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# WEIGHTED $L^p$ ESTIMATES OF KATO SQUARE ROOTS ASSOCIATED TO DEGENERATE ELLIPTIC OPERATORS

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**Abstract:** Let $w$ be a Muckenhoupt $A_2(\mathbb{R}^n)$ weight and $L_w := -w^{-1}\operatorname{div}(A\nabla)$ the degenerate elliptic operator on the Euclidean space $\mathbb{R}^n$, $n \geq 2$. In this article, the authors establish some weighted $L^p$ estimates of Kato square roots associated to the degenerate elliptic operators $L_w$. More precisely, the authors prove that, for $w \in A_p(\mathbb{R}^n)$, $p \in (\frac{2n}{n+1}, 2]$ and any $f \in C_c^\infty(\mathbb{R}^n)$, $\|L_w^{1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \sim \|\nabla f\|_{L^p(w,\mathbb{R}^n)}$, where $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differential functions with compact supports and the implicit equivalent positive constants are independent of $f$.

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## 1. Introduction

The Kato square root problem, which has a long history, was originally posed by Kato [**39**] in 1961. It amounts to identifying the domain of the square root of an abstract maximal accretive operator as the domain of the corresponding sesquilinear form. Although it is known that this problem has an affirmative answer in a few particular cases, in general, the Kato square root problem does not hold true; see, for example, [**42, 43**] for some counterexamples. However, by noticing that Kato posed his problem with the motivation from a special case of elliptic differential operators, McIntosh [**45, 44**] refined the statement of the Kato square root problem in the setting of elliptic operators. More precisely, let $L := -\operatorname{div}(A\nabla)$ be the second order elliptic operator on $\mathbb{R}^n$, with $A$ being an

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$n \times n$ matrix of complex bounded measurable functions on $\mathbb{R}^n$ satisfying the elliptic condition. The refined formulation of the Kato square root problem by McIntosh consists in showing that the domain of the square root $L^{1/2}$ coincides with the Sobolev space $W^{1,2}(\mathbb{R}^n)$ and

$$\|L^{1/2}(f)\|_{L^2(\mathbb{R}^n)} \sim \|\nabla f\|_{L^2(\mathbb{R}^n)} \tag{1.1}$$

with the implicit equivalent positive constants independent of $f$. This problem was completely solved by Auscher et al. [**7, 8, 33**] in the past decade, which consists one of the most celebrated results in harmonic analysis of recent years. For a more complete history of this problem, we refer the reader to the above papers or to the review by Kenig [**41**] and their references.

Observe that (1.1) consists in comparing the $L^2$ norms of $L^{1/2}(f)$ and $\nabla f$. For a general $p \in (1,\infty)$, the $L^p$ theory of square roots has also attracted considerable attention (see [**4, 34**] and the references cited therein). In particular, Auscher [**4**] showed that, for any $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|L^{1/2}(f)\|_{L^p(\mathbb{R}^n)} \sim \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad p \in (p_-(L), 2+\varepsilon(L)), \tag{1.2}$$

here and hereafter, the implicit equivalent positive constants in (1.2) are independent of $f$, $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differential functions with compact supports,

$$p_-(L) := \inf\{p \in [1,\infty] : \nabla L^{-1/2} \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\} \in \left[1, \frac{2n}{n+2}\right)$$

and $\varepsilon(L)$ is a positive constant depending on $L$. Moreover, Hofmann et al. [**36**] generalized the aforementioned result to the range $p \in (\frac{p_-(L)n}{n+p_-(L)}, 2+\varepsilon(L))$, by establishing the Riesz transform characterizations of the Hardy spaces $H_L^p(\mathbb{R}^n)$ associated to the second order elliptic operator $L = -\operatorname{div}(A\nabla)$, namely, for all $f \in H_L^p(\mathbb{R}^n)$,

$$\|f\|_{H_L^p(\mathbb{R}^n)} \sim \|\nabla L^{-1/2}(f)\|_{H^p(\mathbb{R}^n)}, \quad p \in \left(\frac{p_-(L)n}{n+p_-(L)}, 2+\varepsilon(L)\right), \tag{1.3}$$

where $H^p(\mathbb{R}^n)$ denotes the classical Hardy space and the implicit equivalent positive constants in (1.3) are independent of $f$. Noticing that, for all $p \in (p_-(L), 2+\varepsilon(L))$, both $H^p(\mathbb{R}^n)$ and $H_L^p(\mathbb{R}^n)$ coincide with the Lebesgue spaces $L^p(\mathbb{R}^n)$ (see [**36**, Proposition 9.1(v)]), hence (1.3) covers (1.2).

In the present article, we consider the $L^p$ theory of square roots in the case of degenerate elliptic operators. To be precise, let $w \in A_2(\mathbb{R}^n)$ be a Muckenhoupt weight. A matrix $A(x) := (A_{ij}(x))_{i,j=1}^n$ of complex-valued, measurable functions on $\mathbb{R}^n$ is said to satisfy the *degenerate* *elliptic condition* if there exist positive constants $\lambda \le \Lambda$ such that, for almost every $x \in \mathbb{R}^n$ and all $\xi, \eta \in \mathbb{C}^n$,

$$|\langle A(x)\xi, \eta\rangle| \le \Lambda w(x)|\xi||\eta| \tag{1.4}$$

and

$$\Re\langle A(x)\xi, \xi\rangle \ge \lambda w(x)|\xi|^2, \tag{1.5}$$

where $\Re z$ denotes the *real part* of $z$ for any $z \in \mathbb{C}$. For such a matrix $A(x)$, the associated *degenerate elliptic operator* $L_w$ is defined by setting, for all $f \in D(L_w) \subset \mathcal{H}_0^1(w, \mathbb{R}^n)$,

$$L_w f := -\frac{1}{w}\operatorname{div}(A\nabla f), \tag{1.6}$$

which is interpreted in the usual weak sense via the sesquilinear form, where $D(L_w)$ denotes the domain of $L_w$. Here and hereafter, $\mathcal{H}_0^1(w, \mathbb{R}^n)$ denotes the *weighted Sobolev space* which is defined to be the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the *norm*

$$\|f\|_{\mathcal{H}_0^1(w,\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} [|f(x)|^2 + |\nabla f(x)|^2] w(x)\,dx\right\}^{1/2}.$$

The *sesquilinear form* $\mathfrak{a}$, associated with $L_w$, is defined by setting, for all $f,\ g \in \mathcal{H}_0^1(w, \mathbb{R}^n)$,

$$\mathfrak{a}(f, g) := \int_{\mathbb{R}^n} [A(x)\nabla f(x)] \cdot \overline{\nabla g(x)}\,dx.$$

Operators of the form (1.6) and the associated elliptic equations were first studied by Fabes et al. [**31**] and have also been considered by a number of other authors (see, for example, [**18, 19, 17**] and, especially, some recent articles by Cruz-Uribe et al. [**22, 23, 25, 24**]).

Observe that the accretive condition (1.5) enables one to define the square root $L_w^{1/2}$ (see [**39, 40**]). It is a natural question to consider the associated Kato square root problem in the case of the degenerate elliptic operator $L_w$. In particular, Cruz-Uribe et al. [**25**] proved that, for any $f \in H_0^1(w, \mathbb{R}^n)$,

$$\|L_w^{1/2}(f)\|_{L^2(w,\mathbb{R}^n)} \sim \|\nabla f\|_{L^2(w,\mathbb{R}^n)},$$

where the implicit equivalent positive constants are independent of $f$ and $L^2(w, \mathbb{R}^n)$ denotes the *weighted Lebesgue space* equipped with the norm

$$\|f\|_{L^2(w,\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^2 w(x)\,dx\right]^{\frac{1}{2}}.$$

This result solves the Kato square root problem associated to the operator $L_w$. Notice that, when $w\equiv 1$, $L_w$ is just the second order elliptic operator $L$, thus, the results in [**25**] may be seen as generalizations of those in [**7**].

Motivated by the aforementioned results in [**25, 7, 4, 36**], our aim of this article is to study the weighted $L^p$ estimates of Kato square roots associated to $L_w$. To be precise, let $w\in A_\infty(\mathbb{R}^n)$ be a Muckenhoupt weight (see (2.1), (2.2), and (2.3) below for the precise definitions of $A_p(\mathbb{R}^n)$ of Muckenhoupt weights with $p\in[1,\infty]$). For any measurable subset $E$ of $\mathbb{R}^n$ and $p\in(0,\infty)$, $L^p(w,E)$ denotes the *weighted Lebesgue space* equipped with the *(quasi-)norm*

$$\|f\|_{L^p(w,E)}:=\left\{\int_E |f(x)|^p w(x)\,dx\right\}^{\frac{1}{p}}.$$

Let $L_w$ be a degenerate elliptic operator as in (1.6) with $w\in A_2(\mathbb{R}^n)$. The following theorem is the main result of the present article, which is proved in Section 7.

**Theorem 1.1.** *Let $p\in(\frac{2n}{n+1},2]$ and $w\in A_p(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for any $f\in C_c^\infty(\mathbb{R}^n)$,*

$$C^{-1}\|\nabla f\|_{L^p(w,\mathbb{R}^n)}\le \|L_w^{1/2}(f)\|_{L^p(w,\mathbb{R}^n)}\le C\|\nabla f\|_{L^p(w,\mathbb{R}^n)}.$$

This result establishes the weighted $L^p$ estimates of Kato square roots associated to the degenerate elliptic operators $L_w$ for $p\in(\frac{2n}{n+1},2]$. In particular, when $p=2$, Theorem 1.1, together with a density argument, leads to the corresponding result in [**25**].

To prove Theorem 1.1, we use the strategy of establishing the Riesz transform characterizations of the Hardy spaces associated to $L_w$, which is accomplished by Propositions 1.5, 1.6, and 1.7 below. We point out that this idea is inspired by Hofmann et al. [**36**]. Now we introduce some related definitions and notation on the Hardy spaces associated to the degenerate elliptic operator $L_w$. In what follows, let $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,\infty)$. Letting $w\in A_2(\mathbb{R}^n)$ and $L_w$ be as in (1.6), for any $f\in L^2(w,\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, the *square function $\mathcal{S}_{L_w}(f)$, associated with $L_w$*, is defined by setting

$$\mathcal{S}_{L_w}(f)(x):=\left[\iint_{\Gamma(x)}|t^2L_we^{-t^2L_w}(f)(y)|^2w(y)\frac{dy}{w(B(x,t))}\frac{dt}{t}\right]^{1/2},$$

where $B(x,t):=\{y\in\mathbb{R}^n:\,|x-y|<t\}$, $w(B(x,t)):=\int_{B(x,t)}w(y)\,dy$, and

$$\Gamma_\alpha(x):=\{(y,t)\in\mathbb{R}^{n+1}_+:\,|x-y|<\alpha t\}\tag{1.7}$$

denotes the *cone of aperture $\alpha$ with vertex $x$*. In particular, if $\alpha=1$, we write $\Gamma(x)$ instead of $\Gamma_\alpha(x)$.

For any $p \in (0, \infty)$, the Hardy space $H^p_{L_w}(\mathbb{R}^n)$ associated to $L_w$ is defined as follows.

**Definition 1.2.** Let $w \in A_2(\mathbb{R}^n)$ and $L_w$ be the degenerate elliptic operator as in (1.6) with the matrix $A$ satisfying the degenerate elliptic conditions (1.4) and (1.5). For any $p \in (0, 2]$, the *Hardy space* $H^p_{L_w}(\mathbb{R}^n)$, *associated to* $L_w$, is defined as the completion of the space

$$\{f \in L^2(w, \mathbb{R}^n) : \|S_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)} < \infty\}$$

with respect to the *(quasi-)norm*

$$\|f\|_{H^p_{L_w}(\mathbb{R}^n)} := \|S_{L_w}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

For any $p \in (2, \infty)$, define

$$H^p_{L_w}(\mathbb{R}^n) := (H^{p'}_{L_w^*}(\mathbb{R}^n))^*,$$

here and hereafter, $1/p + 1/p' = 1$ and $L_w^*$ denotes the *adjoint operator* of $L_w$ in $L^2(w, \mathbb{R}^n)$.

We point out that the study of the Hardy spaces associated to different operators (for example, the non-negative self-adjoint operator, the second order elliptic operator $-\operatorname{div}(A\nabla)$, and the Schrödinger operator $-\Delta + V$) has attracted considerable attention and the real-variable theory of these spaces has been established in recent years (see, for example, [**6, 29, 30, 11, 49, 37, 35, 36, 28, 27, 16**]).

Moreover, we need to introduce the Hardy space $H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$ associated to the Riesz transform $\nabla L_w^{-1/2}$, which, when $w \equiv 1$, is a special case of that defined in [**36**, p. 728].

**Definition 1.3.** Let $p \in (1, \infty)$, $w \in A_2(\mathbb{R}^n)$, and $L_w$ be the degenerate elliptic operator as in (1.6) with the matrix $A$ satisfying the degenerate elliptic conditions (1.4) and (1.5). The *Hardy space* $H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$ is defined as the completion of the space

$$\{f \in L^2(w, \mathbb{R}^n) : \nabla L_w^{-1/2}(f) \in L^p(w, \mathbb{R}^n)\}$$

with respect to the *norm*

$$\|f\|_{H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)} := \|\nabla L_w^{-1/2}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

*Remark* 1.4. Comparing with Definition 1.3, recall that, when $p \in (0, 1]$ and $w \in A_2(\mathbb{R}^n)$, the Hardy space $H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$ was introduced in [**50**, Definition 1.2], which is defined as the completion of the space

$$\{f \in L^2(w, \mathbb{R}^n) : \nabla L_w^{-1/2}(f) \in H^p_w(\mathbb{R}^n)\}$$

with respect to the *quasi-norm*

$$\|f\|_{H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)} := \|\nabla L_w^{-1/2}(f)\|_{H^p_w(\mathbb{R}^n)},$$

where $H^p_w(\mathbb{R}^n)$ denotes the classical weighted Hardy space. Moreover, in [**50**], the Hardy spaces $H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$ and $H^p_{L_w}(\mathbb{R}^n)$ were proved to coincide when $p \in (\delta, 1]$, where $\delta \in (0,1)$ is some fixed constant.

To prove Theorem 1.1, we first prove the following three propositions.

**Proposition 1.5.** *Let $w \in A_2(\mathbb{R}^n)$. Then, for any given $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$, $H^p_{L_w}(\mathbb{R}^n)$ and $L^p(w, \mathbb{R}^n)$ coincide with equivalent norms.*

Proposition 1.5 is proved in Section 4. In particular, when $w \equiv 1$, $L_w$ is just the usual second order elliptic operator $L = -\operatorname{div}(A\nabla)$ studied in [**36**], where Hofmann et al. proved that, for any $p \in (p_-(L), p_+(L))$, $H^p_L(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$ with equivalent norms. Notice that $1 \le p_-(L) < \frac{2n}{n+1} < \frac{2n}{n-1} < p_+(L) \le \infty$ (see [**36**, p. 4]). Recall that, via the local weighted Sobolev embedding inequality proved in [**31**] (see also Lemma 6.1 below), it was proved in [**51**, Proposition 1.5] that, for any $\frac{2n}{n+1} \le p \le q \le \frac{2n}{n-1}$, the semigroup $\{e^{-tL_w}\}_{t\ge 0}$ satisfies the weighted $L^p - L^q$ off-diagonal estimates on balls (see also Proposition 2.4 below). This is a main tool used in the proof of Proposition 1.5, which restricts the range of $p$ in Proposition 1.5 to the narrower interval $(\frac{2n}{n+1}, \frac{2n}{n-1})$ instead of $(p_-(L), p_+(L))$. It is still unclear whether Proposition 1.5 still holds true or not for a wider range of $p$ than $(\frac{2n}{n+1}, \frac{2n}{n-1})$.

**Proposition 1.6.** (i) *Let $w \in A_2(\mathbb{R}^n)$ and $p \in (\frac{2n}{n+1}, 2]$. Then there exists a positive constant $C$ such that, for any $f \in H^p_{L_w}(\mathbb{R}^n)$,*

$$\|\nabla L_w^{-1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \le C\|f\|_{H^p_{L_w}(\mathbb{R}^n)}.$$

(ii) *Let $w \in A_q(\mathbb{R}^n)$ with $q \in [1, 1+\frac{1}{n})$ and $p \in [1, \frac{2n}{n+1}]$. Then there exists a positive constant $C$ such that, for any $f \in H^p_{L_w}(\mathbb{R}^n)$,*

$$\|\nabla L_w^{-1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \le C\|f\|_{H^p_{L_w}(\mathbb{R}^n)}.$$

Proposition 1.6 is proved in Section 5. We point out that, when $w \equiv 1$, Proposition 1.6 is covered by [**36**, Propositions 5.32 and 5.6], where Hofmann et al. proved that, for any $p \in [1, 2+\varepsilon(L))$, $\nabla L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. We prove Proposition 1.6 by using the local weighted Poincaré inequality in [**31**] (see also Lemma 5.4 below), which implies that, for $w \in A_2(\mathbb{R}^n)$ and any $p \in (\frac{2n}{n+2}, 2]$, $\nabla L_w^{-1/2}$ is bounded on $L^p(w,\mathbb{R}^n)$. This restricts the results of Proposition 1.6 to the narrower interval $p \in [1,2]$ instead of $p \in [1, 2+\varepsilon(L))$.

**Proposition 1.7.** *Let $n \geq 2$, $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$, and $w \in A_p(\mathbb{R}^n) \cap A_2(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for any $f \in L^2(w,\mathbb{R}^n) \cap H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$,*

$$\|f\|_{H^p_{L_w}(\mathbb{R}^n)} \leq C\|\nabla L_w^{-1/2}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

Proposition 1.7 is proved in Section 6. Its proof relies on the weighted off-diagonal estimates on balls for $L_w$ (see Proposition 2.4 below) and the local weighted Poincaré and Sobolev embedding inequalities (see Lemmas 5.4 and 6.1 below), which restrict the results of Proposition 1.7 to $n \geq 2$, $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$, and $w \in A_p(\mathbb{R}^n)$ when $p \leq 2$. Proposition 1.7 is an analogue of [**36**, Proposition 5.34], where Hofmann et al. proved that, if, for some $r \in (1,2]$, the semigroup $\{e^{-tL}\}_{t\geq 0}$ satisfies $L^r - L^2$ off-diagonal estimates, then, for any $p \in (\max\{1, \frac{rn}{n+r}\}, p_+(L))$, there exists a positive constant $C$ such that, for any $h \in L^2(\mathbb{R}^n) \cap H^p_{L,\mathrm{Riesz}}(\mathbb{R}^n)$,

$$\|h\|_{H^p_L(\mathbb{R}^n)} \leq C\|\nabla L^{-1/2}(h)\|_{L^p(\mathbb{R}^n)}.$$

Notice that, for any $r \in (1,2]$, $(\frac{2n}{n+1}, \frac{2n}{n-1}) \subset (\max\{1, \frac{rn}{n+r}\}, p_+(L))$. Thus, when $w \equiv 1$, Proposition 1.7 is covered by [**36**, Proposition 5.34].

The remainder of this article is organized as follows. In Subsection 2.1, we first recall some notions and results on Muckenhoupt weights; in Subsection 2.2, we recall the holomorphic functional calculus of $L_w$; then, in Subsection 2.3, we introduce the weighted off-diagonal estimates for $L_w$, which have been established in [**51**]; in Subsection 2.4, we recall the notion of the weighted tent space and recall some results on their dual and interpolation results. In Section 3, by following the strategy used in [**36**, Section 4], for $p \in (0,\infty)$, we establish the square function characterizations of $H^p_{L_w}(\mathbb{R}^n)$ (see Propositions 3.6 and 3.7 below).

We end this section by making some conventions on notation. Throughout this article, $L_w$ always denotes a degenerate elliptic operator as in (1.6). We denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\dots)}$ to denote a positive constant depending on the parameters $\alpha$, $\beta$, $\dots$ . The *symbol* $f \lesssim g$ means that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$. For any measurable subset $E$ of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \backslash E$. Let $\mathbb{N} := \{1,2,\dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any closed subset $F \subset \mathbb{R}^n$, we let

$$R(F) := \bigcup_{x\in F} \Gamma(x), \tag{1.8}$$

where $\Gamma(x)$, for all $x \in F$, is as in (1.7) with $\alpha = 1$. For any $\mu \in (0,\pi)$, let

$$\Sigma_\mu^0 := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\}. \tag{1.9}$$

For any ball $B := (x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, $\alpha \in (0,\infty)$ and $j \in \mathbb{N}$, we let $\alpha B := B(x_B, \alpha r_B)$,

$$U_0(B) := B \quad \text{and} \quad U_j(B) := (2^j B) \setminus (2^{j-1}B). \tag{1.10}$$

For any $p\in[1,\infty]$, $p'$ denotes its conjugate number, namely, $1/p+1/p'=1$.

## 2. Preliminaries

In this section, we first recall the definition of the Muckenhoupt weights and some of their properties. Then we recall the holomorphic functional calculus of $L_w$, as was introduced by McIntosh [**46**], and the weighted off-diagonal estimates on balls for $L_w$. Finally, we introduce the weighted tent spaces and some of their properties.

**2.1. Muckenhoupt weights.** Let $q \in [1,\infty)$. A nonnegative and locally integrable function $w$ on $\mathbb{R}^n$ is said to belong to the *Muckenhoupt class* $A_q(\mathbb{R}^n)$ if there exists a positive constant $C$ such that, for any ball $B \subset \mathbb{R}^n$, when $q \in (1,\infty)$,

$$\frac{1}{|B|}\int_B w(x)\,dx \left\{\frac{1}{|B|}\int_B [w(x)]^{-\frac{1}{q-1}}\,dx\right\}^{q-1} \le C \tag{2.1}$$

or, when $q = 1$,

$$\frac{1}{|B|}\int_B w(x)\,dx \le C \operatorname*{ess\,inf}_{x\in B} w(x). \tag{2.2}$$

We also let

$$A_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} A_q(\mathbb{R}^n) \tag{2.3}$$

and $w(E) := \int_E w(x)\,dx$ for any measurable subset $E \subset \mathbb{R}^n$.

Let $r \in (1,\infty]$. A nonnegative locally integrable function $w$ is said to belong to the *reverse Hölder class* $RH_r(\mathbb{R}^n)$ if there exists a positive constant $C$ such that, for any ball $B \subset \mathbb{R}^n$,

$$\left\{\frac{1}{|B|}\int_B [w(x)]^r\,dx\right\}^{1/r} \le C\frac{1}{|B|}\int_B w(x)\,dx,$$

where we replace $\{\frac{1}{|B|}\int_B [w(x)]^r\,dx\}^{1/r}$ by $\|w\|_{L^\infty(B)}$ when $r = \infty$.

We recall some properties of Muckenhoupt weights and reverse Hölder classes in the following two lemmas (see, for example, [**26**] for their proofs).

**Lemma 2.1.** (i) *If* $1 \le p \le q \le \infty$, *then* $A_1(\mathbb{R}^n) \subset A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$.
(ii) $A_\infty(\mathbb{R}^n) := \cup_{p\in[1,\infty)} A_p(\mathbb{R}^n) = \cup_{r\in(1,\infty]} RH_r(\mathbb{R}^n)$.

**Lemma 2.2.** *Let* $q \in [1,\infty)$ *and* $r \in (1,\infty]$. *If a nonnegative measurable function* $w \in A_q(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$, *then there exists a constant* $C \in (1,\infty)$ *such that, for any ball* $B \subset \mathbb{R}^n$ *and any measurable subset* $E$ *of* $B$,

$$C^{-1}\left(\frac{|E|}{|B|}\right)^q \le \frac{w(E)}{w(B)} \le C\left(\frac{|E|}{|B|}\right)^{\frac{r-1}{r}}.$$

**2.2. Holomorphic functional calculi for $L_w$.** Let $L_w$ be the degenerate elliptic operator as in (1.6) with the matrix $A$ satisfying the degenerate elliptic conditions (1.4) and (1.5). By [**22**, pp. 291–294], we know that $L_w$ is an operator of *type* $\omega$ with $\omega := \arctan(\Lambda/\lambda) \in (0,\pi/2)$ (see [**46**] for the definition), where $0 < \lambda \le \Lambda < \infty$ are as in (1.4) and (1.5), and $-L_w$ generates a holomorphic semigroup in the sector $\Sigma^0_{\pi/2-\omega}$, where $\Sigma^0_{\pi/2-\omega}$ is as in (1.9) with $\mu$ replaced by $\pi/2-\omega$.

Furthermore, $L_w$ has a bounded holomorphic functional calculus on $L^2(w,\mathbb{R}^n)$ as was defined by McIntosh [**46**] (see also [**1**, Lecture 4]). We now recall some preliminary definitions.

For any $\mu \in (0,\pi/2)$, define

$$H_\infty(\Sigma^0_\mu) := \left\{f\colon \Sigma^0_\mu \to \mathbb{C} \text{ is holomorphic and } \|f\|_{L^\infty(\Sigma^0_\mu)} < \infty\right\}.$$

For any $\alpha,\beta \in (0,\infty)$, let

$$\Psi_{\alpha,\beta}(\Sigma^0_\mu) := \left\{\psi \in H_\infty(\Sigma^0_\mu) : |\psi(z)| \le C\frac{|z|^\alpha}{1+|z|^{\alpha+\beta}},\ \forall\, z \in \Sigma^0_\mu\right\}, \tag{2.4}$$

where $C$ is a positive constant independent of $z \in \Sigma^0_\mu$.

Let $\Psi(\Sigma^0_\mu) := \cup_{\alpha,\beta\in(0,\infty)}\Psi_{\alpha,\beta}(\Sigma^0_\mu)$. For any $\mu \in (\omega,\pi/2)$ and $\psi \in \Psi(\Sigma^0_\mu)$, define

$$\psi(L_w) := \frac{1}{2\pi i}\int_\gamma \psi(\zeta)(\zeta I - L_w)^{-1}\,d\zeta, \tag{2.5}$$

where $\gamma := \{re^{i\nu} : r \in (0,\infty)\} \cup \{re^{-i\nu} : r \in (0,\infty)\}$, $\nu \in (\omega,\mu)$, is a curve consisting of two rays parameterized anti-clockwise.

In general, for any $\psi \in H_\infty(\Sigma^0_\mu)$, $\psi(L_w)$ can be defined by a limiting procedure (see [**1**, Theorem G]).

**2.3. Weighted off-diagonal estimates on balls for $L_w$.** The notion of weighted off-diagonal estimates on balls was first introduced by Auscher and Martell in [**10**].

**Definition 2.3** ([**10**])**.** Let $p, q \in [1, \infty]$ with $p \le q$, $w \in A_\infty(\mathbb{R}^n)$, and $\{T_t\}_{t>0}$ be a family of sublinear operators. The family $\{T_t\}_{t>0}$ is said to satisfy the *weighted $L^p$-$L^q$ off-diagonal estimates on balls*, denoted by $T_t \in \mathcal{O}_w(L^p - L^q)$, if there exist constants $\theta_1, \theta_2 \in [0, \infty)$ and $C, c \in (0, \infty)$ such that, for any $t \in (0, \infty)$, any ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, and any $f \in L^p_{\mathrm{loc}}(w, \mathbb{R}^n)$,

$$\left\{\frac{1}{w(B)}\int_B |T_t(\chi_B f)(x)|^q w(x)\,dx\right\}^{1/q} \le C\left[\Upsilon\left(\frac{r_B}{t^{1/2}}\right)\right]^{\theta_2}\left\{\frac{1}{w(B)}\int_B |f(x)|^p w(x)\,dx\right\}^{1/p}$$

and, for any $j \in \mathbb{N} \cap [3, \infty)$,

$$\left\{\frac{1}{w(2^jB)}\int_{U_j(B)} |T_t(\chi_B f)(x)|^q w(x)\,dx\right\}^{1/q} \le C2^{j\theta_1}\left[\Upsilon\left(\frac{2^j r_B}{t^{1/2}}\right)\right]^{\theta_2} e^{-c\frac{(2^j r_B)^2}{t}}\left\{\frac{1}{w(B)}\int_B |f(x)|^p w(x)\,dx\right\}^{1/p}$$

and

$$\left\{\frac{1}{w(B)}\int_B |T_t(\chi_{U_j(B)} f)(x)|^q w(x)\,dx\right\}^{1/q} \le C2^{j\theta_1}\left[\Upsilon\left(\frac{2^j r_B}{t^{1/2}}\right)\right]^{\theta_2} e^{-c\frac{(2^j r_B)^2}{t}}\left\{\frac{1}{w(2^jB)}\int_{U_j(B)} |f(x)|^p w(x)\,dx\right\}^{1/p},$$

where $U_j(B)$ is as in (1.10) and, for all $s \in (0, \infty)$, $\Upsilon(s) := \max\{s, \frac{1}{s}\}$.

The following proposition is just [**51**, Proposition 1.5].

**Proposition 2.4** ([**51**])**.** *Let $w \in A_2(\mathbb{R}^n)$ and $k \in \mathbb{Z}_+$. Then, for any $\frac{2n}{n+1} \le p \le q \le \frac{2n}{n-1}$, $(tL_w)^k e^{-tL_w} \in \mathcal{O}_w(L^p - L^q)$.*

The following lemma is an analogue of [**36**, Lemma 2.40], whose proof is omitted. In what follows, for any $E, F \subset \mathbb{R}^n$, let

$$d(E, F) := \inf\{|x - y| : x \in E,\ y \in F\}.$$

**Lemma 2.5.** *Let* $\omega := \arctan(\Lambda/\lambda)$*,* $\mu \in (\omega, \pi/2)$*, and* $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0, \infty)$*. Assume that* $\psi \in \Psi_{\sigma_1,\tau_1}(\Sigma_\mu^0)$*,* $\widetilde{\psi} \in \Psi_{\sigma_2,\tau_2}(\Sigma_\mu^0)$*, and* $f \in H_\infty(\Sigma_\mu^0)$*. Then, for any* $a \in (0, \min\{\sigma_1, \tau_2\})$ *and* $b \in (0, \min\{\sigma_2, \tau_1\})$*, there exists a family of sub-linear operators,* $\{T_{s,t}\}_{s,t>0}$*, such that*

$$\psi(L_w) \circ f(L_w) \circ \widetilde{\psi}(L_w) = \min\left\{\left(\frac{s}{t}\right)^a, \left(\frac{t}{s}\right)^b\right\} T_{s,t},$$

*where* $\{T_{s,t}\}_{s,t>0}$ *have the following properties:*

(i) *There exists a positive constant* $C$*, independent of* $s$*, such that, for any* $t \in [s, \infty)$*, any closed subsets* $E$ *and* $F$ *of* $\mathbb{R}^n$*, and* $f \in L^2(w, \mathbb{R}^n)$ *with* $\operatorname{supp} f \subset E$*,*

$$\|T_{s,t}(f)\|_{L^2(w,F)} \le C\|f\|_{L^\infty(\Sigma_\mu^0)} \left[\min\left\{1, \frac{t}{[d(E,F)]^2}\right\}\right]^{\sigma_2+a} \|f\|_{L^2(w,E)}.$$

(ii) *There exists a positive constant* $C$*, independent of* $t$*, such that, for any* $s \in [t, \infty)$*, any closed subsets* $E$ *and* $F$ *of* $\mathbb{R}^n$*, and* $f \in L^2(w, \mathbb{R}^n)$ *with* $\operatorname{supp} f \subset E$*,*

$$\|T_{s,t}(f)\|_{L^2(w,F)} \le C\|f\|_{L^\infty(\Sigma_\mu^0)} \left[\min\left\{1, \frac{s}{[d(E,F)]^2}\right\}\right]^{\sigma_1+b} \|f\|_{L^2(w,E)}.$$

**2.4. Weighted tent spaces.** Let $w \in A_\infty(\mathbb{R}^n)$ and $f$ be a measurable function on $\mathbb{R}^{n+1}_+$. For any $x \in \mathbb{R}^n$, define

$$A(f)(x) := \left[\iint_{\Gamma(x)} |f(y,t)|^2 \frac{w(y)\,dy\,dt}{w(B(x,t))t}\right]^{1/2},$$

where $\Gamma(x)$ is as in (1.7) with $\alpha = 1$. For any $p \in (0, \infty)$, the *weighted tent space* $T^p(w, \mathbb{R}^n)$ is defined to be the space of all measurable functions $f$ on $\mathbb{R}^{n+1}_+$ such that $\|f\|_{T^p(w,\mathbb{R}^n)} := \|A(f)\|_{L^p(w,\mathbb{R}^n)} < \infty$.

For any open subset $O \subset \mathbb{R}^n$, the tent $\widehat{O}$ over $O$ is defined by setting

$$\widehat{O} := \{(x,t) \in \mathbb{R}^{n+1}_+ : d(x, O^\complement) \ge t\},$$

where $d(x, O^\complement) := \inf\{|x - y| : y \in O^\complement\}$ for any $x \in \mathbb{R}^n$.

Let $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$. A measurable function $a$ on $\mathbb{R}^{n+1}_+$ is called a $(w, p, 2)$-*atom* if there exists a ball $B$ of $\mathbb{R}^n$ such that

(i) $\operatorname{supp} a \subset \widehat{B}$;

(ii)

$$\left[\int_0^\infty \int_{\mathbb{R}^n} |a(y,t)|^2 w(y)\,dy \frac{dt}{t}\right]^{\frac{1}{2}} \le [w(B)]^{\frac{1}{2}-\frac{1}{p}}. \tag{2.6}$$

Noticing that, for any $w \in A_\infty(\mathbb{R}^n)$, $(\mathbb{R}^n, |\cdot|, w(x)\,dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [**20, 21**], the following lemma was proved in [**47**, Theorem 1.1], except for the last part concerning the $T^2(w,\mathbb{R}^n)$ convergence. By an argument similar to that used in the proof of [**36**, Proposition 3.25], we can show Lemma 2.6, the details being omitted.

**Lemma 2.6** ([**47**]). *Let $p \in (0,1]$, $w \in A_\infty(\mathbb{R}^n)$, and $f \in T^p(w,\mathbb{R}^n)$. Then there exist a sequence of $(w,p,2)$-atoms, $\{a_j\}_{j\in\mathbb{N}}$, and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that*

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j, \tag{2.7}$$

*where the series converges in $T^p(w,\mathbb{R}^n)$. Moreover, there exist positive constants $\widetilde{C}$ and $C$, independent of $f$, such that*

$$\widetilde{C}\|f\|_{T^p(w,\mathbb{R}^n)} \le \left\{\sum_{j\in\mathbb{N}} |\lambda_j|^p\right\}^{1/p} \le C\|f\|_{T^p(w,\mathbb{R}^n)}.$$

*Furthermore, if $f \in T^p(w,\mathbb{R}^n) \cap T^2(w,\mathbb{R}^n)$, then the series in* (2.7) *converges in both $T^p(w,\mathbb{R}^n)$ and $T^2(w,\mathbb{R}^n)$.*

The following lemma establishes the complex interpolation property of the weighted tent spaces. Noticing that, for any $w \in A_\infty(\mathbb{R}^n)$, $w(x)\,dx$ is a doubling measure on $\mathbb{R}^n$, Lemma 2.7 is just a special case of [**2**, Proposition 3.18]. In what follows, for any $\theta \in [0,1]$, $[\cdot,\cdot]_\theta$ denotes the complex interpolation space (see, for example, [**12**, Chapter 4] for the definition).

**Lemma 2.7** ([**2**]). *Let $w \in A_\infty(\mathbb{R}^n)$. Then, for any $p_0, p_1 \in [1,\infty)$ and $\theta \in [0,1]$, it holds true that*

$$[T^{p_0}(w,\mathbb{R}^n), T^{p_1}(w,\mathbb{R}^n)]_\theta = T^p(w,\mathbb{R}^n),$$

*where $1/p = \theta/p_0 + (1-\theta)/p_1$.*

From [**2**, Proposition 3.10], we deduce the following conclusion.

**Lemma 2.8** ([**2**]). *Let $p \in (1,\infty)$ and $w \in A_\infty(\mathbb{R}^n)$. Then, for any $f \in T^p(w,\mathbb{R}^n)$ and $g \in T^{p'}(w,\mathbb{R}^n)$, the pairing*

$$\langle f, g\rangle := \iint_{\mathbb{R}^{n+1}_+} f(y,t)\overline{g(y,t)}w(y)\,dy\frac{dt}{t}$$

*realizes $T^{p'}(w,\mathbb{R}^n)$ as the dual space of $T^p(w,\mathbb{R}^n)$, up to equivalent norms, where $1/p + 1/p' = 1$.*

## 3. Square function characterizations of $H^p_{L_w}(\mathbb{R}^n)$

In this section, we prove the square function characterizations of $H^p_{L_w}(\mathbb{R}^n)$ and we mainly follow the strategy used in [**36**]. To this end, we first establish some technical lemmas.

Let $w \in A_2(\mathbb{R}^n)$, $\omega := \arctan(\Lambda/\lambda)$, $\mu \in (\omega, \pi/2)$, and $\psi \in \Psi(\Sigma^0_\mu)$. For any $f \in L^2(w, \mathbb{R}^n)$ and $(x,t) \in \mathbb{R}^{n+1}_+$, define

$$Q_{\psi,L_w}(f)(x,t) := \psi(t^2 L_w)(f)(x).$$

By [**1**, Theorem F] and a simple calculation, we find that $Q_{\psi,L_w}$ is bounded from $L^2(w,\mathbb{R}^n)$ to $T^2(w,\mathbb{R}^n)$. For any $\psi \in \Psi(\Sigma^0_\mu)$, $F \in T^2(w,\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, let

$$\pi_{\psi,L_w}(F)(x) := \int_0^\infty \psi(t^2 L_w)(F(\cdot,t))(x)\frac{dt}{t}.$$

Since $L^*_w$ is the adjoint operator of $L_w$ in $L^2(w,\mathbb{R}^n)$, it follows that, for any $f \in L^2(w,\mathbb{R}^n)$ and $G \in T^2(w,\mathbb{R}^n)$,

$$\begin{aligned}\langle Q_{\psi,L^*_w}(f), G\rangle &= \int_0^\infty \int_{\mathbb{R}^n} Q_{\psi,L^*_w}(f)(x,t)\overline{G(x,t)}w(x)\,dx\frac{dt}{t}\\ &= \int_0^\infty \int_{\mathbb{R}^n} \psi(t^2L^*_w)(f)(x)\overline{G(x,t)}w(x)\,dx\frac{dt}{t}\\ &= \int_0^\infty \int_{\mathbb{R}^n} f(x)\overline{\psi(t^2L_w)(G(\cdot,t))(x)}w(x)\,dx\frac{dt}{t}\\ &= \int_{\mathbb{R}^n} f(x)\overline{\pi_{\psi,L_w}(G)(x)}w(x)\,dx =: (f, \pi_{\psi,L_w}(G))_w.\end{aligned} \tag{3.1}$$

This, combined with the fact that $Q_{\psi,L^*_w}$ is bounded from $L^2(w,\mathbb{R}^n)$ to $T^2(w,\mathbb{R}^n)$, implies that $\pi_{\psi,L_w}$ is the adjoint operator of $Q_{\psi,L^*_w}$ and bounded from $T^2(w,\mathbb{R}^n)$ to $L^2(w,\mathbb{R}^n)$.

For any $\psi, \widetilde{\psi} \in \Psi(\Sigma^0_\mu)$, $f \in H_\infty(\Sigma^0_\mu)$, $F \in T^2(w,\mathbb{R}^n)$, and $(x,t) \in \mathbb{R}^n_+$, define

$$\begin{aligned}Q^f(F)(x,t) &:= Q_{\psi,L_w} \circ f \circ \pi_{\widetilde{\psi},L_w}(F)(x,t)\\ &:= \int_0^\infty \psi(t^2L_w)(f(L_w)\widetilde{\psi}(s^2L_w)(F(\cdot,s)))(x)\frac{ds}{s}.\end{aligned}$$

From the above argument and the fact that $L_w$ has a bounded holomorphic functional calculus, it follows that $Q^f$ is bounded from $T^2(w,\mathbb{R}^n)$ to itself. Moreover, by Lemmas 2.7 and 2.8, we have the following conclusion, which is an analogue of [**36**, Proposition 4.4].

**Lemma 3.1.** *Let $w \in A_2(\mathbb{R}^n)$, $\omega := \arctan(\Lambda/\lambda)$, and $\mu \in (\omega, \pi/2)$. Then, for any $\psi, \widetilde{\psi} \in \Psi(\Sigma_\mu^0)$ and $f \in H_\infty(\Sigma_\mu^0)$, the operator $Q^f := Q_{\psi,L_w} \circ f \circ \pi_{\widetilde{\psi},L_w}$ is bounded from $T^p(w,\mathbb{R}^n)$ to itself if*

(i) *$p \in (0,2]$, $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$, and $\widetilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, or*

(ii) *$p \in (2,\infty)$, $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, and $\widetilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$,*

*where $\alpha \in (0,\infty)$ and $\beta \in (n(\max\{\frac{1}{p},1\} - \frac{1}{2}), \infty)$.*

*Proof:* We first prove Lemma 3.1 in case $p \in (0,1]$. By Lemma 2.5, we see that, for any $a \in (0,\alpha)$, $b \in (0,\beta)$, $F \in T^p(w,\mathbb{R}^n)$, and $(x,s) \in \mathbb{R}^{n+1}_+$,

$$(3.2) \quad Q^f(F)(x,s) = \int_0^\infty \min\left\{\left(\frac{s}{t}\right)^{2a}, \left(\frac{t}{s}\right)^{2b}\right\} T_{s^2,t^2}(F(\cdot,t))(x)\frac{dt}{t},$$

where the family $\{T_{s,t}\}_{s,t>0}$ of sublinear operators has the following properties:

(i) For any $t \in [s,\infty)$, any closed subsets $E$ and $F$ of $\mathbb{R}^n$, and $g \in L^2(w,\mathbb{R}^n)$ with $\operatorname{supp} g \subset E$,

$$(3.3) \quad \|T_{s,t}(g)\|_{L^2(w,F)} \lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\left[\min\left\{1, \frac{t}{[d(E,F)]^2}\right\}\right]^{\beta+a}\|g\|_{L^2(w,E)}.$$

(ii) For any $s \in [t,\infty)$, any closed subsets $E$ and $F$ of $\mathbb{R}^n$, and $g \in L^2(w,\mathbb{R}^n)$ with $\operatorname{supp} g \subset E$,

$$(3.4) \quad \|T_{s,t}(g)\|_{L^2(w,F)} \lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\left[\min\left\{1, \frac{s}{[d(E,F)]^2}\right\}\right]^{\alpha+b}\|g\|_{L^2(w,E)}.$$

Fix $b \in (n(\max\{\frac{1}{p},1\} - \frac{1}{2}), \infty)$ and choose a constant

$$M \in (n[\max\{1/p,1\} - 1/2], \min\{\alpha+b, \beta+a\}).$$

Then, from (3.3) and (3.4), it follows that, for any $s,t > 0$, any closed subsets $E$ and $F$ of $\mathbb{R}^n$, and $g \in L^2(w,\mathbb{R}^n)$ with $\operatorname{supp} g \subset E$,

$$(3.5) \quad \|T_{s^2,t^2}(g)\|_{L^2(w,F)} \lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\left[\min\left\{1, \frac{\max\{s^2,t^2\}}{[d(E,F)]^2}\right\}\right]^M \|g\|_{L^2(w,E)}.$$

It is easy to see that $T^2(w,\mathbb{R}^n) \cap T^p(w,\mathbb{R}^n)$ is dense in $T^p(w,\mathbb{R}^n)$ (see the proof of [**36**, Proposition 3.25]). By this, we claim that, to prove Lemma 3.1 in case $p \in (0,1]$, it suffices to prove that, for any $(w,p,2)$-atom $A$,

$$(3.6) \quad \|Q^f(A)\|_{T^p(w,\mathbb{R}^n)} \lesssim 1.$$

Indeed, from Lemma 2.6, we deduce that, for any $F \in T^2(w,\mathbb{R}^n) \cap T^p(w,\mathbb{R}^n)$, there exist a sequence of $(w,p,2)$-atoms, $\{A_j\}_{j\in\mathbb{N}}$, and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that

$$F = \sum_{j\in\mathbb{N}} \lambda_j A_j \quad \text{in} \quad T^2(w,\mathbb{R}^n) \cap T^p(w,\mathbb{R}^n) \tag{3.7}$$

and

$$\left[\sum_{j\in\mathbb{N}} |\lambda_j|^p\right]^{1/p} \sim \|F\|_{L^p(w,\mathbb{R}^n)}. \tag{3.8}$$

For any $N \in \mathbb{N}$, let $S_N := \sum_{j=1}^N \lambda_j A_j$. By (3.7) and the fact that $Q^f$ is bounded on $T^2(w,\mathbb{R}^n)$, we know that there exists a subsequence of $\{S_N\}_{N\in\mathbb{N}}$ (without loss of generality, we may use the same notation as the original sequence) such that, for almost every $(y,t) \in \mathbb{R}^{n+1}_+$, $\lim_{N\to\infty} Q^f(S_N)(y,t) = Q^f(F)(y,t)$. From this, (3.6), and (3.8), it follows that

$$\begin{aligned}\|Q^f(F)\|_{T^p(w,\mathbb{R}^n)} &\le \left[\sum_{j=1}^\infty |\lambda_j|^p \|Q^f(A_j)\|^p_{T^p(w,\mathbb{R}^n)}\right]^{\frac1p}\\ &\lesssim \left[\sum_{j=1}^\infty |\lambda_j|^p\right]^{\frac1p} \sim \|F\|_{T^p(w,\mathbb{R}^n)},\end{aligned}$$

which is the desired conclusion.

Next, we prove (3.6). For any $(w,p,2)$-atom $A$, there exists a ball $B := B(x_B, r_B)$ of $\mathbb{R}^n$, with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, such that $\operatorname{supp} A \subset \widehat{B}$. Let $S_1(\widehat{B}) := \widehat{2B}$ and $S_j(\widehat{B}) := \widehat{2^jB} \setminus \widehat{2^{j-1}B}$ for $j \in \mathbb{N} \cap [2,\infty)$. It is easy to see that $\mathbb{R}^{n+1}_+ = \cup_{j=1}^\infty S_j(\widehat{B})$. Next, for $j \in \mathbb{N}$, we estimate $\|\chi_{S_j(\widehat{B})} Q^f(A)\|_{T^p(w,\mathbb{R}^n)}$.

When $j = 1$, it is easy to see that, for any $(y,t) \in \widehat{2B}$,

$$\{x \in \mathbb{R}^n : |x-y| < t\} \subset 3B.$$

From this, the fact that $p \in (0,1]$, the Hölder inequality, Lemma 2.2, the fact that $Q^f$ is bounded on $T^2(w,\mathbb{R}^n)$, and (2.6), it follows that

$$\|\chi_{\widehat{2B}} Q^f(A)\|_{T^p(w,\mathbb{R}^n)}$$

$$= \left\{\int_{3B}\left[\iint_{\Gamma(x)} |(\chi_{\widehat{2B}} Q^f(A))(y,t)|^2 \frac{w(y)\,dy\,dt}{w(B(x,t))\,t}\right]^{\frac{p}{2}} w(x)\,dx\right\}^{\frac{1}{p}}$$

$$\leq \|\chi_{\widehat{2B}} Q^f(A)\|_{T^2(w,\mathbb{R}^n)}[w(3B)]^{\frac{1}{p}-\frac{1}{2}}$$

$$\lesssim \|A\|_{T^2(w,\mathbb{R}^n)}[w(B)]^{\frac{1}{p}-\frac{1}{2}} \lesssim 1.$$

When $j \in \mathbb{N} \cap [2,\infty)$, it is easy to see that, for any $(y,t) \in S_j(\widehat{B})$,

$$\{x \in \mathbb{R}^n : |x-y| < t\} \subset 2^{j+1}B.$$

By this, the fact that $p \in (0,1]$, the Hölder inequality, the Fubini theorem, and Lemma 2.2, we conclude that

(3.9)

$$\|\chi_{S_j(\widehat{B})} Q^f(A)\|_{T^p(w,\mathbb{R}^n)}$$

$$= \left\{\int_{2^{j+1}B}\left[\iint_{\Gamma(x)} |(\chi_{S_j(\widehat{B})} Q^f(A))(y,s)|^2 \frac{w(y)\,dy\,ds}{w(B(x,s))\,s}\right]^{\frac{p}{2}} w(x)\,dx\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_{2^{j+1}B}\iint_{\Gamma(x)} |(\chi_{S_j(\widehat{B})} Q^f(A))(y,s)|^2 \frac{w(y)\,dy\,ds}{w(B(x,s))\,s} w(x)\,dx\right\}^{\frac{1}{2}}$$

$$\times [w(2^{j+1}B)]^{\frac{1}{p}-\frac{1}{2}}$$

$$\lesssim \left\{\left[\int_0^{2^{j-1}r_B}\int_{\mathbb{R}^n} |(\chi_{S_j(\widehat{B})} Q^f(A))(y,s)|^2 w(y)\,dy\frac{ds}{s}\right]^{\frac{1}{2}}\right.$$

$$\left. + \left[\int_{2^{j-1}r_B}^{2^j r_B}\int_{\mathbb{R}^n}\cdots\right]^{\frac{1}{2}}\right\}$$

$$\times 2^{2nj(\frac{1}{p}-\frac{1}{2})}[w(B)]^{\frac{1}{p}-\frac{1}{2}}$$

$$=: \{\mathrm{I}+\mathrm{II}\}2^{2nj(\frac{1}{p}-\frac{1}{2})}[w(B)]^{\frac{1}{p}-\frac{1}{2}}.$$

For II, from (3.2), (3.5), the Minkowski inequality, and the Hölder inequality, we deduce that

(3.10)

$$
\begin{aligned}
\mathrm{II} &= \left[\int_{2^{j-1}r_B}^{2^j r_B}\int_{\mathbb{R}^n}\chi_{S_j(\widehat{B})}(x,s)\left|\int_0^{r_B}\left(\frac{t}{s}\right)^{2b}T_{s^2,t^2}(A(\cdot,t))(x)\frac{dt}{t}\right|^2 w(x)\,dx\frac{ds}{s}\right]^{\frac12}\\
&\le \int_0^{r_B}\left(\frac{t}{2^j r_B}\right)^{2b}\\
&\qquad\times\left[\int_{2^{j-1}r_B}^{2^j r_B}\int_{\mathbb{R}^n}\chi_{S_j(\widehat{B})}(x,s)|T_{s^2,t^2}(A(\cdot,t))(x)|^2 w(x)\,dx\frac{ds}{s}\right]^{\frac12}\frac{dt}{t}\\
&\lesssim \int_0^{r_B}\left(\frac{t}{2^j r_B}\right)^{2b}\|A(\cdot,t)\|_{L^2(w,B)}\frac{dt}{t}\\
&\lesssim \left\{\int_0^{r_B}\left(\frac{t}{2^j r_B}\right)^{4b}\frac{dt}{t}\right\}^{\frac12}\left\{\int_0^{r_B}\int_B |A(y,t)|^2 w(y)\,dy\frac{dt}{t}\right\}^{\frac12}\\
&\lesssim [w(B)]^{\frac12-\frac1p}2^{-2bj}.
\end{aligned}
$$

For I, by (3.2), and the Minkowski inequality, we have

(3.11)

$$
\begin{aligned}
\mathrm{I} &= \left[\int_0^{2^{j-1}r_B}\int_{\mathbb{R}^n}\chi_{S_j(\widehat{B})}(x,s)\right.\\
&\qquad\times\left.\left|\int_0^{r_B}\psi(s^2L_w)f(L_w)\widetilde{\psi}(t^2L_w)(A(\cdot,t))(x)\frac{dt}{t}\right|^2 w(x)\,dx\frac{ds}{s}\right]^{\frac12}\\
&\le \int_0^{r_B}\left[\int_0^t\int_{\mathbb{R}^n}\left(\frac{s}{t}\right)^{4a}\chi_{S_j(\widehat{B})}(x,s)|T_{s^2,t^2}(A(\cdot,t))(x)|^2 w(x)\,dx\frac{ds}{s}\right]^{\frac12}\frac{dt}{t}\\
&\quad+\int_0^{r_B}\left[\int_t^{2^{j-1}r_B}\int_{\mathbb{R}^n}\left(\frac{t}{s}\right)^{4b}\cdots\right]^{\frac12}\frac{dt}{t}\\
&=: \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}
$$

For $\mathrm{I}_1$, by (3.5), the Hölder inequality, and (2.6), we know that

$$\begin{aligned}\mathrm{I}_1 &\lesssim \int_0^{r_B} \left\{\int_0^t \left(\frac{s}{t}\right)^{4a} \left[\frac{t^2}{(2^j r_B)^2}\right]^{2M} \|A(\cdot,t)\|^2_{L^2(w,B)} \frac{ds}{s}\right\}^{\frac{1}{2}} \frac{dt}{t} \\ &\sim \frac{1}{(2^j r_B)^{2M}} \int_0^{r_B} \|A(\cdot,t)\|_{L^2(w,B)} t^{2M} \frac{dt}{t} \\ &\lesssim \frac{1}{(2^j r_B)^{2M}} \left[\int_0^{r_B}\int_B |A(y,t)|^2 w(y)\,dy \frac{dt}{t}\right]^{\frac{1}{2}} \left[\int_0^{r_B} t^{4M-1}\,dt\right]^{\frac{1}{2}} \\ &\lesssim 2^{-2Mj}[w(B)]^{\frac{1}{2}-\frac{1}{p}}.\end{aligned} \tag{3.12}$$

Similarly, we see that

$$\begin{aligned}\mathrm{I}_2 &\lesssim \int_0^{r_B} \left[\left(\frac{t}{2^j r_B}\right)^{2b} + \left(\frac{t}{2^j r_B}\right)^{2M}\right] \|A(\cdot,t)\|_{L^2(w,B)} \frac{dt}{t} \\ &\lesssim (2^{-2bj} + 2^{-2Mj})[w(B)]^{\frac{1}{2}-\frac{1}{p}}.\end{aligned} \tag{3.13}$$

From (3.9), (3.10), (3.11), (3.12), (3.13), and $M > n(\max\{\frac{1}{p},1\} - \frac{1}{2})$, we deduce that, for any $(w,p,2)$-atom $A$,

$$\|Q^f(A)\|_{T^p(w,\mathbb{R}^n)} \lesssim 1,$$

which completes the proof of Lemma 3.1 in case $p \in (0,1]$. Since we already known that, for any $F \in T^2(w,\mathbb{R}^n)$,

$$\|Q^f(F)\|_{T^2(w,\mathbb{R}^n)} \lesssim \|F\|_{T^2(w,\mathbb{R}^n)},$$

then, by Lemma 2.7 and the well-known property of interpolation spaces (see, for example, [**12**, Theorem 4.1.2]), we find that, for any $p \in (1,2]$ and $F \in T^p(w,\mathbb{R}^n)$,

$$\|Q^f(F)\|_{T^p(w,\mathbb{R}^n)} \lesssim \|F\|_{T^p(w,\mathbb{R}^n)}.$$

This proves Lemma 3.1 in case $p \in (1,2]$.

By the above argument and duality, it is easy to prove Lemma 3.1 in case $p \in (2,\infty)$. This finishes the proof of Lemma 3.1. □

We have the following Calderón reproducing formula.

**Lemma 3.2.** *Let $w \in A_2(\mathbb{R}^n)$, $\omega := \arctan(\Lambda/\lambda)$, and $\mu \in (\omega,\pi/2)$. For any $\psi,\widetilde{\psi} \in \Psi(\Sigma^0_\mu)$ satisfying $\int_0^\infty \psi(t^2)\widetilde{\psi}(t^2)\frac{dt}{t} = 1$, and any $f \in L^2(w,\mathbb{R}^n)$, it holds true that*

$$\pi_{\psi,L_w} \circ Q_{\widetilde{\psi},L_w}(f) = \pi_{\widetilde{\psi},L_w} \circ Q_{\psi,L_w}(f) = f \quad \text{in } L^2(w,\mathbb{R}^n).$$

*Proof:* By a simple calculation, we find that, for any $z \in \Sigma_\mu^0$,

$$\int_0^\infty \psi(t^2z)\widetilde{\psi}(t^2z)\frac{dt}{t} = \int_0^\infty \psi(t^2)\widetilde{\psi}(t^2)\frac{dt}{t} = 1.$$

From (2.5) and the properties of holomorphic functional calculi (see [**1**, Lecture 2]), we deduce that

$$\psi(t^2L_w)\widetilde{\psi}(t^2L_w) = \int_\gamma \psi(t^2z)\widetilde{\psi}(t^2z)(zI - L_w)^{-1}\,dz,$$

where $\gamma := \{re^{i\nu} : r \in (0,\infty)\} \cup \{re^{-i\nu} : r \in (0,\infty)\}$ and $\nu \in (\omega,\mu)$. Hence,

$$\begin{aligned}
\pi_{\psi,L_w} \circ Q_{\widetilde{\psi},L_w} &= \int_0^\infty \psi(t^2L_w)\widetilde{\psi}(t^2L_w)\frac{dt}{t} \\
&= \int_0^\infty \int_\gamma \psi(t^2z)\widetilde{\psi}(t^2z)(zI - L_w)^{-1}\,dz\frac{dt}{t} \\
&= \int_\gamma \int_0^\infty \psi(t^2z)\widetilde{\psi}(t^2z)\frac{dt}{t}(zI - L_w)^{-1}\,dz \\
&= \int_\gamma (zI - L_w)^{-1}\,dz = I.
\end{aligned}$$

By changing the roles of $\psi$ and $\widetilde{\psi}$, we obtain $\pi_{\widetilde{\psi},L_w} \circ Q_{\psi,L_w} = I$. This finishes the proof of Lemma 3.2. □

*Remark* 3.3. For any $\psi \in \Psi(\Sigma_\mu^0)$, $\psi \not\equiv 0$, and $z \in \Sigma_\mu^0$, by taking

$$\widetilde{\psi}(z) := \frac{2\overline{\psi(z)}}{\int_0^\infty |\psi(t)|^2\,\frac{dt}{t}},$$

we find that

$$\int_0^\infty \psi(t^2)\widetilde{\psi}(t^2)\frac{dt}{t} = \frac{2\int_0^\infty |\psi(t^2)|^2\,\frac{dt}{t}}{\int_0^\infty |\psi(t)|^2\,\frac{dt}{t}} = 1.$$

Thus, $\psi$ and $\widetilde{\psi}$ satisfy the assumptions of Lemma 3.2.

**Definition 3.4.** Let $w \in A_2(\mathbb{R}^n)$, $\omega := \arctan(\Lambda/\lambda)$, and $\mu \in (\omega, \pi/2)$. Let

(i) $p \in (0,2]$ and $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$, or
(ii) $p \in (2,\infty)$ and $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$,

where $\alpha \in (0,\infty)$ and $\beta \in (n[\max\{\frac{1}{p},1\}-\frac{1}{2}],\infty)$. The Hardy space $H^p_{\psi,L_w}(\mathbb{R}^n)$ is defined as the completion of the space

$$\{f \in L^2(w,\mathbb{R}^n) : Q_{\psi,L_w}(f) \in T^p(w,\mathbb{R}^n)\}$$

with respect to the (quasi-)norm

$$\|f\|_{H^p_{\psi,L_w}(\mathbb{R}^n)} := \|Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)}.$$

**Lemma 3.5.** *Let $w \in A_2(\mathbb{R}^n)$, $\omega := \arctan(\Lambda/\lambda)$, and $\mu \in (\omega,\pi/2)$.*

(i) *Let $p \in (0,2]$ and $\psi,\psi_0 \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$, or $p \in (2,\infty)$ and $\psi,\psi_0 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$, where $\alpha \in (0,\infty)$, $\beta \in (n[\max\{\frac{1}{p},1\}-\frac{1}{2}],\infty)$, and $\psi,\psi_0 \not\equiv 0$. Then there exists a positive constant $C$ such that, for any $f \in H^p_{\psi_0,L_w}(\mathbb{R}^n)$,*

$$\|Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)} \le C\|f\|_{H^p_{\psi_0,L_w}(\mathbb{R}^n)}. \tag{3.14}$$

(ii) *Let $p \in (0,2]$, $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$, and $\psi_0 \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$, or $p \in (2,\infty)$, $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$, and $\psi_0 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$, where $\alpha \in (0,\infty)$, $\beta \in (n[\max\{\frac{1}{p},1\}-\frac{1}{2}],\infty)$, and $\psi,\psi_0 \not\equiv 0$. Then there exists a positive constant $C$ such that, for any $f \in T^p(w,\mathbb{R}^n)$,*

$$\|\pi_{\psi,L_w}(f)\|_{H^p_{\psi_0,L_w}(\mathbb{R}^n)} \le C\|f\|_{T^p(w,\mathbb{R}^n)}. \tag{3.15}$$

*Proof:* We first prove (i). If $p \in (0,2]$ and $\psi,\psi_0 \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$, choose a function $\widetilde{\psi}_0 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ such that $\int_0^\infty \widetilde{\psi}_0(t^2)\psi_0(t^2)\frac{dt}{t} = 1$. Hence, by Lemma 3.2, we know that, for any $f \in L^2(w,\mathbb{R}^n)$,

$$f = \pi_{\widetilde{\psi}_0,L_w} \circ Q_{\psi_0,L_w}(f) \quad \text{in } L^2(w,\mathbb{R}^n).$$

From this and Lemma 3.1(i), it follows that, for any $f \in L^2(w,\mathbb{R}^n) \cap H^p_{\psi_0,L_w}(\mathbb{R}^n)$,

$$\begin{aligned}\|Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)} &\sim \|Q_{\psi,L_w} \circ \pi_{\widetilde{\psi}_0,L_w} \circ Q_{\psi_0,L_w}(f)\|_{T^p(w,\mathbb{R}^n)}\\ &\lesssim \|Q_{\psi_0,L_w}(f)\|_{T^p(w,\mathbb{R}^n)} \sim \|f\|_{H^p_{\psi_0,L_w}(\mathbb{R}^n)}.\end{aligned}$$

Since $L^2(w,\mathbb{R}^n) \cap H^p_{\psi_0,L_w}(\mathbb{R}^n)$ is dense in $H^p_{\psi_0,L_w}(\mathbb{R}^n)$, it then follows that (3.14) holds true in this case. If $p \in (2,\infty)$ and $\psi,\psi_0 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$, by Lemma 3.1(ii) and an argument similar to that used as above, we find that (3.14) also holds true in this case.

Next, we prove (ii). If $p \in (0,2]$, $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$, and $\psi_0 \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$, from Lemma 3.1(i), we deduce that, for any $f \in T^p(w,\mathbb{R}^n) \cap T^2(w,\mathbb{R}^n)$,

$$\|\pi_{\psi,L_w}(f)\|_{H^p_{\psi_0,L_w}(\mathbb{R}^n)} = \|Q_{\psi_0,L_w} \circ \pi_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)} \lesssim \|f\|_{T^p(w,\mathbb{R}^n)}.$$

Since $T^p(w,\mathbb{R}^n)\cap T^2(w,\mathbb{R}^n)$ is dense in $T^p(w,\mathbb{R}^n)$, we then obtain (3.15) in this case. If $p\in(2,\infty)$, $\psi\in\Psi_{\alpha,\beta}(\Sigma^0_\mu)$, and $\psi_0\in\Psi_{\beta,\alpha}(\Sigma^0_\mu)$, similarly, by Lemma 3.1(ii) and a density argument, we find that (3.15) also holds true in this case. This finishes the proof of Lemma 3.5. □

**Proposition 3.6.** *Let $p\in(0,2]$, $w\in A_2(\mathbb{R}^n)$, $\omega:=\arctan(\Lambda/\lambda)$, and $\mu\in(\omega,\pi/2)$. For any $\psi\in\Psi_{\alpha,\beta}(\Sigma^0_\mu)$, $\psi\not\equiv0$, where $\alpha\in(0,\infty)$ and $\beta\in(n[\max\{\frac{1}{p},1\}-\frac{1}{2}],\infty)$, $H^p_{L_w}(\mathbb{R}^n)$ and $H^p_{\psi,L_w}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

*Proof:* Take $\psi_0(z):=ze^{-z}$ for any $z\in\Sigma^0_\mu$. It is easy to show that $\psi_0\in\Psi_{\alpha,\beta}(\Sigma^0_\mu)$ with $\alpha\in(0,\infty)$ and $\beta\in(n[\max\{\frac{1}{p},1\}-\frac{1}{2}],\infty)$. Then, from Definition 1.2 and (3.14), we deduce that, for any $f\in H^p_{L_w}(\mathbb{R}^n)\cap L^2(w,\mathbb{R}^n)$,

$$\|f\|_{H^p_{\psi,L_w}(\mathbb{R}^n)}=\|Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)}\lesssim\|f\|_{H^p_{\psi_0,L_w}(\mathbb{R}^n)}\sim\|f\|_{H^p_{L_w}(\mathbb{R}^n)},$$

which implies that

$$[H^p_{L_w}(\mathbb{R}^n)\cap L^2(w,\mathbb{R}^n)]\subset[H^p_{\psi,L_w}(\mathbb{R}^n)\cap L^2(w,\mathbb{R}^n)]. \tag{3.16}$$

Next, we prove the reverse inclusion. To this end, we only need to show that, for any $f\in H^p_{\psi,L_w}(\mathbb{R}^n)\cap L^2(w,\mathbb{R}^n)$,

$$\|f\|_{H^p_{L_w}(\mathbb{R}^n)}\lesssim\|f\|_{H^p_{\psi,L_w}(\mathbb{R}^n)}.$$

Choose a function $\widetilde{\psi}\in\Psi_{\beta,\alpha}(\Sigma^0_\mu)$ such that $\int_0^\infty\widetilde{\psi}(t^2)\psi(t^2)\frac{dt}{t}=1$. Then, by Lemma 3.2, we know that, for any $f\in L^2(w,\mathbb{R}^n)$,

$$\pi_{\widetilde{\psi},L_w}\circ Q_{\psi,L_w}(f)=f\quad\text{in } L^2(w,\mathbb{R}^n).$$

This, together with Lemma 3.1(i), implies that

$$\begin{aligned}\|f\|_{H^p_{L_w}(\mathbb{R}^n)}&=\|Q_{\psi_0,L_w}\|_{T^p(w,\mathbb{R}^n)}\sim\|Q_{\psi_0,L_w}\circ\pi_{\widetilde{\psi},L_w}\circ Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)}\\&\lesssim\|Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)}\sim\|f\|_{H^p_{\psi,L_w}(\mathbb{R}^n)}.\end{aligned}$$

Therefore,

$$[H^p_{\psi,L_w}(\mathbb{R}^n)\cap L^2(w,\mathbb{R}^n)]\subset[H^p_{L_w}(\mathbb{R}^n)\cap L^2(w,\mathbb{R}^n)].$$

This, combined with (3.16) and a density argument, then finishes the proof of Proposition 3.6. □

**Proposition 3.7.** *Let $p\in(2,\infty)$, $w\in A_2(\mathbb{R}^n)$, $\omega:=\arctan(\Lambda/\lambda)$, and $\mu\in(\omega,\pi/2)$. For any $\psi\in\Psi_{\beta,\alpha}(\Sigma^0_\mu)$, $\psi\not\equiv0$, where $\alpha\in(0,\infty)$ and $\beta\in(n[\max\{\frac{1}{p},1\}-\frac{1}{2}],\infty)$, $H^p_{L_w}(\mathbb{R}^n)$ and $H^p_{\psi,L_w}(\mathbb{R}^n)$ coincide with equivalent norms.*

*Proof:* We first prove the following inclusion:

$$H^p_{\psi,L_w}(\mathbb{R}^n) \subset (H^{p'}_{L_w^*}(\mathbb{R}^n))^* = H^p_{L_w}(\mathbb{R}^n). \tag{3.17}$$

Take a function $\widetilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$ such that $\int_0^\infty \psi(t^2)\widetilde{\psi}(t^2)\frac{dt}{t} = 1$. Then, for any $f \in L^2(w,\mathbb{R}^n) \cap H^p_{\psi,L_w}(\mathbb{R}^n)$ and $g \in L^2(w,\mathbb{R}^n) \cap H^{p'}_{L_w^*}(\mathbb{R}^n)$, by Lemma 3.2 and (3.1), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\overline{g(x)}w(x)\,dx &= \int_{\mathbb{R}^n} \pi_{\widetilde{\psi},L_w} \circ Q_{\psi,L_w}(f)(x)\overline{g(x)}w(x)\,dx\\ &= \int_0^\infty \int_{\mathbb{R}^n} Q_{\psi,L_w}(f)(x,t)\overline{Q_{\widetilde{\psi},L_w^*}(g)(x,t)}w(x)\,dx\frac{dt}{t}.\end{aligned}$$

Hence, from this, Lemmas 2.8 and 3.5, and Proposition 3.6, it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}w(x)\,dx\right| &\lesssim \|Q_{\psi,L_w}(f)\|_{T^p(w,\mathbb{R}^n)}\|Q_{\widetilde{\psi},L_w^*}(g)\|_{T^{p'}(w,\mathbb{R}^n)}\\ &\sim \|f\|_{H^p_{\psi,L_w}(\mathbb{R}^n)}\|g\|_{H^{p'}_{L_w^*}(\mathbb{R}^n)}.\end{aligned}$$

This, together with a density argument, implies (3.17).

Next, we prove the reverse inclusion of (3.17). Take a function $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ with $\alpha \in (0,\infty)$ and $\beta \in (n[\max\{\frac{1}{p},1\} - \frac{1}{2}],\infty)$. Then, for any $F \in T^{p'}(w,\mathbb{R}^n)$, by Lemma 3.5 and Proposition 3.6, we obtain $\|\pi_{\psi,L_w^*}(F)\|_{H^{p'}_{L_w^*}(\mathbb{R}^n)} \lesssim \|F\|_{T^{p'}(w,\mathbb{R}^n)}$, which implies $\pi_{\psi,L_w^*}(F) \in H^{p'}_{L_w^*}(\mathbb{R}^n)$. For any $l \in (H^{p'}_{L_w^*}(\mathbb{R}^n))^*$ and $F \in T^{p'}(w,\mathbb{R}^n)$, define

$$\langle l,F\rangle := l(\pi_{\psi,L_w^*}(F)).$$

Then we find that, for any $F \in T^{p'}(w,\mathbb{R}^n)$,

$$|\langle l,F\rangle| \le \|l\|_{(H^{p'}_{L_w^*}(\mathbb{R}^n))^*}\|\pi_{\psi,L_w^*}F\|_{H^{p'}_{L_w^*}(\mathbb{R}^n)} \lesssim \|l\|_{(H^{p'}_{L_w^*}(\mathbb{R}^n))^*}\|F\|_{T^{p'}(w,\mathbb{R}^n)}.$$

This implies $l \in (T^{p'}(w,\mathbb{R}^n))^*$. Since $(T^{p'}(w,\mathbb{R}^n))^* = T^p(w,\mathbb{R}^n)$ (see Lemma 2.8), we know that there exists a function $G_l \in T^p(w,\mathbb{R}^n)$ such that, for any $F \in T^{p'}(w,\mathbb{R}^n)$,

$$\langle l,F\rangle = l(\pi_{\psi,L_w^*}(F)) = \int_0^\infty \int_{\mathbb{R}^n} F(x,t)\overline{G_l(x,t)}w(x)\frac{dx\,dt}{t} = \langle F,G_l\rangle \tag{3.18}$$

and

$$\|G_l\|_{T^p(w,\mathbb{R}^n)} \sim \|l\|_{(H^{p'}_{L_w^*}(\mathbb{R}^n))^*}. \tag{3.19}$$

Choose a function $\widetilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$

$$\text{with } \alpha \in (0,\infty) \text{ and } \beta \in \left(n\left[\max\left\{\frac{1}{p},1\right\}-\frac{1}{2}\right],\infty\right)$$

such that $\int_0^\infty \psi(t^2)\widetilde{\psi}(t^2)\frac{dt}{t}=1$. By Lemma 3.5, we have

$$\|\pi_{\widetilde{\psi},L_w}(G_l)\|_{H^p_{\psi,L_w}(\mathbb{R}^n)} \lesssim \|G_l\|_{T^p(w,\mathbb{R}^n)}. \tag{3.20}$$

Next, we prove that, for any $f\in H^{p'}_{L_w^*}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\overline{\pi_{\widetilde{\psi},L_w}(G_l)(x)}w(x)\,dx = l(f) \tag{3.21}$$

and

$$\|\pi_{\widetilde{\psi},L_w}(G_l)\|_{H^p_{\psi,L_w}(\mathbb{R}^n)} \sim \|l\|_{(H^{p'}_{L_w^*}(\mathbb{R}^n))^*}. \tag{3.22}$$

Since $T^2(w,\mathbb{R}^n)\cap T^p(w,\mathbb{R}^n)$ is dense in $T^p(w,\mathbb{R}^n)$, from (3.1) and a density argument, it follows that, for any $f\in L^2(w,\mathbb{R}^n)\cap H^{p'}_{L_w^*}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\overline{\pi_{\widetilde{\psi},L_w}(G_l)(x)}w(x)\,dx &= \int_0^\infty\int_{\mathbb{R}^n} Q_{\widetilde{\psi},L_w^*}(f)(x,t)\overline{G_l(x,t)}w(x)\,dx\frac{dt}{t}\\ &= \langle Q_{\widetilde{\psi},L_w^*}(f),G_l\rangle\end{aligned}$$

which, combined with (3.18) and Lemma 3.2, further implies that, for any $f\in L^2(w,\mathbb{R}^n)\cap H^{p'}_{L_w^*}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\overline{\pi_{\widetilde{\psi},L_w}(G_l)(x)}w(x)\,dx = l(\pi_{\psi,L_w}\circ Q_{\widetilde{\psi},L_w^*}(f)) = l(f).$$

By the fact that $L^2(w,\mathbb{R}^n)\cap H^{p'}_{L_w^*}(\mathbb{R}^n)$ is dense in $H^{p'}_{L_w^*}(\mathbb{R}^n)$, we obtain (3.21). From (3.19) and (3.20), it follows that $\|\pi_{\widetilde{\psi},L_w}(G_l)\|_{H^p_{\psi,L_w}(\mathbb{R}^n)} \lesssim \|l\|_{(H^{p'}_{L_w^*}(\mathbb{R}^n))^*}$. By (3.21), we further see that

$$\|l\|_{(H^{p'}_{L_w^*}(\mathbb{R}^n))^*} \lesssim \|\pi_{\widetilde{\psi},L_w}(G_l)\|_{H^p_{\psi,L_w}(\mathbb{R}^n)}.$$

Therefore, (3.22) holds true. This implies that

$$H^p_{L_w}(\mathbb{R}^n) = (H^p_{L_w^*}(\mathbb{R}^n))^* \subset H^p_{\psi,L_w}(\mathbb{R}^n),$$

which, together with (3.17), then completes the proof of Proposition 3.7. $\square$

## 4. Proof of Proposition 1.5

In this section, we show Proposition 1.5.

Let $w \in A_2(\mathbb{R}^n)$. For any $k \in \mathbb{N}$, $f \in L^2(w, \mathbb{R}^n)$, and $x \in \mathbb{R}^n$, define

$$S_{L_w,k}(f)(x) := \left[\iint_{\Gamma(x)} |(t^2 L_w)^k e^{-t^2 L_w}(f)(y)|^2 w(y) \frac{dy}{w(B(x,t))} \frac{dt}{t}\right]^{\frac{1}{2}}.$$

Noticing that, for any $w \in A_2(\mathbb{R}^n)$, $w(x)\,dx$ is a doubling measure on $\mathbb{R}^n$, by Proposition 2.4 and [**15**, Theorem 2.13], we know that, for any given $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$, there exists a positive constant $C$ such that, for any $f \in L^p(w, \mathbb{R}^n)$,

$$\|S_{L_w,k}(f)\|_{L^p(w,\mathbb{R}^n)} \le C\|f\|_{L^p(w,\mathbb{R}^n)}. \tag{4.1}$$

*Proof of Proposition 1.5:* For $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$, by Propositions 3.6 and 3.7, we see that $L^2(w, \mathbb{R}^n) \cap H^p_{L_w}(\mathbb{R}^n)$ is dense in $H^p_{L_w}(\mathbb{R}^n)$. Since $L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$ is also dense in $L^p(w, \mathbb{R}^n)$, to prove Proposition 1.5, we only need to show that

$$[L^2(w, \mathbb{R}^n) \cap H^p_{L_w}(\mathbb{R}^n)] = [L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)] \tag{4.2}$$

and, for any $f \in L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$,

$$\|f\|_{H^p_{L_w}(\mathbb{R}^n)} \sim \|f\|_{L^p(w,\mathbb{R}^n)}. \tag{4.3}$$

For any $k \in \mathbb{N}$ with $k > n(\max\{\frac{1}{p}, 1\} - \frac{1}{2})$, take $\psi(z) := z^k e^{-z}$ for all $z \in \Sigma^0_\mu$. From (2.4), it is easy to see that, for any $\alpha \in (0, \infty)$, $\psi \in \Psi_{k,\alpha}(\Sigma^0_\mu)$. Then, for any $f \in L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$, we have

$$\begin{aligned}\|f\|_{H^p_{L_w}(\mathbb{R}^n)} &\sim \left\|\left\{\iint_{\Gamma(\cdot)} |\psi(t^2 L_w)(f)(y)|^2 w(y) \frac{dy}{w(B(\cdot,t))} \frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_{L^p(w,\mathbb{R}^n)} \\ &\sim \|S_{L_w,k}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|f\|_{L^p(w,\mathbb{R}^n)}.\end{aligned} \tag{4.4}$$

On the other hand, taking an appropriate $\widetilde{\psi} \in \Psi(\Sigma^0_\mu)$ and using (3.1), Lemmas 3.2, 2.8, and 3.5, Propositions 3.6 and 3.7, we conclude that, for any $f \in L^2(w, \mathbb{R}^n) \cap H^p_{L_w}(\mathbb{R}^n)$ and $g \in L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$ with

$\|g\|_{L^{p'}(w,\mathbb{R}^n)} = 1$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}w(x)\,dx\right| &\sim \left|\int_{\mathbb{R}^n} \pi_{\psi,L_w}\circ Q_{\widetilde{\psi},L_w}(f)(x)\overline{g(x)}w(x)\,dx\right|\\ &\lesssim \|Q_{\widetilde{\psi},L_w}(f)\|_{T^p(w,\mathbb{R}^n)}\|Q_{\widetilde{\psi},L_w^*}(g)\|_{T^{p'}(w,\mathbb{R}^n)}\\ &\lesssim \|f\|_{H^p_{L_w}(\mathbb{R}^n)}\|S_{L_w,k}(g)\|_{L^{p'}(w,\mathbb{R}^n)}\\ &\lesssim \|f\|_{H^p_{L_w}(\mathbb{R}^n)}\|g\|_{L^{p'}(w,\mathbb{R}^n)} \lesssim \|f\|_{H^p_{L_w}(\mathbb{R}^n)}.\end{aligned}$$

This implies $\|f\|_{L^p(w,\mathbb{R}^n)} \lesssim \|f\|_{H^p_{L_w}(\mathbb{R}^n)}$. By this and (4.4), we obtain (4.2) and (4.3), which then completes the proof of Proposition 1.5. □

## 5. Proof of Proposition 1.6

In this section, we show Proposition 1.6. To this end, we first establish some technical lemmas.

**Lemma 5.1.** *Let $w \in A_2(\mathbb{R}^n)$. For any $\theta \in (0,1)$ and $p_0, p_1 \in [1,\infty)$, it holds true that*

$$[H^{p_0}_{L_w}(\mathbb{R}^n), H^{p_1}_{L_w}(\mathbb{R}^n)]_\theta = H^p_{L_w}(\mathbb{R}^n),$$

*where $1/p = (1-\theta)/p_0 + \theta/p_1$.*

*Proof:* It is easy to see that $H^{p_0}_{L_w}(\mathbb{R}^n)$ and $H^{p_1}_{L_w}(\mathbb{R}^n)$ are *compatible*, namely, $H^{p_0}_{L_w}(\mathbb{R}^n)$ and $H^{p_1}_{L_w}(\mathbb{R}^n)$ are subspaces of $H^{p_0}_{L_w}(\mathbb{R}^n) + H^{p_1}_{L_w}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{H^{p_0}_{L_w}(\mathbb{R}^n)+H^{p_1}_{L_w}(\mathbb{R}^n)} := \inf\Big\{\|f_0\|_{H^{p_0}_{L_w}(\mathbb{R}^n)}+\|f_1\|_{H^{p_1}_{L_w}(\mathbb{R}^n)} : f_0+f_1=f,\\ f_0 \in H^{p_0}_{L_w}(\mathbb{R}^n),\ f_1 \in H^{p_1}_{L_w}(\mathbb{R}^n)\Big\}.$$

By Lemmas 3.5 and 3.2, we know that, for $i \in \{0,1\}$ and suitable $\psi, \widetilde{\psi} \in \Psi(\Sigma^0_\mu)$, $Q_{\psi,L_w}$ is bounded from $H^{p_i}_{L_w}(\mathbb{R}^n)$ to $T^{p_i}(w,\mathbb{R}^n)$, $\pi_{\widetilde{\psi},L_w}$ is bounded from $T^{p_i}(w,\mathbb{R}^n)$ to $H^{p_i}_{L_w}(\mathbb{R}^n)$, and $\pi_{\widetilde{\psi},L_w}\circ Q_{\psi,L_w} = I$ on $H^{p_i}_{L_w}(\mathbb{R}^n)$. This implies that $\{H^{p_0}_{L_w}(\mathbb{R}^n), H^{p_1}_{L_w}(\mathbb{R}^n)\}$ is a *retract* of $\{T^{p_0}(w,\mathbb{R}^n), T^{p_1}(w,\mathbb{R}^n)\}$ (see [**38**, p. 151] for the definition). Since $H^{p_i}_{L_w}(\mathbb{R}^n)$, $i \in \{0,1\}$, is a Banach space, we know that $H^{p_i}_{L_w}(\mathbb{R}^n)$, $i \in \{0,1\}$, is *analytic convex* (see [**38**, p. 145] for the definition).

Hence, from the above argument, [**38**, Lemma 7.11], and Lemma 2.7, it follows that, for some suitable $\widetilde{\psi} \in \Psi(\Sigma^0_\mu)$, any $\theta \in (0,1)$ and $1/p =$

$(1-\theta)/p_0+\theta/p_1$,

$$[H^{p_0}_{L_w}(\mathbb{R}^n),H^{p_1}_{L_w}(\mathbb{R}^n)]_\theta=\pi_{\widetilde{\psi},L_w}([T^{p_0}(w,\mathbb{R}^n),T^{p_1}(w,\mathbb{R}^n)]_\theta)$$
$$=\pi_{\widetilde{\psi},L_w}(T^p(w,\mathbb{R}^n))=H^p_{L_w}(\mathbb{R}^n).$$

This finishes the proof of Lemma 5.1. □

The following lemma is an analogue of [**25**, Lemma 2.10].

**Lemma 5.2.** *Let $E$ and $F$ be two closed subsets of $\mathbb{R}^n$. Then there exist positive constants $C$ and $c$ such that, for any $t\in(0,\infty)$ and $f\in L^2(w,\mathbb{R}^n)$ with* $\operatorname{supp} f\subset E$,

$$\|e^{-tL_w}(f)\|_{L^2(w,F)}\le Ce^{-\frac{[d(E,F)]^2}{ct}}\|f\|_{L^2(w,E)}, \tag{5.1}$$

$$\|\sqrt{t}\nabla e^{-tL_w}(f)\|_{L^2(w,F)}\le Ce^{-\frac{[d(E,F)]^2}{ct}}\|f\|_{L^2(w,E)} \tag{5.2}$$

*and, for any $t\in(0,\infty)$ and $\boldsymbol{f}:=(f_1,\dots,f_n)$ with $f_i\in L^2(w,\mathbb{R}^n)$,* $\operatorname{supp} f_i\subset E$, $i\in\{1,\dots,n\}$,

$$\left\|\sqrt{t}e^{-tL_w}\left(\frac{1}{w}\operatorname{div}(w\boldsymbol{f})\right)\right\|_{L^2(w,F)}\le Ce^{-\frac{[d(E,F)]^2}{ct}}\|\boldsymbol{f}\|_{L^2(w,E)}. \tag{5.3}$$

*Proof:* Noticing (5.1) and (5.2) have been proved, respectively, in [**22**, Theorem 1.6] and [**51**, Proposition 2.7], to prove Lemma 5.2, we only need to show (5.3).

Indeed, for any $g\in L^2(w,F)$ with $\operatorname{supp} g\subset F$ and $\|g\|_{L^2(w,F)}=1$, by the Hölder inequality and (5.2), we have

$$\begin{aligned}
&\left|\int_F\sqrt{t}e^{-tL_w}\left(\frac{1}{w}\operatorname{div}(w\boldsymbol{f})\right)(x)g(x)w(x)\,dx\right|\\
&\quad=\left|\int_{\mathbb{R}^n}\sqrt{t}\frac{1}{w(x)}\operatorname{div}(w\boldsymbol{f})(x)e^{-tL^*_w}(g)(x)w(x)\,dx\right|\\
&\quad=\left|\int_{\mathbb{R}^n}\sqrt{t}w(x)\boldsymbol{f}(x)\cdot\nabla e^{-tL^*_w}(g)(x)\,dx\right|\\
&\quad\le\int_E|\boldsymbol{f}(x)||\sqrt{t}\nabla e^{-tL^*_w}(g)(x)|w(x)\,dx\\
&\quad\le\left[\int_E|\sqrt{t}\nabla e^{-tL^*_w}(g)(x)|^2w(x)\,dx\right]^{\frac12}\left[\int_E|\boldsymbol{f}(x)|^2w(x)\,dx\right]^{\frac12}\\
&\quad\lesssim e^{-\frac{[d(E,F)]^2}{ct}}\|\boldsymbol{f}\|_{L^2(w,F)},
\end{aligned}$$

which, combined with a dual argument, further implies that (5.3) holds true. This finishes the proof of Lemma 5.2. □

By Lemma 5.2, we obtain the following lemma.

**Lemma 5.3.** *Let $m \in \mathbb{N}$ and $E$, $F$ be closed subsets of $\mathbb{R}^n$. Then there exist positive constants $C$ and $c$ such that, for any $t \in (0,\infty)$ and $\boldsymbol{f} = (f_1, \ldots, f_n)$, with $f_i \in L^2(w,\mathbb{R}^n)$, $\operatorname{supp} f_i \subset E$, $i \in \{1,\ldots,n\}$,*

$$\left\|\sqrt{t}\nabla L_w^{-1/2}(I-e^{-tL_w})^m\left(\frac{1}{w}\operatorname{div}(w\boldsymbol{f})\right)\right\|_{L^2(w,F)} \leq C\left(\frac{[d(E,F)]^2}{t}\right)^{-\left(m+\frac{1}{2}\right)}\|\boldsymbol{f}\|_{L^2(w,E)}$$

*and*

$$\left\|\sqrt{t}\nabla(\nabla L_w^{-1/2}(I-e^{-tL_w})^m)^*(\boldsymbol{f})\right\|_{L^2(w,F)} \leq C\left(\frac{[d(E,F)]^2}{t}\right)^{-\left(m+\frac{1}{2}\right)}\|\boldsymbol{f}\|_{L^2(w,E)}.$$

The proof of Lemma 5.3 is a complete analogue of that of [**34**, Lemma 2.2], the details being omitted.

The following local weighted Poincaré inequality is just [**31**, Theorem (1.2)].

**Lemma 5.4** ([**31**]). *Let $n \geq 2$. For any given $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$, there exist positive constants $C$ and $\delta$ such that, for any ball $B \equiv B(x_B, r_B)$ of $\mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, any Lipschitz continuous function $u$ on $\bar{B}$, and any number $k \in [1, \frac{n}{n-1}+\delta)$,*

$$\left[\frac{1}{w(B)}\int_B |u(x)-u_B|^{kp}w(x)\,dx\right]^{\frac{1}{kp}} \leq Cr_B\left[\frac{1}{w(B)}\int_B |\nabla u(x)|^p w(x)\,dx\right]^{\frac{1}{p}},$$

*where*

$$u_B := \frac{1}{w(B)}\int_B u(x)w(x)\,dx. \tag{5.4}$$

Let $w \in A_\infty(\mathbb{R}^n)$. For any $p \in (0,\infty)$, the *weighted weak-$L^p$ space* $L^{p,\infty}(w,\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\infty}(w,\mathbb{R}^n)} := \sup_{\alpha\in(0,\infty)} \alpha[w(\{x\in\mathbb{R}^n : |f(x)|>\alpha\})]^{\frac{1}{p}} < \infty.$$

For $L^{p,\infty}(w,\mathbb{R}^n)$, we have the following Fatou lemma (see [**32**, Exercise 1.1.12] for the outline of its proof).

**Lemma 5.5.** *Let $w \in A_\infty(\mathbb{R}^n)$ and $p \in (0, \infty)$. Then there exists a positive constant $C_{(w,p)}$, depending on $w$ and $p$, such that, for all measurable functions $\{g_k\}_{k=1}^\infty$ on $\mathbb{R}^n$,*

$$\Big\|\liminf_{k\to\infty} |g_k|\Big\|_{L^{p,\infty}(w,\mathbb{R}^n)} \le C_{(w,p)} \liminf_{k\to\infty} \|g_k\|_{L^{p,\infty}(w,\mathbb{R}^n)}.$$

By Lemmas 5.2, 5.3, and 5.4, we obtain the following theorem which establishes the boundedness of the Riesz transform $\nabla L_w^{-1/2}$ on $L^q(w, \mathbb{R}^n)$. Theorem 5.6 is an analogue of [**34**, Theorem 1.2].

**Theorem 5.6.** *Let $p := \frac{2n}{n+1}$ and $w \in A_2(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for any $\alpha \in (0, \infty)$ and $f \in L^p(w, \mathbb{R}^n)$,*

$$w\left(\left\{x \in \mathbb{R}^n : |\nabla L_w^{-1/2}(f)(x)| > \alpha\right\}\right) \le \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx. \tag{5.5}$$

*Moreover, for any given $q \in (p, 2]$, there exists a positive constant $C$ such that, for any $f \in L^q(w, \mathbb{R}^n)$,*

$$\|\nabla L_w^{-1/2}(f)\|_{L^q(w,\mathbb{R}^n)} \le C\|f\|_{L^q(w,\mathbb{R}^n)}.$$

*Proof:* To prove Theorem 5.6, by the Marcinkiewicz interpolation theorem and a density argument, we only need to show that, for any $f \in L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$, (5.5) holds true. Indeed, since $L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$ is dense in $L^p(w, \mathbb{R}^n)$, we know that, for any $f \in L^p(w, \mathbb{R}^n)$, there exists a family of functions, $\{f_k\}_{k\in\mathbb{N}} \subset [L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)]$, such that $\lim_{k\to\infty} \|f_k - f\|_{L^p(w,\mathbb{R}^n)} = 0$. Thus, $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^p(w, \mathbb{R}^n)$. By (5.5), we know that $\{\nabla L_w^{-1/2}(f_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in measure $w(x)\, dx$. From [**32**, Theorem 1.1.13], it follows that there exists a subsequence of $\{\nabla L_w^{-1/2}(f_k)\}_{k\in\mathbb{N}}$ (without loss of generality, we may use the same notation as the original sequence) such that, for almost every $x \in \mathbb{R}^n$, $\lim_{k\to\infty} \nabla L_w^{-1/2}(f_k)(x)$ exists. For almost every $x \in \mathbb{R}^n$, let

$$\nabla L_w^{-1/2}(f)(x) := \lim_{k\to\infty} \nabla L_w^{-1/2}(f_k)(x).$$

It is easy to see that $\nabla L_w^{-1/2}(f)$ is well defined. By Lemma 5.5, we further find that, for any $f \in L^p(w, \mathbb{R}^n)$, (5.5) holds true.

Next, we prove that, for any $f \in L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$, (5.5) holds true. To this end, for any $f \in L^1_{\mathrm{loc}}(w, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$M_w(f)(x) := \sup_{B\ni x} \frac{1}{w(B)} \int_B |f(y)|w(y)\, dy, \tag{5.6}$$

where the supremum is taken over all balls containing $x$. For any $f \in L^2(w, \mathbb{R}^n) \cap L^p(w, \mathbb{R}^n)$, by the generalized Calderón–Zygmund decomposition [**48**, p. 17, Theorem 2] and its proof, we know that there exist positive constants $C$ and $N$ such that, for any $\alpha \in (0, \infty)$, there exist a collection of balls, $\{B_k\}_{k=1}^{\infty} := \{B(x_k, r_k)\}_{k=1}^{\infty}$ of $\mathbb{R}^n$ with $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ and $\{r_k\}_{k=1}^{\infty} \subset (0, \infty)$, a family $\{b_k\}_{k=1}^{\infty}$ of functions and an almost everywhere bounded function $g$ such that the following properties hold true:

$$f(x) = g(x) + \sum_{k=1}^{\infty} b_k(x) \quad \text{for almost every } x \in \mathbb{R}^n; \tag{5.7}$$

$$|g(x)| \leq C\alpha \quad \text{for almost every } x \in \mathbb{R}^n; \tag{5.8}$$

$$\begin{aligned} &\text{for any } k \in \mathbb{N}, \quad \operatorname{supp} b_k \subset B_k, \quad \int_{B_k} b_k(x) w(x)\, dx = 0, \\ &\qquad \text{and } \left[\frac{1}{w(B_k)} \int_{B_k} |b_k(x)|^p w(x)\, dx\right]^{\frac{1}{p}} \leq C\alpha; \end{aligned} \tag{5.9}$$

$$\sum_{k=1}^{\infty} w(B_k) \leq C\alpha^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx; \tag{5.10}$$

$$\sum_{k=1}^{\infty} \chi_{B_k}(x) \leq N. \tag{5.11}$$

Let $b := \sum_{k=1}^{\infty} b_k$. By (5.7), we have

$$\begin{aligned} &w\left(\left\{x \in \mathbb{R}^n : |\nabla L_w^{-1/2}(f)(x)| > 3\alpha\right\}\right) \\ &\quad \leq w\left(\left\{x \in \mathbb{R}^n : |\nabla L_w^{-1/2}(g)(x)| > \alpha\right\}\right) \\ &\qquad + w\left(\left\{x \in \mathbb{R}^n : |\nabla L_w^{-1/2}(b)(x)| > 2\alpha\right\}\right) =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We first estimate I. By the fact that $\nabla L_w^{-1/2}$ is bounded on $L^2(w, \mathbb{R}^n)$ (see [**25**, Theorem 1.1]), $p = \frac{2n}{n+1} < 2$, (5.8), (5.9), (5.10), and (5.11), we see that

$$
\begin{aligned}
\mathrm{I} &\leq \frac{1}{\alpha^2}\int_{\mathbb{R}^n} |\nabla L_w^{-1/2}(g)(x)|^2 w(x)\,dx \lesssim \frac{1}{\alpha^2}\int_{\mathbb{R}^n} |g(x)|^2 w(x)\,dx \\
&\lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} |g(x)|^p w(x)\,dx \\
&\lesssim \frac{1}{\alpha^p}\left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx + \sum_{k=1}^{\infty}\int_{B_k} |b_k(x)|^p w(x)\,dx\right\} \\
&\lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.
\end{aligned}
\tag{5.12}
$$

Next, we prove

$$
\mathrm{II} := w\left(\left\{x\in\mathbb{R}^n : |\nabla L_w^{-1/2}(b)(x)| > 2\alpha\right\}\right) \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{5.13}
$$

We claim that, to prove (5.13), it suffices to show that there exists a positive constant $C_{(w,p)}$, depending on $w$ and $p$, such that, for any $N \in \mathbb{N}$ and $\alpha \in (0,\infty)$,

$$
w\left(\left\{x \in \mathbb{R}^n : \left|\nabla L_w^{-1/2}\left(\sum_{k=1}^{N} b_k\right)(x)\right| > 2\alpha\right\}\right) \leq \frac{C_{(w,p)}}{\alpha^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{5.14}
$$

Indeed, for any $f \in L^2(w,\mathbb{R}^n) \cap L^p(w,\mathbb{R}^n)$, by the proof of [**48**, p. 17, Theorem 2], it is easy to see that $b = \lim_{N\to\infty}\sum_{k=1}^N b_k$ in $L^2(w,\mathbb{R}^n)$. Let $S_N := \sum_{k=1}^N b_k$. By the fact that $\nabla L_w^{-1/2}$ is bounded on $L^2(w,\mathbb{R}^n)$, we know that there exists a subsequence of $\{\nabla L_w^{-1/2}(S_N)\}_{N=1}^\infty$ (without loss of generality, we may use the same notation as the original sequence) such that, for almost every $x \in \mathbb{R}^n$,

$$
\lim_{N\to\infty} \nabla L_w^{-1/2}(S_N)(x) = \nabla L_w^{-1/2}(b)(x).
$$

By this, Lemma 5.5, and (5.14), we see that

$$
\begin{aligned}
&\alpha^p w\left(\left\{x \in \mathbb{R}^n : |\nabla L_w^{-1/2}(b)(x)| > 2\alpha\right\}\right) \\
&\quad\leq \left\| |\nabla L_w^{-1/2}(b)| \right\|_{L^{p,\infty}(w,\mathbb{R}^n)}^p = \left\| \lim_{N\to\infty} |\nabla L_w^{-1/2}(S_N)| \right\|_{L^{p,\infty}(w,\mathbb{R}^n)}^p \\
&\quad\leq \liminf_{N\to\infty} \|\nabla L_w^{-1/2}(S_N)\|_{L^{p,\infty}(w,\mathbb{R}^n)}^p \lesssim \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx,
\end{aligned}
$$

which implies (5.13).

Next, we prove (5.14). Let $T := \nabla L_w^{-1/2}$. Fix some $m \in \mathbb{N}$ satisfying $m > \frac{n-1}{2}$. For any $k \in \mathbb{N}$, let $T_k := T(I - e^{-t_k L_w})^m$ and $B_k^* := 2B_k$, where $t_k := r_k^2$ and $r_k \in (0, \infty)$ denotes the radius of $B_k$. For any $N \in \mathbb{N}$ and almost every $x \in \mathbb{R}^n$, we write

$$T(S_N)(x) = \sum_{k=1}^N T(b_k)(x) = \sum_{k=1}^N T_k(b_k)(x) + \sum_{k=1}^N (T - T_k)(b_k)(x).$$

Hence,

$$\begin{aligned}
& w(\{x \in \mathbb{R}^n : |T(S_N)(x)| > 2\alpha\}) \\
&\quad \le w\left(\left\{x \in \mathbb{R}^n : \left|\sum_{k=1}^N T_k(b_k)(x)\right| > \alpha\right\}\right) \\
&\qquad + w\left(\left\{x \in \mathbb{R}^n : \left|\sum_{k=1}^N (T - T_k)(b_k)(x)\right| > \alpha\right\}\right) \\
&\quad \le w\left(\bigcup_{k=1}^N B_k^*\right) \qquad (5.15)\\
&\qquad + w\left(\left\{x \in \left(\bigcup_{k=1}^N B_k^*\right)^{\complement} : \left|\sum_{k=1}^N T_k(b_k)(x)\right| > \alpha\right\}\right) \\
&\qquad + w\left(\left\{x \in \mathbb{R}^n : \left|\sum_{k=1}^N (T - T_k)(b_k)(x)\right| > \alpha\right\}\right) \\
&\quad =: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.
\end{aligned}$$

We first estimate $\mathrm{II}_1$. By the fact that $w \in A_2(\mathbb{R}^n)$, Lemma 2.2, and (5.10), we see that

$$\mathrm{II}_1 \le \sum_{k=1}^N w(B_k^*) \lesssim \sum_{k=1}^N w(B_k) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}}^n |f(x)|^p w(x)\, dx. \tag{5.16}$$

For $\mathrm{II}_3$, by the Chebyshev inequality and the fact that $T$ is bounded on $L^2(w, \mathbb{R}^n)$, we have

$$\begin{aligned}
\mathrm{II}_3 &= w\left(\left\{x \in \mathbb{R}^n : \left|T\left(\sum_{k=1}^N [I - (I - e^{-t_k L_w})^m](b_k)\right)(x)\right| > \alpha\right\}\right) \\
&\lesssim \frac{1}{\alpha^2} \left\|\sum_{k=1}^N [I - (I - e^{-t_k L_w})^m](b_k)\right\|_{L^2(w, \mathbb{R}^n)}^2 .
\end{aligned}$$

From this and the fact that

$$I-(I-e^{-t_kL_w})^m = I-\sum_{j=0}^{m}\binom{m}{j}e^{-jt_kL_w} = -\sum_{j=1}^{m}\binom{m}{j}e^{-jt_kL_w},$$

where $\binom{m}{j}$ denotes the binomial coefficients, it follows that

$$\mathrm{II}_3 \lesssim \frac{1}{\alpha^2}\sum_{j=1}^{m}\left\|\sum_{k=1}^{N}e^{-jt_kL_w}(b_k)\right\|^2_{L^2(w,\mathbb{R}^n)}. \tag{5.17}$$

For any $k,l\in\mathbb{N}$, let $S(l,k):=2^{l+1}B_k\setminus 2^lB_k$ and $S(0,k):=2B_k$. For any $h\in L^2(w,\mathbb{R}^n)$ with $\|h\|_{L^2(w,\mathbb{R}^n)}=1$, let $h_{(l,k)}:=h\chi_{S(l,k)}$. Then, for any $j\in\{1,\dots,m\}$, from (5.9), the Hölder inequality, Lemmas 5.4 and 5.2, we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\sum_{k=1}^{N}e^{-jt_kL_w}(b_k)(x)h(x)w(x)\,dx\right|\\
&\quad=\left|\sum_{k=1}^{N}\sum_{l=0}^{\infty}\int_{B_k}b_k(x)e^{-jt_kL_w^*}(h_{(l,k)})(x)w(x)\,dx\right|\\
&\quad=\left|\sum_{k=1}^{N}\sum_{l=0}^{\infty}\int_{B_k}b_k(x)[e^{-jt_kL_w^*}(h_{(l,k)})(x)-(e^{-jt_kL_w^*}(h_{(l,k)}))_{B_k}]w(x)\,dx\right|\\
&\quad\le\sum_{k=1}^{N}\sum_{l=0}^{\infty}\|b_k\|_{L^p(w,B_k)}\|e^{-jt_kL_w^*}(h_{(l,k)})-(e^{-jt_kL_w^*}(h_{(l,k)}))_{B_k}\|_{L^{p'}(w,B_k)}\\
&\quad\lesssim\alpha\sum_{k=1}^{N}\sum_{l=0}^{\infty}[w(B_k)]^{\frac1p}[w(B_k)]^{\frac{1}{p'}-\frac12}\|\sqrt{t_k}\nabla e^{-jt_kL_w^*}(h_{(l,k)})\|_{L^2(w,\mathbb{R}^n)}\\
&\quad\lesssim\alpha\sum_{k=1}^{N}\sum_{l=0}^{\infty}[w(B_k)]^{\frac12}e^{-\frac{[d(S(l,k),B_k)]^2}{cjt_k}}\|h_{(l,k)}\|_{L^2(w,S(l,k))}\\
&\quad\lesssim\alpha\sum_{k=1}^{N}\sum_{l=0}^{\infty}[w(B_k)]^{\frac12}e^{-c4^l}\|h\|_{L^2(w,S(l,k))},
\end{aligned}$$

where $(e^{-jt_kL_w^*}(h_{(l,k)}))_{B_k}$ is as in (5.4) with $u$ replaced by $e^{-jt_kL_w^*}(h_{(l,k)})$ and $B$ replaced by $B_k$. By this, Lemma 2.2, the Kolmogorov lemma (see, for example, [**26**, Lemma 5.16]), and the fact that $\|h\|_{L^2(w,\mathbb{R}^n)}=1$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\sum_{k=1}^{N} e^{-jt_kL_w}(b_k)(x)h(x)w(x)\,dx\right|\\
&\quad\lesssim \alpha\sum_{k=1}^{N}\sum_{l=0}^{\infty}[w(B_k)]^{\frac12}e^{-c4^l}\|h\|_{L^2(w,S(l,k))}\\
&\quad\lesssim \alpha\sum_{k=1}^{N}\sum_{l=0}^{\infty}[w(B_k)]^{\frac12}e^{-c4^l}[w(2^{l+1}B_k)]^{\frac12}\\
&\qquad\qquad\times\left[\frac{1}{w(2^{l+1}B_k)}\int_{2^{l+1}B_k}|h(x)|^2w(x)\,dx\right]^{\frac12}\\
&\quad\lesssim \alpha\sum_{k=1}^{N}w(B_k)\operatorname{ess\,inf}_{y\in B_k}[M_w(|h|^2)(y)]^{\frac12}\sum_{l=0}^{\infty}e^{-c4^l}2^{nl}\\
&\quad\lesssim \alpha\sum_{k=1}^{N}\int_{B_k}\operatorname{ess\,inf}_{y\in B_k}[M_w(|h|^2)(y)]^{\frac12}w(x)\,dx\\
&\quad\lesssim \alpha\int_{\cup_{k=1}^{\infty}B_k}[M_w(|h|^2)(x)]^{\frac12}w(x)\,dx\\
&\quad\lesssim \alpha\left[w\left(\bigcup_{k=1}^{\infty}B_k\right)\right]^{\frac12}\||h|^2\|_{L^1(w,\mathbb{R}^n)}^{\frac12}\lesssim \alpha\left[w\left(\bigcup_{k=1}^{\infty}B_k\right)\right]^{\frac12},
\end{aligned}$$

where $M_w$ is as in (5.6). This, together with (5.10) and (5.11), implies that, for any $j\in\{1,\dots,m\}$,

$$\left\|\sum_{k=1}^{N}e^{-jt_kL_w}(b_k)\right\|_{L^2(w,\mathbb{R}^n)}^2\lesssim\frac{1}{\alpha^{p-2}}\int_{\mathbb{R}^n}|f(x)|^pw(x)\,dx.$$

By this and (5.17), we know that

$$\mathrm{II}_3\lesssim\frac{1}{\alpha^p}\int_{\mathbb{R}^n}|f(x)|^pw(x)\,dx. \tag{5.18}$$

Next, we estimate $\mathrm{II}_2$. For any $N \in \mathbb{N}$, let $E_N^* := (\cup_{k=1}^N B_k^*)^{\complement}$. Then it is easy to see that

$$\mathrm{II}_2 \le \frac{1}{\alpha^2} \left\| \sum_{k=1}^N T_k(b_k) \right\|_{L^2(w,E_N^*)}^2. \tag{5.19}$$

For any $k \in \{1, \ldots, N\}$, let $T_k^*$ be the adjoint operator of $T_k$, namely,

$$T_k^* = (\nabla L_w^{-1/2}(I - e^{-tL_w})^m)^*.$$

For any $\boldsymbol{h}{:=}(h_1, \ldots, h_n)$ satisfying $\|\boldsymbol{h}\|_{L^2(w,\mathbb{R}^n)}{:=}\|[\sum_i^n |h_i|^2]^{1/2}\|_{L^2(w,\mathbb{R}^n)}{=}1$ with $\operatorname{supp} h_i \subset E_N^*$, $i \in \{1, \ldots, n\}$, any $l \in \mathbb{Z}_+$ and $k \in \{1, \ldots, N\}$, let $\boldsymbol{h}_{(l,k)} := \boldsymbol{h}\chi_{S(l,k)}$. Since, for any $k \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, n\}$, $\operatorname{supp} h_i \subset E_N^*$ and $E_N^* \subset (B_k^*)^{\complement}$, it follows that $\boldsymbol{h}_{(0,k)} = \mathbf{0}$. By this, the Hölder inequality, Lemmas 5.4, 5.3, and 2.2, the fact that $m > \frac{n-1}{2}$, the Kolmogorov lemma (see, for example, [**26**, Lemma 5.16]), and the fact that $\|\boldsymbol{h}\|_{L^2(w,\mathbb{R}^n)} = 1$, we conclude that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \left[ \sum_{k=1}^N T_k(b_k)(x) \right] \cdot \boldsymbol{h}(x) w(x)\, dx \right| \\
&\quad = \left| \sum_{k=1}^N \sum_{l=1}^\infty \int_{\mathbb{R}^n} [T_k(b_k)(x)] \cdot \boldsymbol{h}_{(l,k)}(x) w(x)\, dx \right| \\
&\quad = \left| \sum_{k=1}^N \sum_{l=1}^\infty \int_{B_k} b_k(x) T_k^*(\boldsymbol{h}_{(l,k)})(x) w(x)\, dx \right| \\
&\quad = \left| \sum_{k=1}^N \sum_{l=1}^\infty \int_{B_k} b_k(x) [T_k^*(\boldsymbol{h}_{(l,k)})(x) - (T_k^*(\boldsymbol{h}_{(l,k)}))_{B_k}] w(x)\, dx \right| \\
&\quad \le \sum_{k=1}^N \sum_{l=1}^\infty \|b_k\|_{L^p(w,B_k)} \|T_k^*(\boldsymbol{h}_{(l,k)}) - (T_k^*(\boldsymbol{h}_{(l,k)}))_{B_k}\|_{L^{p'}(w,\mathbb{R}^n)} \\
&\quad \lesssim \alpha \sum_{k=1}^N \sum_{l=1}^\infty [w(B_k)]^{\frac{1}{p}} [w(B_k)]^{\frac{1}{p'} - \frac{1}{2}} \|\sqrt{t_k} \nabla T_k^*(\boldsymbol{h}_{(l,k)})\|_{L^2(w,B_k)} \\
&\quad \lesssim \alpha \sum_{k=1}^N \sum_{l=1}^\infty [w(B_k)]^{\frac{1}{2}} \left( \frac{[d(B_k, S(l,k))]^2}{t_k} \right)^{-(m+\frac{1}{2})} \|\boldsymbol{h}\|_{L^2(w,S(l,k))} \\
&\quad \lesssim \alpha \sum_{k=1}^N \sum_{l=1}^\infty [w(B_k)]^{\frac{1}{2}} 2^{-2l(m+\frac{1}{2})} [w(2^{l+1}B_k)]^{\frac{1}{2}} \\
&\qquad\qquad \times \left[ \frac{1}{w(2^{l+1}B_k)} \int_{2^{l+1}B_k} |\boldsymbol{h}(y)|^2 w(y)\, dy \right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \alpha \sum_{k=1}^{N} w(B_k) \operatorname{ess\,inf}_{y\in B_k} [M_w(|\boldsymbol{h}|^2)(y)]^{\frac{1}{2}} \sum_{l=1}^{\infty} 2^{-2l(m+\frac{1}{2}-\frac{n}{2})} \\
&\lesssim \alpha \int_{\cup_{k=1}^{\infty} B_k} [M_w(|\boldsymbol{h}|^2)(x)]^{\frac{1}{2}} w(x)\,dx \\
&\lesssim \alpha \left[ w\left( \bigcup_{k=1}^{\infty} B_k \right) \right]^{\frac{1}{2}} \||\boldsymbol{h}|^2\|_{L^1(w,\mathbb{R}^n)}^{\frac{1}{2}} \lesssim \alpha \left[ w\left( \bigcup_{k=1}^{\infty} B_k \right) \right]^{\frac{1}{2}},
\end{aligned}
$$

where $M_w$ is as in (5.6) and $(T_k^*(\boldsymbol{h}_{(l,k)}))_{B_k}$ is as in (5.4) with $u$ and $B$ replaced by $T_k^*(\boldsymbol{h}_{(l,k)})$ and $B_k$, respectively. This, combined with (5.10) and (5.11), implies that

$$
\left\| \sum_{k=1}^{N} T_k(b_k) \right\|_{L^2(w,E_N^*)}^2 \lesssim \frac{1}{\alpha^{p-2}} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.
$$

Combining this and (5.19), we have

$$
\mathrm{II}_2 \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.
$$

This, together with (5.18), (5.16), and (5.15), implies (5.14). Hence, (5.13) holds true. Combining (5.13) and (5.12), we then complete the proof of Theorem 5.6. □

We are now in a position to prove Proposition 1.6.

*Proof of Proposition 1.6:* We first prove (i). Indeed, from Theorem 5.6, it follows that, for any $f \in L^p(w,\mathbb{R}^n)$,

$$
\|\nabla L_w^{-1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|f\|_{L^p(w,\mathbb{R}^n)}.
$$

This, combined with Proposition 1.5, implies that, for any $p \in (\frac{2n}{n+1}, 2]$ and $f \in H^p_{L_w}(\mathbb{R}^n)$,

$$
\|\nabla L_w^{-1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|f\|_{H^p_{L_w}(\mathbb{R}^n)}. \tag{5.20}
$$

Next, we prove (ii). From [**51**, Theorem 1.6], we deduce that, for any $w \in A_q(\mathbb{R}^n)$ with $q \in [1, 1+\frac{1}{n})$ and $f \in H^1_{L_w}(\mathbb{R}^n)$,

$$
\|\nabla L_w^{-1/2}(f)\|_{L^1(w,\mathbb{R}^n)} \lesssim \|f\|_{H^1_{L_w}(\mathbb{R}^n)}.
$$

Combining this, (5.20), Lemma 5.1, and the fact that

$$
[L^1(w,\mathbb{R}^n), L^{p_0}(w,\mathbb{R}^n)]_\theta = L^p(w,\mathbb{R}^n),
$$

where $\theta \in (0,1)$, $1/p = (1-\theta) + \theta/p_0$, and $p_0 \in (1,\infty)$ (see, for example, [**12**, Theorem 5.5.1]), by the well-known properties of interpolation spaces (see, for example, [**12**, Theorem 4.1.2]), we obtain (5.20) in case $p \in [1, \frac{2n}{n+1}]$. This finishes the proof of Proposition 1.6. □

## 6. Proof of Proposition 1.7

To prove Proposition 1.7, we need the following local weighted Sobolev embedding theorem (see [**31**, Theorem (1.2)]).

**Lemma 6.1** ([**31**]). *Let $n \geq 2$. For any given $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, there exist positive constants $C$ and $\delta$ such that, for any number $k_0 \in [1, \frac{n}{n-1} + \delta]$, any ball $B \equiv B(x_B, r_B)$ of $\mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, and any $u \in C_c^\infty(B)$,*

$$\left[\frac{1}{w(B)} \int_B |u(x)|^{k_0 p} w(x)\, dx\right]^{\frac{1}{k_0 p}} \leq C r_B \left[\frac{1}{w(B)} \int_B |\nabla u(x)|^p w(x)\, dx\right]^{\frac{1}{p}}.$$

We are now in a position to prove Proposition 1.7.

*Proof of Proposition 1.7:* Let $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$ and $w \in A_p(\mathbb{R}^n) \cap A_2(\mathbb{R}^n)$. We first show that, for any given $p \in [2, \frac{2n}{n-1})$ and any $h \in L^2(w, \mathbb{R}^n) \cap H^p_{L_w, \mathrm{Riesz}}(\mathbb{R}^n)$,

$$\|h\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\nabla L_w^{-1/2}(h)\|_{L^p(w,\mathbb{R}^n)}. \tag{6.1}$$

Indeed, by [**25**, Theorem 1.1], the fact that $(L_w^{1/2})^* = (L_w^*)^{1/2}$, and an argument similar to that used in the proof of [**4**, Lemma 2.2], we find that, for any $u, v \in H_0^1(w, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} L_w^{1/2}(u)(x)\overline{(L_w^*)^{1/2}(v)(x)} w(x)\, dx = \int_{\mathbb{R}^n} [A(x)\nabla u(x)] \cdot \overline{\nabla v(x)}\, dx,$$

where $A$ is the complex-valued matrix associated to $L_w$, which satisfies the degenerate elliptic conditions (1.4) and (1.5). By this, we see that, for any $f \in H_0^1(w, \mathbb{R}^n)$ and $g \in L^2(w, \mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} L_w^{1/2}(f)(x)\overline{g(x)} w(x)\, dx &= \int_{\mathbb{R}^n} L_w^{1/2}(f)(x)\overline{(L_w^*)^{1/2}((L_w^*)^{-1/2}(g))(x)} w(x)\, dx \\ &= \int_{\mathbb{R}^n} [A(x)\nabla f(x)] \cdot \overline{\nabla (L_w^*)^{-1/2}(g)(x)}\, dx.\end{aligned}$$

From this, (1.4), and the Hölder inequality, it follows that, for any given $p \in [2, \frac{2n}{n-1})$, any $f \in H_0^1(w, \mathbb{R}^n)$, and $g \in L^2(w, \mathbb{R}^n) \cap L^{p'}(w, \mathbb{R}^n)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} L_w^{1/2}(f)(x)\overline{g(x)} w(x)\, dx\right| \\ &\quad \lesssim \int_{\mathbb{R}^n} |\nabla f(x)||\nabla (L_w^*)^{-1/2}(g)(x)| w(x)\, dx \\ &\quad \lesssim \|\nabla f\|_{L^p(w,\mathbb{R}^n)} \|\nabla (L_w^*)^{-1/2}(g)\|_{L^{p'}(w,\mathbb{R}^n)}.\end{aligned} \tag{6.2}$$

Observing that $p' \in (\frac{2n}{n+1}, 2]$, by Theorem 5.6, we see that

$$\|\nabla(L_w^*)^{-1/2}(g)\|_{L^{p'}(w,\mathbb{R}^n)} \lesssim \|g\|_{L^{p'}(w,\mathbb{R}^n)}.$$

By this and (6.2), we conclude that, for any given $p \in [2, \frac{2n}{n-1})$ and any $f \in H_0^1(w, \mathbb{R}^n)$,

$$\|L_w^{1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(w,\mathbb{R}^n)},$$

which further implies (6.1).

Therefore, to complete the proof of Proposition 1.7, we only need to prove (6.1) in case $p \in (\frac{2n}{n+1}, 2)$. To this end, we first recall some well-known results. Let $\mathcal{S}(\mathbb{R}^n)$ denote the *space of all Schwartz functions* and $\mathcal{S}'(\mathbb{R}^n)$ the *space of all Schwartz distributions.* For any $p \in [1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, the *weighted Sobolev space* $\dot{W}^{1,p}(w, \mathbb{R}^n)$ is defined by setting

$$\dot{W}^{1,p}(w, \mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathbb{C} : \sum_{k=1}^n \|\partial_k f\|_{L^p(w,\mathbb{R}^n)} < \infty \right\},$$

where, for any $k \in \{1, \dots, n\}$, $\partial_k f$ denotes the distributional derivative of $f$. From [**14**, Theorem 2.8(ii) and Remark 4.5(i)], it follows that, for any $p \in [1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\dot{F}_{p,2}^{1,w}(\mathbb{R}^n) = \dot{W}^{1,p}(w, \mathbb{R}^n),$$

where $\dot{F}_{p,2}^{1,w}(\mathbb{R}^n)$ denotes the homogeneous weighted Triebel spaces (see [**14**, p. 583] for the definition). By this and [**13**, Theorem 6.2], we conclude that, for any $\theta \in (0, 1)$, $1 \le p_0 \le p_1 < \infty$, and $w \in A_{p_0}(\mathbb{R}^n)$,

$$\begin{aligned}[\dot{W}^{1,p_0}(w, \mathbb{R}^n), \dot{W}^{1,p_1}(w, \mathbb{R}^n)]_\theta &= [\dot{F}_{p_0,2}^{1,w}(\mathbb{R}^n), \dot{F}_{p_1,2}^{1,w}(\mathbb{R}^n)]_\theta \\ &= \dot{F}_{p,2}^{1,w}(\mathbb{R}^n) = \dot{W}^{1,p}(w, \mathbb{R}^n),\end{aligned} \tag{6.3}$$

where $1/p = (1 - \theta)/p_0 + \theta/p_1$.

To prove (6.1) in case $p \in (\frac{2n}{n+1}, 2)$ and $w \in A_p(\mathbb{R}^n)$, we claim that it suffices to show that, for any $\alpha \in (0, \infty)$ and any $f \in \dot{W}^{1,p}(w, \mathbb{R}^n)$,

$$w\left(\left\{x\in\mathbb{R}^n : |S_1(\sqrt{L_w}(f))(x)| > \alpha\right\}\right) \lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)\, dx, \tag{6.4}$$

where, for any $h \in L^2(w, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$S_1(h)(x) := \left[ \iint_{\Gamma(x)} |t\sqrt{L_w} e^{-t^2 L_w}(h)(y)|^2 w(y) \frac{dy}{w(B(x,t))} \frac{dt}{t} \right]^{\frac{1}{2}}.$$

Indeed, since $L_w$ has a bounded $H_\infty$ functional calculus in $L^2(w, \mathbb{R}^n)$, we know that $S_1$ is bounded on $L^2(w, \mathbb{R}^n)$ (see, for example, [**3**, p. 487] or [**1**]). Thus, by this and [**25**, Theorem 1.1], we know that, for any $f \in \mathcal{S}(\mathbb{R}^n) \subset H^1_0(w,\mathbb{R}^n)$,

$$(6.5)\qquad \|S_1(\sqrt{L_w}(f))\|_{L^2(w,\mathbb{R}^n)} \lesssim \|\sqrt{L_w}(f)\|_{L^2(w,\mathbb{R}^n)} \sim \|\nabla f\|_{L^2(w,\mathbb{R}^n)}.$$

Since $\mathcal{S}(\mathbb{R}^n) \cap \dot{F}^{1,w}_{2,2}(\mathbb{R}^n)$ is dense in $\dot{F}^{1,w}_{2,2}(\mathbb{R}^n)$ (see [**13**, p. 153]), we know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{W}^{1,2}(w,\mathbb{R}^n)$. From this and a limiting procedure, we deduce that, for all $f \in \dot{W}^{1,2}(w,\mathbb{R}^n)$, (6.5) holds true. By [**12**, Theorem 5.3.1], we find that, for any $p_0 \in [1,\infty)$ and $\theta \in (0,1)$,

$$[L^{p_0,\infty}(w,\mathbb{R}^n), L^{2,\infty}(w,\mathbb{R}^n)]_\theta = L^p(w,\mathbb{R}^n),$$

where $1/p = (1-\theta)/p_0 + \theta/2$. Combining this, (6.5), (6.3), and (6.4), by the well-known properties of interpolation spaces (see, for example, [**12**, Theorem 4.1.2]), we see that, for any $q \in (p,2)$ with $p \in (\frac{2n}{n+1},2)$ and $f \in L^q(w,\mathbb{R}^n)$,

$$\|S_1(\sqrt{L_w}(f))\|_{L^q(w,\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^q(w,\mathbb{R}^n)}.$$

This, together with Proposition 3.6, implies that, for all $q \in (\frac{2n}{n+1},2)$ and $h \in L^2(w,\mathbb{R}^n \cap H^q_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$,

$$\|h\|_{H^q_{L_w}(\mathbb{R}^n)} \sim \|S_1(h)\|_{L^q(w,\mathbb{R}^n)} \lesssim \|\nabla L_w^{-1/2}(h)\|_{L^q(w,\mathbb{R}^n)}.$$

Next, we prove that, for any $f \in \mathcal{S}(\mathbb{R}^n)$, (6.4) holds true. Then, by Lemma 5.5 and a density argument, we further know that, for any $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, (6.4) holds true. For any $f \in \mathcal{S}(\mathbb{R}^n)$, by the Calderón–Zygmund decomposition of weighted Sobolev spaces (see, for example, [**5**, Proposition 1.1] or [**9**, Lemma 6.6]), we conclude that there exist positive constants $C$ and $N$ such that, for any $\alpha \in (0,\infty)$, there exist a collection $\{B_i\}_{i=1}^\infty$ of balls of $\mathbb{R}^n$, a family of functions, $\{b_i\}_{i=1}^\infty \subset C^1(\mathbb{R}^n)$, and an almost everywhere Lipschitz function $g$ such that the following properties hold true:

$$f(x) = g(x) + \sum_{i=1}^\infty b_i(x) \quad \text{for almost every } x \in \mathbb{R}^n;$$

$$(6.6)\qquad \|\nabla g\|_{L^p(w,\mathbb{R}^n)} \le C\|\nabla f\|_{L^p(w,\mathbb{R}^n)},\ |\nabla g(x)| \le C\alpha \quad \text{for almost every } x \in \mathbb{R}^n;$$

$$(6.7)\qquad \text{for any } i\in\mathbb{N},\ \operatorname{supp} b_i \subset B_i \text{ and } \int_{B_i} |\nabla b_i(x)|^p w(x)\,dx \le C\alpha^p w(B_i);$$

$$(6.8)\qquad \sum_{i=1}^\infty w(B_i) \le C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)\,dx;$$

$$(6.9)\qquad \sum_{i=1}^\infty \chi_{B_i}(x) \le N,$$

here and hereafter, for any $k \in \mathbb{N}$, $C^k(\mathbb{R}^n)$ denotes the space of all functions possessing continuous derivatives up to order $k$ on $\mathbb{R}^n$. Moreover, by the proof of [**5**, Proposition 1.1], we further see that, for any $i \in \mathbb{N}$,

$$b_i = (f - f_{B_i})\zeta_i, \tag{6.10}$$

where $0 \le \zeta_i \le 1$, $\zeta_i \in C^1(\mathbb{R}^n)$ with $\operatorname{supp}\zeta_i \subset B_i$, and $f_{B_i}$ is as in (5.4) with $u$ and $B$ replaced by $f$ and $B_i$, respectively. By this, (6.9), and the fact that $f \in \mathcal{S}(\mathbb{R}^n) \subset L^2(w,\mathbb{R}^n)$, it is easy to see that $\sum_{i=1}^\infty b_i \in L^2(w,\mathbb{R}^n)$. From this and the fact that $L_w e^{-t^2 L_w}$ is bounded on $L^2(w,\mathbb{R}^n)$ for any $t \in (0,\infty)$, it follows that

$$\lim_{N\to\infty}\sum_{i=1}^N tL_w e^{-t^2L_w}(b_i) = tL_w e^{-t^2L_w}\left(\sum_{i=1}^\infty b_i\right) \quad \text{in } L^2(w,\mathbb{R}^n). \tag{6.11}$$

For any $N \in \mathbb{N}$, let $S_N := \sum_{i=1}^N tL_w e^{-t^2L_w}(b_i)$. By (6.11), we further know that there exists a subsequence of $\{S_N\}_{N\in\mathbb{N}}$ (without loss of generality, we may use the same notation as the original sequence) such that, for almost every $x \in \mathbb{R}^n$,

$$tL_w e^{-t^2L_w}\left(\sum_{i=1}^\infty b_i\right)(x) = \lim_{N\to\infty} S_N(x) = \sum_{i=1}^\infty tL_w e^{-t^2L_w}(b_i)(x). \tag{6.12}$$

Moreover, from (6.10) and Lemma 5.4, we deduce that, for any $i \in \mathbb{N}$,

$$\int_{B_i} \frac{|b_i(x)|^2}{r_{B_i}^2} w(x)\,dx \lesssim \int_{B_i} \frac{|f(x)-f_{B_i}|^2}{r_{B_i}^2} w(x)\,dx \lesssim \int_{B_i} |\nabla f(x)|^2 w(x)\,dx,$$

which, combined with (6.9), implies that $\sum_{i=1}^\infty \frac{b_i}{r_{B_i}} \in L^2(w,\mathbb{R}^n)$. By an argument similar to that used in the proof of (6.12), we find that, for any $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$,

$$tL_w e^{-t^2L_w}\left(\sum_{i=1}^\infty \frac{b_i}{r_{B_i}}\right)(x) = \sum_{i=1}^\infty tL_w e^{-t^2L_w}\left(\frac{b_i}{r_{B_i}}\right)(x).$$

By this, (6.12) and the Minkowski inequality, for any $x \in \mathbb{R}^n$, we find that

$$
\begin{aligned}
&S_1(\sqrt{L_w}(f))(x) \\
&\quad \le S_1(\sqrt{L_w}(g))(x) \\
&\qquad + \left[\iint_{\Gamma(x)} \left|\sum_{i=1}^{\infty} tL_w e^{-t^2 L_w}(b_i)(y)\chi_{(0,r_{B_i})}(t)\right|^2 \frac{w(y)\,dy}{w(B(x,t))}\frac{dt}{t}\right]^{\frac{1}{2}} \\
&\qquad + \left[\iint_{\Gamma(x)} \left|\sum_{i=1}^{\infty} tL_w e^{-t^2 L_w}(b_i)(y)\chi_{[r_{B_i},\infty)}(t)\right|^2 \frac{w(y)\,dy}{w(B(x,t))}\frac{dt}{t}\right]^{\frac{1}{2}} \\
&\quad \le S_1(\sqrt{L_w}(g))(x) \\
&\qquad + \sum_{i=1}^{\infty}\left[\int_0^{r_{B_i}} \int_{B(x,t)} |tL_w e^{-t^2 L_w}(b_i)(y)|^2 \frac{w(y)\,dy}{w(B(x,t))}\frac{dt}{t}\right]^{\frac{1}{2}} \\
&\qquad + \left[\int_0^{\infty} \int_{B(x,t)} \left|t^2 L_w e^{-t^2 L_w}\left(\sum_{i=1}^{\infty}\frac{b_i}{r_{B_i}}\right)(y)\right|^2 \frac{w(y)\,dy}{w(B(x,t))}\frac{dt}{t}\right]^{\frac{1}{2}} \\
&\quad =: \mathrm{I}_1(x) + \sum_{i=1}^{\infty}\mathrm{I}_{2,i}(x) + \mathrm{I}_3(x).
\end{aligned}
$$

From this, we deduce that

$$
\begin{aligned}
&w\left(\left\{x \in \mathbb{R}^n : |S_1(\sqrt{L_w}(f))(x)| > \alpha\right\}\right) \\
&\quad \le w\left(\left\{x \in \mathbb{R}^n : \mathrm{I}_1(x) > \frac{\alpha}{3}\right\}\right) \\
&\qquad + w\left(\left\{x \in \mathbb{R}^n : \sum_{i=1}^{\infty}\mathrm{I}_{2,i}(x) > \frac{\alpha}{3}\right\}\right) \qquad (6.13) \\
&\qquad + w\left(\left\{x \in \mathbb{R}^n : \mathrm{I}_3(x) > \frac{\alpha}{3}\right\}\right) \\
&\quad =: \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3.
\end{aligned}
$$

We first estimate $\mathrm{A}_1$. Using the Chebyshev inequality, (6.5), and (6.6), we have

$$\begin{aligned}\mathrm{A}_1 &\lesssim \frac{1}{\alpha^2}\int_{\mathbb{R}^n} |S_1(\sqrt{L_w}(g))(x)|^2 w(x)\,dx \lesssim \frac{1}{\alpha^2}\int_{\mathbb{R}^n} |\nabla g(x)|^2 w(x)\,dx \\ &\lesssim \frac{1}{\alpha^2}\int_{\mathbb{R}^n} \alpha^{2-p}|\nabla g(x)|^p w(x)\,dx \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)\,dx.\end{aligned} \tag{6.14}$$

Next, we estimate $\mathrm{A}_3$. By the Chebyshev inequality, (4.1), and (6.9), we conclude that

$$\begin{aligned}\mathrm{A}_3 &\lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} \left| S_{L_w}\left(\sum_{i=1}^{\infty} \frac{b_i}{r_{B_i}}\right)(x)\right|^p w(x)\,dx \\ &\lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} \left|\sum_{i=1}^{\infty} \frac{b_i(x)}{r_{B_i}}\right|^p w(x)\,dx \\ &\lesssim \frac{1}{\alpha^p}\sum_{i=1}^{\infty}\int_{B_i} \frac{|b_i(x)|^p}{r_{B_i}^p} w(x)\,dx.\end{aligned} \tag{6.15}$$

Observing that $b_i \in C^1(\mathbb{R}^n)$ with $\operatorname{supp} b_i \subset B_i$, by Lemma 6.1, (6.7), and (6.8), we see that

$$\left[\int_{B_i} |b_i(x)|^p w(x)\,dx\right]^{\frac{1}{p}} \lesssim r_{B_i}\left[\int_{B_i} |\nabla b_i(x)|^p w(x)\,dx\right]^{\frac{1}{p}} \lesssim r_{B_i}\alpha[w(B_i)]^{\frac{1}{p}}. \tag{6.16}$$

This, together with (6.15), implies that

$$\mathrm{A}_3 \lesssim \sum_{i=1}^{\infty} w(B_i) \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)\,dx. \tag{6.17}$$

Finally, we estimate $\mathrm{A}_2$. From (6.13), Lemma 2.2, the Chebyshev inequality, and (6.8), it follows that

(6.18)
$$\begin{aligned}
\mathrm{A}_2 &= w\left(\left\{x \in \mathbb{R}^n : \sum_{i=1}^{\infty} \mathrm{I}_{2,i}(x) > \frac{\alpha}{3}\right\}\right) \\
&\le \sum_{i=1}^{\infty} w(4B_i) + w\left(\left\{x \in \left(\bigcup_{i=1}^{\infty} 4B_i\right)^{\complement} : \sum_{i=1}^{\infty} \mathrm{I}_{2,i}(x) > \frac{\alpha}{3}\right\}\right) \\
&\lesssim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)\,dx + \frac{1}{\alpha^2} \int_{\mathbb{R}^n} \left|\sum_{i=1}^{\infty} \mathrm{I}_{2,i}(x) \chi_{(4B_i)^{\complement}}(x)\right|^2 w(x)\,dx \\
&\sim \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)\,dx \\
&\quad + \frac{1}{\alpha^2}\left\{\sup_{\|u\|_{L^2(w,\mathbb{R}^n)}=1} \left|\int_{\mathbb{R}^n} \left[\sum_{i=1}^{\infty} \mathrm{I}_{2,i}(x)\chi_{(4B_i)^{\complement}}(x)\right] u(x) w(x)\,dx\right|\right\}^2.
\end{aligned}$$

For any $u \in L^2(w, \mathbb{R}^n)$ with $\|u\|_{L^2(w,\mathbb{R}^n)} = 1$, by the Hölder inequality, and the Fubini theorem, we find that

(6.19)
$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \left[\sum_{i=1}^{\infty} \mathrm{I}_{2,i}(x)\chi_{(4B_i)^{\complement}}(x)\right] u(x) w(x)\,dx\right| \\
&\qquad \le \sum_{i=1}^{\infty} \int_{(4B_i)^{\complement}} |\mathrm{I}_{2,i}(x)||u(x)|w(x)\,dx \\
&\qquad = \sum_{i=1}^{\infty} \sum_{j=3}^{\infty} \int_{U_j(B_i)} |\mathrm{I}_{2,i}(x)||u(x)|w(x)\,dx \\
&\qquad \le \sum_{i=1}^{\infty} \sum_{j=3}^{\infty} \left[\int_{U_j(B_i)} \int_0^{r_{B_i}} \int_{B(x,t)} |t^2 L_w e^{-t^2 L_w}(b_i)(y)|^2 \right. \\
&\qquad\qquad \left. \times \frac{w(y)\,dy}{w(B(x,t))} \frac{dt}{t^3} w(x)\,dx\right]^{\frac{1}{2}} \|u\|_{L^2(w,U_j(B_i))}
\end{aligned}$$

$$\lesssim \sum_{i=1}^{\infty}\sum_{j=3}^{\infty}\left[\iint_{(y,t)\in R(U_j(B_i)),t\in(0,r_{B_i})}|t^2L_we^{-t^2L_w}(b_i)(y)|^2w(y)\,dy\frac{dt}{t^3}\right]^{\frac{1}{2}}$$
$$\times\|u\|_{L^2(w,U_j(B_i))}$$
$$\lesssim \sum_{i=1}^{\infty}\sum_{j=3}^{\infty}\left[\int_0^{r_{B_i}}\int_{2^{j+1}B_i\setminus 2^{j-2}B_i}|t^2L_we^{-t^2L_w}(b_i)(y)|^2w(y)\,dy\frac{dt}{t^3}\right]^{\frac{1}{2}}$$
$$\times\|u\|_{L^2(w,U_j(B_i))},$$

where $U_j(B_i)$ is as in (1.10) with $B$ replaced by $B_i$ and $R(U_j(B_i))$ is as in (1.8) with $F$ replaced by $U_j(B_i)$. From Proposition 2.4, Lemma 2.2, and (6.16), it follows that there exist positive constants $c$, $\widetilde{c}$, $\theta_1$, $\theta_2$, and $\theta$ such that, for all $t\in(0,r_{B_i})$,

$$\left[\int_{2^{j+1}B_i\setminus 2^{j-2}B_i}|t^2L_we^{-t^2L_w}(b_i)(y)|^2w(y)\,dy\right]^{\frac{1}{2}}$$
$$\lesssim 2^{j\theta_1}\left[\Upsilon\left(\frac{2^jr_{B_i}}{t}\right)\right]^{\theta_2}e^{-c\left(\frac{2^jr_{B_i}}{t}\right)^2}[w(2^jB_i)]^{\frac{1}{2}}\left[\frac{1}{w(B_i)}\int_{B_i}|b_i(y)|^pw(y)\,dy\right]^{\frac{1}{p}}$$
$$\lesssim 2^{j\theta}e^{-\widetilde{c}\left(\frac{2^jr_{B_i}}{t}\right)^2}[w(B_i)]^{\frac{1}{2}-\frac{1}{p}}r_{B_i}\alpha[w(B_i)]^{\frac{1}{p}}$$
$$\lesssim 2^{j\theta}r_{B_i}\alpha e^{-\widetilde{c}\frac{(2^jr_{B_i})^2}{t^2}}[w(B_i)]^{\frac{1}{2}}.$$

From this and (6.19), via choosing a positive constant $N\in(\max\{2\theta,3\},\infty)$, we deduce that, for any $u\in L^2(w,\mathbb{R}^n)$ with $\|u\|_{L^2(w,\mathbb{R}^n)}=1$,

(6.20)
$$\left|\int_{\mathbb{R}^n}\left[\sum_{i=1}^{\infty}\mathrm{I}_{2,i}(x)\chi_{(4B_i)^{\complement}}(x)\right]u(x)w(x)\,dx\right|$$
$$\lesssim \sum_{i=1}^{\infty}\sum_{j=3}^{\infty}\alpha[w(B_i)]^{\frac{1}{2}}\left[\int_0^{r_{B_i}}2^{j(2\theta-N)}e^{-2\widetilde{c}\left(\frac{2^jr_{B_i}}{t}\right)^2}\right.$$
$$\left.\times\left(\frac{2^jr_{B_i}}{t}\right)^Nr_{B_i}^{2-N}t^{N-3}\,dt\right]^{\frac{1}{2}}\|u\|_{L^2(w,U_j(B_i))}$$

$$\lesssim \sum_{i=1}^{\infty}\sum_{j=3}^{\infty} 2^{-j(\frac{N}{2}-\theta)}\alpha[w(B_i)]^{\frac{1}{2}}\|u\|_{L^2(w,U_j(B_i))}$$

$$\lesssim \sum_{i=1}^{\infty}\sum_{j=3}^{\infty} 2^{-j(\frac{N}{2}-\theta)}\alpha w(B_i)\left[\frac{1}{w(2^jB_i)}\int_{2^jB_i}|u(y)|^2w(y)\,dy\right]^{\frac{1}{2}}$$

$$\lesssim \sum_{i=1}^{\infty}\sum_{j=3}^{\infty} 2^{-j(\frac{N}{2}-\theta)}\alpha w(B_i)\inf_{z\in B_i}[M_w(|u|^2)(z)]^{\frac{1}{2}}$$

$$\lesssim \sum_{i=1}^{\infty}\alpha\int_{B_i}[M_w(|u|^2)(z)]^{\frac{1}{2}}w(z)\,dz \lesssim \alpha\int_{\cup_{i=1}^{\infty}B_i}[M_w(|u|^2)(z)]^{\frac{1}{2}}w(z)\,dz,$$

where the Hardy–Littlewood maximal function $M_w$ is as in (5.6). Using the Kolmogorov lemma (see, for example, [**26**, Lemma 5.16]), we obtain

$$\int_{\cup_{i=1}^{\infty}B_i}[M_w(|u|^2)(z)]^{\frac{1}{2}}w(z)\,dz \lesssim \left[w\left(\bigcup_{i=1}^{\infty}B_i\right)\right]^{1-\frac{1}{2}}\||u|^2\|_{L^1(w,\mathbb{R}^n)}^{\frac{1}{2}}.$$

This, combined with (6.20), (6.18), (6.8), (6.9), and the fact that $\|u\|_{L^2(w,\mathbb{R}^n)}=1$, implies that

$$\mathrm{A}_2 \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n}|\nabla f(x)|^p w(x)\,dx + w\left(\bigcup_{i=1}^{\infty}B_i\right) \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n}|\nabla f(x)|^p w(x)\,dx. \tag{6.21}$$

Combining (6.13), (6.14), (6.17), and (6.21), we find that, for any $f\in\mathcal{S}(\mathbb{R}^n)$,

$$w\left(\left\{x\in\mathbb{R}^n : |S_1(\sqrt{L_w}(f))(x)|>\alpha\right\}\right) \lesssim \frac{1}{\alpha^p}\int_{\mathbb{R}^n}|\nabla f(x)|^p w(x)\,dx,$$

which further implies (6.4). This finishes the proof of Proposition 1.7. □

## 7. Proof of Theorem 1.1

In this section, we show that Propositions 1.5, 1.6, and 1.7 imply Theorem 1.1.

*Proof of Theorem 1.1:* Let $p\in(\frac{2n}{n+1},2]$ and $w\in A_p(\mathbb{R}^n)$. For any $f\in C_c^\infty(\mathbb{R}^n)$, by [**25**, Theorem 1.1], we know that $L_w^{1/2}(f)\in L^2(w,\mathbb{R}^n)$. Moreover, it is easy to see that, for any $p\in(\frac{2n}{n+1},2]$ and any $f\in C_c^\infty(\mathbb{R}^n)$,

$$\|\nabla L_w^{-1/2}(L_w^{1/2}(f))\|_{L^p(w,\mathbb{R}^n)} = \|\nabla f\|_{L^p(w,\mathbb{R}^n)} < \infty,$$

which implies that $L_w^{1/2}(f) \in H^p_{L_w,\mathrm{Riesz}}(\mathbb{R}^n)$. Thus, by Propositions 1.7 and 1.5, we conclude that, for any $p \in (\frac{2n}{n+1}, 2]$ and any $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|L_w^{1/2}(f)\|_{L^p(w,\mathbb{R}^n)} \lesssim \|\nabla L_w^{-1/2}(L_w^{1/2}(f))\|_{L^p(w,\mathbb{R}^n)} \sim \|\nabla f\|_{L^p(w,\mathbb{R}^n)}, \tag{7.1}$$

which further implies that $L_w^{1/2}(f) \in L^p(w,\mathbb{R}^n)$. From this, Propositions 1.5 and 1.6, we deduce that, for any $p \in (\frac{2n}{n+1}, 2]$ and any $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|\nabla f\|_{L^p(w,\mathbb{R}^n)} = \|\nabla L_w^{-1/2}(L_w^{1/2}(f))\|_{L^p(w,\mathbb{R}^n)} \lesssim \|L_w^{1/2}(f)\|_{L^p(w,\mathbb{R}^n)}.$$

This, together with (7.1), then finishes the proof of Theorem 1.1. □

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