^p(\overline{w})} \lesssim \prod_{k=1}^m\left\|\sum_{j_k}\lambda_{k,j_k}\chi_{Q_{k,j_k}}\right\|_{L^{p_k}(w_k)}.$$

We now estimate the norm of G_1. Since $\overline{w} \in A_\infty$ by Lemma 2.1, we can choose $q > \max(1,p)$ such that $\overline{w} \in A_q$. Then by Lemma 3.5 we have that

$$\left(\frac{1}{\overline{w}(R_{j_1,\ldots,j_m})}\int_{R_{j_1,\ldots,j_m}}|T(a_{1,j_1},\ldots,a_{m,j_m})|^q(x)\overline{w}(x)\,dx\right)^{\frac{1}{q}}$$
$$\lesssim \prod_{k=1}^m \inf_{z\in R_{j_1,\ldots,j_m}} M(\chi_{Q_{k,j_k}})(z)^{\frac{n+N+1}{mn}}.$$

If we combine this inequality, Lemma 3.4, Hölder's inequality, and the Fefferman–Stein vector-valued inequality (again see Remark 2.3), we get the following estimate:

$$\|G_1\|_{L^p(\overline{w})} \lesssim \left\|\sum_{j_1\ldots,j_m}\prod_{k=1}^m \lambda_{k,j_k}\left(\frac{1}{\overline{w}(R_{j_1,\ldots,j_m})}\right.\right.$$
$$\left.\left.\times\int_{R_{j_1,\ldots,j_m}}|T(a_{1,j_1},\ldots,a_{m,j_m})|^q(x)\overline{w}(x)\,dx\right)^{\frac{1}{q}}\chi_{R_{j_1,\ldots,j_m}}\right\|_{L^p(\overline{w})}$$
$$\lesssim \left\|\sum_{j_1\ldots,j_m}\left(\prod_{k=1}^m \lambda_{k,j_k}\right)\right.$$
$$\left.\times\left(\prod_{k=1}^m \inf_{z\in R_{j_1,\ldots,j_m}} M(\chi_{Q_{k,j_k}})(z)^{\frac{n+N+1}{mn}}\right)\chi_{R_{j_1,\ldots,j_m}}\right\|_{L^p(\overline{w})}$$
$$\lesssim \left\|\prod_{k=1}^m\left(\sum_{j_k}\lambda_{k,j_k} M(\chi_{Q_{k,j_k}})^{\frac{n+N+1}{mn}}\right)\right\|_{L^p(\overline{w})}$$

$$\lesssim \prod_{k=1}^{m} \left\| \sum_{j_k} \lambda_{k,j_k} M(\chi_{Q_{k,j_k}})^{\frac{n+N+1}{mn}} \right\|_{L^{p_k}(w_k)}$$

$$\lesssim \prod_{k=1}^{m} \left\| \sum_{j_k} \lambda_{k,j_k} \chi_{Q_{k,j_k}} \right\|_{L^{p_k}(w_k)}.$$

If we combine the estimates for G_1 and G_2, we get the desired inequality.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1. Instead of estimating the norm of T, we will estimate the norm of $M_\phi \circ T$, where M_ϕ is the non-tangential maximal operator

$$M_\phi f(x) = \sup_{0 < t < \infty} \sup_{|y-x| < t} |\phi_t * f(y)|,$$

where $\phi \in C_0^\infty$ and $\text{supp}(\phi) \subset B(0, 1)$. We will use the that the Hardy space can be characterized by using the non-tangential maximal function M_ϕ with the norm

$$\|f\|_{H^p(w)} \approx \|M_\phi f\|_{L^p(w)}.$$

See [31]; this equivalence is guaranteed by our choice of N_0 sufficiently large. Throughout this section we fix a choice of ϕ.

In this section, we fix the smooth approximate identity ϕ supported in the unit ball. The following lemma was first proved in [20]; it is the essential part in the proof of Theorem 1.2 and so we repeat the proof here for the convenience of the reader. Hereafter, given a cube Q, let $Q^{**} = 4nQ$.

Lemma 5.1. *For $1 \le k \le m$, let a_k be (N, ∞) atoms with $\text{supp}(a_k) \subset Q_k$. Suppose that Q_1 is such that $\ell(Q_1) = \min\{\ell(Q_k) : 1 \le k \le m\}$. Then, for all $x \notin Q_1^{**}$, we have*

$$(5.1) \quad M_\phi T(a_1, \ldots, a_m)(x) \lesssim \prod_{l=1}^{m} M(\chi_{Q_l})(x)^{\frac{n+N+1}{mn}}$$

$$+ M(\chi_{Q_1})(x)^{\frac{n+s_{\overline{w}}+1}{n}} \prod_{l=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{N-s_{\overline{w}}}{mn}},$$

where T is the operator in Theorem 1.1.

Proof: Fix $x \in (Q_1^{**})^c$, $0 < t < \infty$, and $y \in \mathbb{R}^n$ such that $|y - x| < t$. To prove (5.1) it will suffice to show that

$$|\phi_t * T(a_1, \ldots, a_m)(y)| \lesssim \prod_{l=1}^{m} M(\chi_{Q_l})(x)^{\frac{n+N+1}{mn}}$$

$$+ M(\chi_{Q_1})(x)^{\frac{n+s_{\overline{w}}+1}{n}} \prod_{l=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{N-s_{\overline{w}}}{mn}},$$

where the implicit constant does not depend on x, y, and t. We will consider two cases.

Case 1: $t > \frac{1}{1000 n^2} |x - c_1|$. We will exploit the cancellation in (1.2) to show that

$$(5.2) \quad |\phi_t * T(a_1, \ldots, a_m)(y)| \lesssim M(\chi_{Q_1})(x)^{\frac{n+s_{\overline{w}}+1}{n}} \prod_{l=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{N-s_{\overline{w}}}{mn}}.$$

By (1.2) we have

$$\phi_t * T(a_1, \ldots, a_m)(y) = \int \phi_t(y - z) T(a_1, \ldots, a_m)(z) \, dz$$

$$= \int \left(\phi_t(y - z) - \sum_{|\alpha| \le s_{\overline{w}}} \frac{\partial^\alpha [\phi_t](y - c_1)}{\alpha!} (c_1 - z)^\alpha \right) T(a_1, \ldots, a_m)(z) \, dz.$$

Note that, by Taylor's theorem,

$$\left| \phi_t(y - z) - \sum_{|\alpha| \le s_{\overline{w}}} \frac{\partial^\alpha [\phi_t](y - c_1)}{\alpha!} (c_1 - z)^\alpha \right| \lesssim \frac{|z - c_1|^{s_{\overline{w}}+1}}{t^{n+s_{\overline{w}}+1}}$$

for all $y, z \in \mathbb{R}^n$ and all $t \in (0, \infty)$. Since $t \gtrsim |x - c_1|$ and $x \notin Q_1^{**}$, we have

$$|\phi_t * T(a_1, \ldots, a_m)(y)| \lesssim \int \frac{|z - c_1|^{s_{\overline{w}}+1}}{t^{n+s_{\overline{w}}+1}} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$\lesssim \left(\frac{\ell(Q_1)}{|x - c_1|} \right)^{n+s_{\overline{w}}+1} \frac{1}{\ell(Q_1)^{n+s_{\overline{w}}+1}} \int |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$\lesssim M(\chi_{Q_1})(x)^{\frac{n+s_{\overline{w}}+1}{n}} \frac{1}{\ell(Q_1)^{n+s_{\overline{w}}+1}} \int |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz.$$

Hence, to prove (5.2) it remains to show that

$$(5.3) \quad \frac{1}{\ell(Q_1)^{n+s_{\overline{w}}+1}} \int |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$\lesssim \prod_{l=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{N-s_{\overline{w}}}{mn}}.$$

If we split the integral on the left-hand side of (5.3) over Q_1^* and $(Q_1^*)^c$, we can estimate as follows:

$$\int |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$\lesssim \int_{Q_1^*} |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$+ \int_{(Q_1^*)^c} |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$\lesssim \ell(Q_1)^{s_{\overline{w}}+1} \int_{Q_1^*} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$+ \int_{(Q_1^*)^c} |z - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(z)| \, dz.$$

By (3.7), we can estimate the first integral in the last inequality by

$$(5.4) \quad \ell(Q_1)^{s_{\overline{w}}+1} \int_{Q_1^*} |T(a_1, \ldots, a_m)(z)| \, dz$$

$$\lesssim \ell(Q_1)^{n+s_{\overline{w}}+1} \prod_{l=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{n+N+1}{mn}}.$$

To estimate the second integral, we need to exploit carefully the smoothness of the kernel. Recall the representation of $T(a_1, \ldots, a_m)(z)$ in (3.8). Denote

$$J = \{2 \le l \le m : Q_1^{**} \cap Q_l^{**} = \emptyset\}.$$

For $z \notin Q_1^*$, $\xi_1 \in Q_1$, we have $|z - \xi_1| \approx |z - c_1| \ge \ell(Q_1)$. Also, for $l \in J$ and $z_l \in Q_l^*$,

$$|z - \xi_1| + |z - z_l| \ge |\xi_1 - z_l| \gtrsim |c_1 - c_l|.$$

We now estimate $K^1(z, z_1, \ldots, z_m)$ in (3.9) to get

$$|T(a_1, \ldots, a_m)(z)|$$
$$\lesssim \int_{(\mathbb{R}^n)^m} \frac{\ell(Q_1)^{N+1} \chi_{Q_1}(z_1) \, dz_1 \cdots dz_m}{\left(\ell(Q_1) + |z - c_1| + \sum_{l \in J} |c_1 - c_l| + \sum_{l=2}^{m} |z - z_l| \right)^{mn+N+1}}$$

for all $z \in (Q_1^*)^c$. Thus

$$\int_{(Q_1^*)^c} |y - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(y)| \, dy$$

$$\lesssim \int_{\mathbb{R}^n \times (\mathbb{R}^n)^m} \frac{|y - c_1|^{s_{\overline{w}}+1} \ell(Q_1)^{N+1} \chi_{Q_1}(y_1) \, dy_1 \cdots dy_m \, dy}{\left(\ell(Q_1) + |y - c_1| + \sum_{l \in J} |c_1 - c_l| + \sum_{l=2}^{m} |y - y_l| \right)^{mn+N+1}}$$

$$\lesssim \ell(Q_1)^{n+s_{\overline{w}}+1} \prod_{l \in J} \left(\frac{\ell(Q_l)}{\ell(Q_1) + |c_1 - c_l|} \right)^{\frac{N - s_{\overline{w}}}{m}}.$$

Note that $1 \lesssim \inf_{z \in Q_1} M(\chi_{Q_l})(z)$ if $Q_1^{**} \cap Q_l^{**} \neq \emptyset$ and, for all $l \in J$,

$$\frac{\ell(Q_l)}{\ell(Q_1) + |c_1 - c_l|} \lesssim \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{1}{n}}.$$

Therefore

$$(5.5) \quad \int_{(Q_1^*)^c} |y - c_1|^{s_{\overline{w}}+1} |T(a_1, \ldots, a_m)(y)| \, dy$$

$$\lesssim \ell(Q_1)^{n+s_{\overline{w}}+1} \prod_{l=1}^{m} \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{N - s_{\overline{w}}}{mn}}.$$

Now we combine (5.4) and (5.5) we get (5.3), which completes the proof of Case 1.

Case 2: $t \leq \frac{1}{1000n^2} |x - c_1|$. In this case, we will show that

$$(5.6) \quad |\phi_t * T(a_1, \ldots, a_m)(y)| \lesssim \prod_{l=1}^{m} M(\chi_{Q_l})(x)^{\frac{n+N+1}{mn}}.$$

Since $\mathrm{supp}(\phi) \subset B(0, 1)$ and $|y - x| < t$,

$$|\phi_t * T(a_1, \ldots, a_m)(y)| \leq \int_{B(y,t)} t^{-n} |\phi(t^{-1}(y - z)) T(a_1, \ldots, a_m)(z)| \, dz$$

$$(5.7) \qquad\qquad \lesssim \sup_{z \in B(y,t)} |T(a_1, \ldots, a_m)(z)|$$

$$\lesssim \sup_{z \in B(x,2t)} |T(a_1, \ldots, a_m)(z)|.$$

Let $\Lambda = \{1 \le l \le m : x \notin Q_k^{**}\}$. For $z \in B(x, 2t)$, $\xi_1 \in Q_1$, we have

$$|x - c_1| \le |x - z| + |z - c_1| \le 2t + |\xi_1 - c_1| + |z - \xi_1|$$
$$\le \frac{1}{500n^2}|x - c_1| + \frac{1}{2}|x - c_1| + |z - \xi_1|;$$

hence

$$t \lesssim |x - c_1| \lesssim |z - \xi_1|.$$

For $l \in \Lambda$ and $z_l \in Q_l$, since $x \notin Q_l^{**}$,

$$|x - c_l| \le 2|x - z_l| \le 2|x - z| + 2|z - z_l| \le 4t + 2|z - z_k| \lesssim |z - \xi_1| + |z - z_k|.$$

Recall the formula for $T(a_1, \ldots, a_m)(z)$ in (3.8); we estimate $K^1(z, z_1, \ldots, z_m)$ in (3.9) to get

$$|T(a_1, \ldots, a_m)(z)| \lesssim \int_{(\mathbb{R}^n)^m} \frac{\ell(Q_1)^{N+1} \chi_{Q_1}(z_1)\, dz_1 \cdots dz_m}{\left(\sum_{l=2}^m |z - z_l| + \sum_{k \in \Lambda} |x - c_k|\right)^{mn+N+1}}$$

for all $z \in B(x, 2t)$. From this we get that

$$\begin{aligned}
\sup_{z \in B(x,2t)} |T(a_1, \ldots, a_m)(z)| &\lesssim \prod_{l \in \Lambda} \frac{\ell(Q_l)^{\frac{n+N+1}{|\Lambda|}} \chi_{(Q_l^{**})^c}(x)}{|x - c_l|^{\frac{n+N+1}{|\Lambda|}}} \prod_{k \notin \Lambda} \chi_{Q_k^{**}}(x) \\
&\lesssim \prod_{l=1}^m M(\chi_{Q_l})(x)^{\frac{n+N+1}{mn}}.
\end{aligned}$$

(5.8)

Combining (5.7) and (5.8) gives (5.6). This completes Case 2 and so completes the proof. $\qquad\square$

The next lemma is an immediate consequence of Lemma 3.7 and the fact that M_ϕ is bounded on $L^q(w)$ if $w \in A_q$ (since it is controlled pointwise by the Hardy–Littlewood maximal operator; cf. [**15**]).

Lemma 5.2. *Given $w \in A_q$, $1 \le q < \infty$, for $1 \le k \le m$ let a_k be an (N, ∞) atom supported in Q_k. Suppose Q_1 is the cube such that $\ell(Q_1) = \min\{\ell(Q_k) : 1 \le k \le m\}$. Then*

$$\|M_\phi T(a_1, \ldots, a_m) \chi_{Q_1^{**}}\|_{L^q(w)} \lesssim w(Q_1)^{\frac{1}{q}} \prod_{l=1}^m \inf_{z \in Q_1} M(\chi_{Q_l})(z)^{\frac{n+N+1}{mn}}.$$

Proof of Theorem 1.2: Fix $w_k \in A_\infty$, $1 \le k \le m$, and define $\overline{w} = \prod_{k=1}^m w_k^{\frac{p}{p_k}}$. Fix $f_k \in H^{p_k}(w_k) \cap \mathcal{O}_N(\mathbb{R}^n)$, $1 \le k \le m$. We will show that

$$(5.9) \qquad \|M_\phi T(f_1, \ldots, f_m)\|_{L^p(\overline{w})} \lesssim \|f_1\|_{H^{p_1}(w_1)} \cdots \|f_m\|_{H^{p_m}(w_m)}.$$

Form the atomic decompositions of the functions f_k as in the proof of Theorem 1.1 to get (4.1) and (4.2). Then to prove (5.9), it is enough to show that

$$(5.10) \qquad \|M_\phi T(f_1, \ldots, f_m)\|_{L^p(\overline{w})} \lesssim \prod_{k=1}^m \left\| \sum_{j_k} \lambda_{k,j_k} \chi_{Q_{k,j_k}} \right\|_{L^{p_k}(w_k)}.$$

Since $M_\phi \circ T$ is multi-sublinear, we can write

$$M_\phi T(f_1, \ldots, f_m)(x) \leq G_1(x) + G_2(x),$$

where

$$G_1(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \ldots \lambda_{m,j_m} M_\phi T(a_{1,j_1}, \ldots, a_{m,j_m}) \chi_{R_{j_1,\ldots,j_m}}(x)$$

and

$$G_2(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} M_\phi T(a_{1,j_1}, \ldots, a_{m,j_m}) \chi_{(R_{j_1,\ldots,j_m})^c}(x).$$

Here $R_{j_1,\ldots,j_m}$ is the smallest cube among $Q^{**}_{1,j_1}, \ldots, Q^{**}_{m,j_m}$.

A similar argument as in the proof of Theorem 1.1 with Lemma 5.2 in place of Lemma 3.7 gives

$$(5.11) \qquad \|G_1\|_{L^p(\overline{w})} \lesssim \prod_{k=1}^m \left\| \sum_{j_k} \lambda_{k,j_k} \chi_{Q_{k,j_k}} \right\|_{L^{p_k}(w_k)}.$$

We now estimate the norm of G_2. By Lemma 5.1 we get that

$$G_2(x) \lesssim G_{21}(x) + G_{22}(x),$$

where

$$G_{21}(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} \prod_{k=1}^m M(\chi_{Q_{k,j_k}})(x)^{\frac{n+N+1}{mn}}$$

and

$$G_{22}(x) = \sum_{j_1} \cdots \sum_{j_m} \lambda_{1,j_1} \cdots \lambda_{m,j_m} M(\chi_{R_{j_1,\ldots,j_m}})(x)^{\frac{n+s_{\overline{w}}+1}{n}}$$

$$\times \prod_{l=1}^m \inf_{z \in R_{j_1,\ldots,j_m}} M(\chi_{Q_{l,j_l}})(z)^{\frac{N-s_{\overline{w}}}{mn}}.$$

The function G_{21} can be estimated by essentially the same argument used for G_1 to get

$$(5.12) \qquad \|G_{21}\|_{L^p(\overline{w})} \lesssim \prod_{k=1}^m \left\| \sum_{j_k} \lambda_{k,j_k} \chi_{Q_{k,j_k}} \right\|_{L^{p_k}(w_k)}.$$

To estimate G_{22}, since $\frac{(n+s_{\overline{w}}+1)p}{n} > 1$, we use (1.4) and the Fefferman–Stein vector-valued inequality (cf. Remark 2.3) to get

$$\|G_{22}\|_{L^p(\overline{w})} \lesssim \left\|\sum_{j_1}\cdots\sum_{j_m} \lambda_{1,j_1}\cdots\lambda_{m,j_m}\chi_{R_{j_1,\ldots,j_m}}\right.$$

$$\left.\times \prod_{l=1}^{m} \inf_{z\in R_{j_1,\ldots,j_m}} M(\chi_{Q_{l,j_l}})(z)^{\frac{N-s_{\overline{w}}}{mn}}\right\|_{L^p(\overline{w})}$$

$$(5.13) \qquad \leq \left\|\sum_{j_1}\cdots\sum_{j_m} \lambda_{1,j_1}\cdots\lambda_{m,j_m}\prod_{k=1}^{m} M(\chi_{Q_{k,j_k}})^{\frac{N-s_{\overline{w}}}{mn}}\right\|_{L^p(\overline{w})}$$

$$\leq \prod_{k=1}^{m}\left\|\sum_{j_k}\lambda_{k,j_k} M(\chi_{Q_{k,j_k}})^{\frac{N-s_{\overline{w}}}{mn}}\right\|_{L^{p_k}(w_k)}$$

$$\lesssim \prod_{k=1}^{m}\left\|\sum_{j_k}\lambda_{k,j_k}\chi_{Q_{k,j_k}}\right\|_{L^{p_k}(w_k)}.$$

If we combine (5.11), (5.12), and (5.13), we get (5.10) and this completes the proof. □

6. Variable Hardy spaces: Proof of Theorems 1.7 and 1.8

In this section we prove Theorems 1.7 and 1.8. In fact, we will prove two more general results that include these theorems as special cases. To do so, we first recall some basic facts about the variable Lebsesgue spaces. For complete information we refer the reader to [6].

Let $\mathcal{P}_0(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot)\colon \mathbb{R}^n \to (0,\infty)$. Define

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

Given $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ define $L^{p(\cdot)} = L^{p(\cdot)}(\mathbb{R}^n)$ to be the set of all measurable functions f such that, for some $\lambda > 0$,

$$\rho(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This becomes a quasi-Banach space with the "norm"

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho(f/\lambda) \leq 1\}.$$

If $p_- \geq 1$, $L^{p(\cdot)}$ is a Banach space; if $p(\cdot) = p$ a constant, then $L^{p(\cdot)} = L^p$ with equality of norms.

If the maximal operator is bounded on $L^{p(\cdot)}$ we write that $p(\cdot) \in \mathcal{B}$. A necessary condition for this to be the case is that $p_- > 1$. A sufficient condition is that $1 < p_- \leq p_+ < \infty$ and $p(\cdot)$ is log-Hölder continuous: i.e., (1.6) and (1.7) hold. However, this continuity condition is not necessary: see [**6**] for a detailed discussion of this problem.

Given $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, the variable Hardy space $H^p(\cdot)$ is defined to be the set of all distributions f such that $\mathcal{M}_{N_0} f \in L^{p(\cdot)}$. Again, we here assume $N_0 > 0$ is a sufficiently large constant so that all the standard definitions of the classical Hardy spaces are equivalent. These spaces were examined in detail in [**12**] (see also [**28**]).

A very important tool for proving norm inequalities in spaces of variable exponents is the extension of the Rubio de Francia theory of extrapolation to the scale of variable Lebesgue spaces. For the history and application of this approach for linear operators, see [**6, 7**]. To prove Theorems 1.7 and 1.8 we will use a multilinear version due to the first author and Naibo [**10**]. They only stated their proof for the bilinear case, but the same proof immediately extends to the general multilinear setting.

Theorem 6.1. *Let $\mathcal{F} = \{(f_1, \ldots, f_m, F)\}$ be a family of $(m+1)$-tuples of non-negative, measurable functions on $\mathbb{R}^n$. Suppose that there exist indices $0 < p_1, \ldots, p_m, p < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ such that, for all weights $w_k \in A_1$, $1 \leq k \leq m$, and $\overline{w} = \prod_{k=1}^{m} w_k^{\frac{p}{p_k}}$,*

$$(6.1) \qquad \|F\|_{L^p(\overline{w})} \lesssim \|f_1\|_{L^{p_1}(w_1)} \cdots \|f_m\|_{L^{p_m}(w_m)}$$

for all $(f_1, \ldots, f_m, F)$ such that $F \in L^p(\overline{w})$, and where the implicit constant depends only on n, p_k, and $[w_k]_{A_1}$, $1 \leq k \leq m$. Let $q_1(\cdot), \ldots, q_m(\cdot)$, $q(\cdot) \in \mathcal{P}_0$ be such that

$$\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)},$$

$p_k < (q_k)_-$, $1 \leq k \leq m$, and $q_k(\cdot)/p_k \in \mathcal{B}$. Then

$$(6.2) \qquad \|F\|_{L^{q(\cdot)}} \lesssim \|f_1\|_{L^{q_1(\cdot)}} \cdots \|f_m\|_{L^{q_m(\cdot)}}$$

provided $\|F\|_{L^{q(\cdot)}} < \infty$. The implicit constant only depends on n and $q_k(\cdot)$, $1 \leq k \leq m$.

Remark 6.2. In [**10**], the hypothesis on the exponents $q_k(\cdot)$ was stated as $(q_k(\cdot)/p_k)' \in \mathcal{B}$, where this exponent is the conjugate exponent, defined pointwise by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. It was stated in this way for technical reasons related to the proof. However, these two hypotheses are equivalent: see [**6**, Corollary 4.64].

The one technical obstacle in applying Theorem 6.1 is constructing the family $\mathcal{F}$ to satisfy the hypotheses that the left-hand sides of (6.1) and (6.2) are finite and that the resulting family is large enough that the desired result can be proved via a density argument. In our case we will use the atomic decomposition in the weighted and variable Hardy spaces. As we noted in Section 2, given $w \in A_\infty$ and $0 < p < \infty$, every $f \in H^p(w)$ can be written as the sum

$$(6.3) \qquad f = \sum_k \lambda_k a_k,$$

where $\lambda_k \geq 0$ and the a_k are (N, ∞) atoms, provided $N \geq s_w$. Moreover, this series converges both in the sense of distributions and in $H^p(w)$. (See [**31**, Chapter VIII].) The same is true in the variable Hardy spaces. More precisely: suppose $p(\cdot) \in \mathcal{P}_0$ is such that there exists $0 < p_0 < p_-$ with $p(\cdot)/p_0 \in \mathcal{B}$. Then given $N > n(p_0^{-1} - 1)$, if $f \in H^{p(\cdot)}$, there exists a sequence of (N, ∞) atoms a_k and constants λ_k such that (6.3) holds, and the series converges both in the sense of distributions and in $H^{p(\cdot)}$. (See [**12**, Theorem 6.3]; here we have slightly modified the definition of atoms, but the change is immediate.) It follows immediately from these two results that finite sums of (N, ∞) atoms, for N sufficiently large, are dense in $H^p(w)$ and $H^{p(\cdot)}$.

Remark 6.3. In applying the density of finite sums of atoms, we are not making use of the finite atomic decomposition norm (as in Theorem 2.6 for weighted spaces or in the corresponding result for variable Hardy spaces in [**12**]). We will only use that these sums are dense with respect to the given Hardy space norm.

Theorem 6.4. *Let $q_1(\cdot), \ldots, q_m(\cdot), q(\cdot) \in \mathcal{P}_0$ be such that $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)}$ and $0 < (q_k)_- \leq (q_k)_+ < \infty$, $1 \leq k \leq m$. Suppose further that there exist $0 < p_1, \ldots, p_m < \infty$, $0 < p_k < (q_k)_-$, and $q_k(\cdot)/p_k \in \mathcal{B}$. If T is an m-CZO as in Theorem 1.1 satisfying (1.1) for all $|\alpha| \leq N$, where*

$$N \geq \max\left\{ \left\lfloor mn\left(\frac{1}{p_k} - 1\right) \right\rfloor_+, 1 \leq k \leq m \right\} + (m-1)n,$$

then

$$T \colon H^{q_1(\cdot)} \times \cdots \times H^{q_1(\cdot)} \to L^{q(\cdot)}.$$

Remark 6.5. Theorem 1.7 follows at once since if $q_k(\cdot)$ is log-Hölder continuous, then $q_k(\cdot)/p_k \in \mathcal{B}$.

Proof: Fix an integer K_0 such that

$$K_0 > \max\left\{\left\lfloor n\left(\frac{1}{p_k} - 1\right)\right\rfloor_+, 1 \le k \le m\right\}.$$

Define the family $\mathcal{F} = \{(f_1, \ldots, f_m, F)\}$, where, for each $1 \le k \le m$,

$$f_k = \sum_{j=1}^{L} \lambda_j a_j$$

is a finite linear combination of (K_0, ∞) atoms, and

$$F = \min\{|T(f_1, \ldots, f_m)|, R\}\chi_{B(0,R)},$$

where $0 < R < \infty$.

Now fix any collection of weights $w_1, \ldots, w_m \in A_1$. Then for $1 \le k \le m$, $r_{w_k} = 1$, so $K_0 > s_{w_k}$. Therefore, given any $(m+1)$-tuple $(f_1, \ldots, f_m, F) \in \mathcal{F}$, $f_k \in H^{p_k}(w_k)$, and by Theorem 1.1,

$$\|F\|_{L^p(\overline{w})} \le \|T(f_1, \ldots, f_m)\|_{L^p(\overline{w})} \lesssim \|f_1\|_{H^{p_1}(w_1)} \cdots \|f_m\|_{H^{p_m}(w_m)} < \infty.$$

Moreover, we have that $f_k \in H^{q_k(\cdot)}$ and

$$\|F\|_{q(\cdot)} \le R\|\chi_{B(0,R)}\|_{q(\cdot)} < \infty.$$

Hence, by Theorem 6.1 we have that

$$\|F\|_{q(\cdot)} \lesssim \|f_1\|_{H^{q_1(\cdot)}} \cdots \|f_m\|_{H^{q_m(\cdot)}} < \infty.$$

By Fatou's lemma in the scale of variable Lebesgue spaces [**6**, Theorem 2.61], we get

$$\|T(f_1, \ldots, f_m)\|_{q(\cdot)} \lesssim \|f_1\|_{H^{q_1(\cdot)}} \cdots \|f_m\|_{H^{q_m(\cdot)}} < \infty.$$

Since finite sums of (K_0, ∞) atoms are dense in $H^{q_k(\cdot)}$, $1 \le k \le m$, a standard density argument shows that this inequality holds for all $f_k \in H^{q_k(\cdot)}$, $1 \le k \le m$. This completes the proof. $\square$

The proof of the following result is identical to the proof of Theorem 6.4, except that in the definition of the family $\mathcal{F}$ we replace T by $\mathcal{M}_{N_0}T$ (for N_0 sufficiently large) and use Theorem 1.2 instead of Theorem 1.1. Theorem 1.8 again follows as an immediate corollary.

Theorem 6.6. *Given* $q_1(\cdot), \ldots, q_m(\cdot), q(\cdot)$ *and* $p_1, \ldots, p_m$ *as in Theorem 6.4, define* p *by*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Let T *be an* m-*CZO as in Theorem 1.1 satisfying* (1.1) *for all* $|\alpha| \leq N$, *where*

$$N \geq \left\lfloor mn\left(\frac{1}{p} - 1\right)\right\rfloor_+ + \max\left\{\left\lfloor mn\left(\frac{1}{p_k} - 1\right)\right\rfloor_+, 1 \leq k \leq m\right\} + mn.$$

Suppose further that T *satisfies* (1.2) *for all* $|\alpha| \leq \lfloor n(1/p - 1)\rfloor_+$. *Then*

$$T \colon H^{q_1(\cdot)} \times \cdots \times H^{q_1(\cdot)} \to H^q(\cdot).$$

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