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GRÖBNER TECHNIQUES AND RIBBONS

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Abstract. We use Gröbner basis techniques to study the balanced canonical ribbon in each odd genus $g \geq 5$. We obtain equations and syzygies of the ribbon, give a Gröbner interpretation of part of Alper, Fedorchuk, and Smyth's proof of finite Hilbert stability for canonical curves, and discuss the obstacles in using ribbons to give a new proof of Generic Green's Conjecture (Voisin's Theorem).

1. Introduction

A canonical rational ribbon is a double structure on $\mathbb{P}^1$ with a very ample dualizing line bundle. As Bayer and Eisenbud show in their seminal paper [2], canonical rational ribbons arise as flat limits in families of canonically embedded curves specializing to a hyperelliptic curve in moduli. In [7], Fong proved that every canonically embedded rational ribbon can be smoothed to a canonical curve with the same Clifford index as the ribbon. Conversely, if one performs stable reduction on a family of smooth curves specializing to a ribbon, one will obtain as the stable limit a Deligne-Mumford stable curve in the closure of the hyperelliptic locus. Hence, it

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is useful to think of ribbons as possible replacements of hyperelliptic curves in the Hilbert scheme of canonical curves.

In [2], Bayer and Eisenbud studied ribbons with a view toward Generic Green's Conjecture (now Voisin's Theorem) on the graded Betti numbers of canonically embedded curves. By Fong's Theorem and the upper semicontinuity of Betti numbers, proving Green's Conjecture for ribbons would establish the result for a general canonical curve as well. However, it seems that the approach suggested in [2] has never been completed.

More recently, ribbons have appeared in the log minimal model program (LMMP) for the pair $(\overline{\mathcal{M}}_g, \delta)$, where $\overline{\mathcal{M}}_g$ is the moduli space of Deligne-Mumford stable curves, and δ is the divisor of nodal curves. Geometric invariant theory (GIT) calculations suggest that at a certain stage of the LMMP, the locus of hyperelliptic curves in $\overline{\mathcal{M}}_g$ will be flipped to a locus of canonically embedded A_{2g}-curves (see [6, Section 4]). While ribbons lie in codimension 2 inside the locus of A_{2g}-curves, their GIT stability analysis is simplified by the fact that some canonically embedded ribbons admit a $\mathbb{G}_m$-action.

It is conjectured that GIT quotients of the Hilbert scheme of canonical curves are log canonical models of $\overline{\mathcal{M}}_g$ that appear at later stages of the LMMP (see [16]). In [1], Alper, Fedorchuk, and Smyth prove that a general odd genus canonical ribbon is indeed GIT semistable in this setup by proving semistability of a special canonical ribbon with $\mathbb{G}_m$-action, called *the balanced ribbon*. We recall the definition of the balanced ribbon in Section 4. In Section 5.2, we reinterpret certain results of [1] in terms of Gröbner bases to gain further understanding of the combinatorics involved.

The outline of this paper is as follows. In Section 2, we describe the two problems we study using ribbons: Generic Green's Conjecture (Voisin's Theorem) and finite Hilbert stability of canonical curves. Section 3 is devoted to a detailed example of using Gröbner techniques to analyze rational normal curves. We included this as a model of how one can use Gröbner basis techniques to analyze ribbons. In Section 4, we describe balanced ribbons in detail and obtain their equations and first syzygies. The main result of the paper is Theorem 4.4. In Section 5, we discuss applications of Gröbner basis techniques for ribbons.

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2. Two problems involving ribbons

2.1. **Ribbons.** We begin with the most general definition of ribbons:

Definition 2.1 ([2, §1])**.** *A* ribbon *on D is a scheme C equipped with an isomorphism $D \simeq C_{\mathrm{red}}$ such that the ideal sheaf $\mathcal{I}$ of D in C satisfies $\mathcal{I}^2 = 0$, and $\mathcal{I}$ is a line bundle on D.*

In the sequel, we will only consider the case $D = \mathbb{P}^1$. In fact, we shall only consider a very special family of ribbons on $\mathbb{P}^1$, one in each odd genus, called balanced ribbons; see Definition 4.1.

Our motivation for studying ribbons is to gain insight into two problems in algebraic geometry: Generic Green's Conjecture (Voisin's Theorem) and finite Hilbert stability. In this section, we state these two problems.

2.2. **Generic Green's Conjecture.** Let $S = \mathbb{K}[x_0, \ldots, x_k]$, and let M be a finitely generated graded S-module. Let

$$\cdots \to \mathbf{F}_2 \to \mathbf{F}_1 \to \mathbf{F}_0 \to M \to 0$$

be the minimal graded free resolution of M. Since it is a graded free resolution, we have for each i that

$$\mathbf{F}_i = \bigoplus S(-j)^{\oplus \beta_{i,j}}.$$

The numbers $\beta_{i,j}$ are called the *graded Betti numbers* of M. By the definition of Tor, we also have $\beta_{i,j} = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(M, \mathbb{K})_j$. (This observation will be important in the sequel; since the numbers $\beta_{i,j}$ are dimensions of cohomology groups, they are upper semicontinuous in flat families.)

The Betti table of M is the collection of Betti numbers. By convention, the entry in row j column i is $\beta_{i,i+j}$ so that the table looks as follows:

$$\begin{array}{ccccccc} \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} & \beta_{4,4} & \cdots \\ \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} & \beta_{4,5} & \cdots \\ \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \beta_{3,5} & \beta_{4,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Definition 2.2. *A free resolution is* pure *if there is at most one nonzero $\beta_{i,j}$ for each $\mathbf{F}_i$. We will say that a homogeneous ideal in S has* pure Betti table *if its minimal graded free resolution is pure.*

Under the conventions for displaying Betti tables, purity corresponds to the property that there is at most one nonzero entry in each column.

We may now state the generic version of the famous Green's Conjecture [11], which was proven by Voisin in [21] for even genus and in [22] for odd genus.

Theorem 2.3 (Generic Green's Conjecture)**.** *The homogeneous coordinate ring of a general canonical curve of odd genus has a pure Betti table.*

2.3. **Finite Hilbert stability.** In the 1960's, Mumford developed geometric invariant theory (GIT) to construct the moduli space of smooth curves $\mathcal{M}_g$ as a quasi-projective variety. GIT has been an important (but difficult to wield) tool in algebraic geometry ever since.

One of Mumford's foundational insights was that GIT quotients should depend on two ingredients:

(1) a scheme X with an action of a reductive algebraic group G;
(2) a linearization of the group action, that is, a lifting of the group action to the action on sections of an ample line bundle L.

Given these two ingredients, the GIT quotient $X/\!\!/_L G$ is defined as $\operatorname{Proj}(R)$, where R is the ring of invariants of the section ring of L. There is a rational map from X to $X/\!\!/_L G$ which is defined at a point $x \in X$ if there exists an invariant section of a power of L that does not vanish at x. Such points are said to be *semistable*.

In the early 1980's, Gieseker built on Mumford's work and gave a GIT construction of the moduli space of stable curves $\overline{\mathcal{M}}_g$ [8, 9]. We describe his setup in more detail now:

Consider $X \subset \mathbb{P}^k$. Let $S = \mathbb{K}[x_0, \ldots, x_k]$ and let I be the ideal of X. We call the point in the appropriate Grassmannian parameterizing the subspace $I_m \subseteq S_m$ the m^{th} *inner Hilbert point of* I, and if $S_m \to \mathrm{H}^0(X, \mathcal{O}_X(m))$ is surjective, we call the point in the appropriate Grassmannian parameterizing this quotient the m^{th} *outer Hilbert point of* X. (The adjectives "inner" and "outer" will be explained below in Section 3.4.) In each case, the Plücker line bundle on the relevant Grassmannian yields a GIT linearization of the natural $\mathrm{SL}(k+1)$-action.

The terminology *Hilbert points* comes from the fact that for sufficiently large m the sequence

$$0 \to I_m \to S_m \to \mathrm{H}^0(X, \mathcal{O}_X(m)) \to 0$$

is exact for all subschemes X with a fixed Hilbert polynomial $P(t)$. Therefore, the assignment

$$X \subset \mathbb{P}^k \mapsto [S_m \to \mathrm{H}^0(X, \mathcal{O}_X(m))] \in \mathbf{Gr}(P(m), S_m)$$

embeds the Hilbert scheme in the Grassmannian of $P(m)$-dimensional quotients of S_m.

Recently, Hassett, Hyeon, and many others have extended Gieseker's work with the aim of carrying out the log minimal model program for $\overline{\mathcal{M}}_g$. Specifically, Gieseker proved GIT stability of smooth curves of sufficiently high degree when $m \gg 0$, while the more recent work has focused on quotients when the curves are canonically or bicanonically embedded, and when the linearization parameter m is small. For example, the interesting values of m in the bicanonical case are $2 \leq m \leq 6$ [16]. We will refer to the GIT stability problem for small fixed m as *finite Hilbert stability* (in contrast with Gieseker's *asymptotic Hilbert stability*).

Ribbons play an important role in the proof of finite Hilbert stability. In [1], Alper, Fedorchuk, and Smyth show that in each odd genus, there is a ribbon called the balanced ribbon whose m^{th} Hilbert point is semistable for any $m \geq 2$. This implies that m^{th} Hilbert point of a general odd genus canonical curve is also Hilbert semistable. See Sections 4 and 5 for more details.

3. Motivating example: rational normal curves

In this section, we use Gröbner techniques to analyze the Betti tables and finite Hilbert stability of rational normal curves. The calculations below are presented as a model of what could be done for balanced ribbons. Some parts of the calculations below are standard exercises in commutative algebra. Also, one can give much more conceptual proofs of the two main results below using some of the additional good properties of rational normal curves. However, in this section, we use Gröbner basis calculations because these tools are available for ribbons, too.

3.1. **Parametrization.** Recall that the rational normal curve of degree k is the closure of the morphism $\operatorname{Spec} \mathbb{K}[t] \to \mathbb{P}^k$ given by

$$t \mapsto [1 : t : t^2 : \cdots : t^k].$$

3.2. **Elimination.** To obtain equations for the rational normal curve of degree k, we eliminate t from the parameterization above. Let $x_0, \ldots, x_k$ be coordinates on $\mathbb{P}^k$. The parameterization above yields the equations $tx_i - x_{i+1}$ for $i = 0, \ldots, k-1$. Hence, the elimination ideal is

$$I_E = \langle tx_i - x_{i+1} \mid i = 0, \ldots, k-1 \rangle.$$

For the elimination order, we use the Bayer-Stillman 1-elimination order with t first, followed by grevlex on the variables $x_0, \ldots, x_k$.

Theorem 3.1. *The following quadrics form a Gröbner basis with respect to the Bayer-Stillman elimination term order for the elimination ideal I_E of the rational normal curve of degree k:*

(1) $\{tx_i - x_{i+1} \mid i = 0, \ldots, k-1\}$
(2) $\{x_{i+1}x_j - x_i x_{j+1} \mid 0 \leq i < j \leq k-1\}$

Proof. We use Buchberger's algorithm to show that Type (1) and Type (2) quadrics indeed form a Gröbner basis with respect to the specified term order. First, we compute the S-pairs for a pair of Type (1) generators listed above. Without loss of generality, suppose that $i < j$. We have

$$\begin{aligned} S(\underline{tx_i} - x_{i+1}, \underline{tx_j} - x_{j+1}) &= x_j(\underline{tx_i} - x_{i+1}) - x_i(\underline{tx_j} - x_{j+1}) \\ &= -\underline{x_j x_{i+1}} + x_i x_{j+1}. \end{aligned}$$

This cannot be further reduced using the Gröbner basis elements of the form $tx_\ell - x_{\ell+1}$, so we add $\underline{x_{i+1}x_j} - x_i x_{j+1}$ to the Gröbner basis.

Next, we consider the S-pairs between a generator of the form $\underline{tx_i} - x_{i+1}$ and a generator of the form $\underline{x_{a+1}x_b} - x_a x_{b+1}$. The leading terms are coprime unless $i = a+1$ or $i = b$. Suppose first that $i = a+1$. Then

$$\begin{aligned} S(\underline{tx_i} - x_{i+1}, \underline{x_i x_b} - x_{i-1}x_{b+1}) &= x_b(\underline{tx_i} - x_{i+1}) - t(\underline{x_i x_b} - x_{i-1}x_{b+1}) \\ &= -x_{i+1}x_b + tx_{i-1}x_{b+1}. \end{aligned}$$

Subtracting $x_{b+1}(tx_{i-1} - x_i)$ yields

$$-x_{i+1}x_b + x_i x_{b+1},$$

which is already in the Gröbner basis.

Similarly, we can argue that if $i = b$, the S-pair reduces to 0 under the Gröbner basis.

Finally, we consider the S-pairs between two generators of the form $\underline{x_{i+1}x_j} - x_i x_{j+1}$ and $\underline{x_{a+1}x_b} - x_a x_{b+1}$. The leading terms are coprime unless $i = a$, $j = a+1$, $i+1 = b$, or $j = b$. Suppose first that $i = a$. Then

$$\begin{aligned} S(\underline{x_{i+1}x_j} - x_i x_{j+1}, \underline{x_{i+1}x_b} - x_i x_{b+1}) &= x_b(\underline{x_{i+1}x_j} - x_i x_{j+1}) - x_j(\underline{x_{i+1}x_b} - x_i x_{b+1}) \\ &= -x_i x_{j+1}x_b + x_i x_j x_{k+1}. \end{aligned}$$

This reduces to 0 if we add $x_i(x_{j+1}x_b - x_j x_{b+1})$. The other cases ($j = a+1$, $i+1 = b$, $j = b$) are similar. □

Corollary 3.2. *The generators $\{x_{i+1}x_j - x_i x_{j+1} \mid 0 \leq i < j \leq k-1\}$ form a Gröbner basis with respect to the grevlex term order for the ideal of the rational normal curve of degree k.*

Definition 3.3. *For each subset* $\{p,q,r\} \subset \{0,\dots,k-1\}$, *we define*

$$\begin{aligned} S'_{p,q,r} := {} & x_r(x_p x_{q+1} - x_{p+1}x_q) \\ & - x_q(x_p x_{r+1} - x_{p+1}x_r) \\ & + x_p(x_q x_{r+1} - x_{q+1}x_r). \end{aligned} \tag{3.4}$$

$$\begin{aligned} S''_{p,q,r} := {} & x_{r+1}(x_p x_{q+1} - x_{p+1}x_q) \\ & - x_{q+1}(x_p x_{r+1} - x_{p+1}x_r) \\ & + x_{p+1}(x_q x_{r+1} - x_{q+1}x_r). \end{aligned} \tag{3.5}$$

Corollary 3.6. *A Gröbner basis for the module of linear syzygies between the quadrics of the rational normal curve of degree* k *is given by* $S'_{p,q,r}$ *and* $S''_{p,q,r}$ *for each subset* $\{p,q,r\} \subset \{0,\dots,k-1\}$.

Proof. This follows from the calculations in the proof of Theorem 3.1 and Schreyer's Theorem [4, Theorem 15.10]. □

3.3. **Purity of the Betti table.** In the proposition below, we outline one approach to computing the Betti numbers of the rational normal curve. It is based on a theorem of Hochster for computing the graded Betti numbers of squarefree monomial ideals.

Proposition 3.7.

(1) *The generators* $\{x_i x_{j+1} - x_{i+1}x_j \mid 0 \le i < j \le k-1\}$ *form a Gröbner basis with respect to the lex term order for the ideal of the rational normal curve of degree* k.
(2) *The lex initial ideal of the rational normal curve of degree* k *is* $\{x_i x_{j+1} \mid 0 \le i < j \le k-1\}$. *In particular, it is squarefree.*
(3) *The Stanley-Reisner complex* Δ *of* $\operatorname{in}_{\text{lex}} I$ *can be identified with the interval* $[0,k]$.
(4) *The nonzero Betti numbers of* $\operatorname{in}_{\text{lex}} I$ *are* $\beta_{0,0} = 1$ *and* $\beta_{i,i+1} = i\binom{k}{i}$.
(5) *The nonzero Betti numbers of* I *are* $\beta_{0,0} = 1$ *and* $\beta_{i,i+1} = i\binom{k}{i}$.

Proof. All five statements above are exercises using standard results in combinatorial commutative algebra. We give some hints. For Part (1), run Buchberger's algorithm with the lex term order. Part (2) follows immediately from Part (1).

For part (3), see [15, Ch. 1] for the relevant definitions. For part (4), we use Hochster's Theorem. A reference for Hochster's Theorem is [15, Corollary 5.12], where the notation is also explained. Hochster's Theorem states that the nonzero Betti numbers of S/I_Δ lie only in squarefree multidegrees σ, and

$$\beta_{i,\sigma}(S/I_\Delta) = \dim_{\mathbb{K}} \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; \mathbb{K}).$$

Since Δ is one-dimensional and contractible, the only nonzero cohomology of any $\Delta|_\sigma$ is in degrees -1 or 0. The cohomology in degree -1 gives the first row of the Betti table, and we can easily show that β_{ii} is 1 if $i = 0$ and is 0 if $i \neq 0$. The cohomology in degree 0 gives the second row of the Betti table. Here, $\dim_{\mathbb{K}} \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma; \mathbb{K})$ is the number of connected components of $\Delta|_\sigma$ minus 1. We use a formula adapted from [10, p. 55]: Let $\Delta = [0,k]$. The number of subsets $\sigma \subset \{0,\dots,k\}$ such that $|\sigma| = i+1$ and $\Delta|_\sigma$ has $i-m+1$ connected components is

$$c(m, k+1, i+1) = \binom{i}{m}\binom{k-i+1}{i-m+1}.$$

Thus

$$\begin{aligned}\beta_{i,i+1} &= \sum_{|\sigma|=i+1} \beta_{i,\sigma}(S/I_\Delta) \\ &= \sum_{m=0}^{i} (i-m)\binom{i}{m}\binom{k-i+1}{i-m+1}.\end{aligned}$$

We then use the following combinatorial identity: let k be an arbitrary positive integer, and let i be an integer such that $1 \leq i \leq k$. Then

$$\sum_{m=0}^{i} (i-m)\binom{i}{m}\binom{k-i+1}{i-m+1} = i\binom{k}{i+1}.$$

Finally, for the last part, since the Betti table of the lex initial ideal is pure, the Betti table of the rational normal curve is pure, also. Furthermore, these two ideals have the same Hilbert function. But the Hilbert function of an ideal with a pure Betti table determines the graded Betti numbers, and so the graded Betti numbers of the rational normal curve are equal to the graded Betti numbers of the lex initial ideal. □

Remark 3.8. *A more standard way to compute the Betti numbers of a rational normal curve is to use the fact that it is a determinantal variety and to use the Eagon-Northcott complex.*

3.4. **Finite Hilbert semistability.** Next, we seek to prove finite Hilbert semistability of a rational normal curve. We follow an approach first proposed by Bayer and Morrison that uses the *state polytope* of an ideal.

We first discuss state polytopes of points in a Grassmannian. Let $T \simeq \mathbb{G}_m^r$ be a torus. We identify the characters of T with $\mathbb{Z}^r$. Suppose V is a T-representation with a basis $\{v_1, \ldots, v_n\}$ diagonalizing the T-action. Let $\{\chi_i\}_{i=1}^n$ be the characters of T corresponding to $\{v_i\}_{i=1}^n$. For any $0 \leq p \leq \dim(V)$, the Grassmannian (of p-dimensional quotients) $\mathbf{Gr}(p, V)$ admits a T-action, which is linearized by the Plücker coordinates

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} \mid i_1 < \cdots < i_p\}.$$

The T-state of a Plücker coordinate $v_{i_1} \wedge \cdots \wedge v_{i_p}$ is the associated character $\sum_{j=1}^p \chi_{i_j} \in \mathbb{Z}^r$ of T.

Definition 3.9 (State Polytopes for Grassmannian)**.** *Consider $Q \in \mathbf{Gr}(p, V)$. The T-state associated to a nonzero Plücker coordinate of Q is called a T-state of Q. The state polytope* State(Q) *of Q is defined to be the convex hull in $\mathbb{Z}^r$ of all T-states of Q.*

Remark 3.10. *T-states of $Q = [V \to W \to 0]$ come from nonzero Plücker coordinates of Q diagonalizing the T-action. These in turn correspond to subsets $\{v_{i_1}, \ldots, v_{i_p} \mid i_1 < \cdots < i_p\} \subset V$ such that the images of $\{v_{i_1}, \ldots, v_{i_p}\}$ span W. By a slight abuse of language, we will call such a subset a T-basis of W.*

We proceed to give a description of the state polytope of an ideal, since this is what we shall actually use. We refer to [3] and [17] for more details on state polytopes of ideals, and the original motivation for its definition.

Definition 3.11. *The m^{th}* inner state of a monomial ideal J *is the sum of the exponent vectors of the degree m monomials in J:*

$$\sum_{\mathbf{x}^{\mathbf{a}}\in J:\ \deg(\mathbf{x}^{\mathbf{a}})=m} \mathbf{a},$$

The m^{th} inner state polytope of an ideal $I \subset \mathbb{K}[x_0,\dots,x_k]$ *is the convex hull of the m^{th} inner states of the initial ideals of I.*

Similarly, we define the m^{th} outer state of a monomial ideal J *as the sum of the exponent vectors of the degree m monomials outside J, and the m^{th}* outer state polytope of an ideal *as the convex hull of the m^{th} outer states of the initial ideals of I.*

We now explain the relation between Definitions 3.9 and 3.11. To begin, let $S = \mathbb{K}[x_0,\dots,x_k]$. Consider a subscheme $X \subset \mathbb{P}^k$ defined by homogeneous ideal I and with Hilbert polynomial $P(t)$. For an integer m such that $\mathrm{H}^1(\mathbb{P}^k, I(m)) = 0$, the m^{th} (inner or outer) Hilbert point of X is specified by the short exact sequence

$$0 \to I(m) \to S_m \to \mathrm{H}^0(X, \mathcal{O}_X(m)) \to 0.$$

Then for m large enough, the state polytope of the outer m^{th} Hilbert point of X considered as a point in the Grassmannian $\mathbf{Gr}(P(m), S_m)$ is the m^{th} outer state polytope of I; see [3] and [17].

Since for a fixed m, the union of the monomials inside and outside a monomial ideal must be all the degree m monomials, it follows that the inner and outer states of a monomial ideal are related by an affine linear transformation. Precisely, the sum of the m^{th} inner and outer states is

$$\left(\frac{m\binom{k+m}{k}}{k+1},\dots,\frac{m\binom{k+m}{k}}{k+1}\right).$$

This allows us to phrase most of the results below in terms of either the inner or the outer state polytope, whichever is more convenient.

Let $P(t)$ denote the Hilbert polynomial of S/I as before. That is, $P(m) = \dim_{\mathbb{K}}((S/I)_m)$. Then the m^{th} outer state polytope as we defined it above lies in an affine hyperplane in $\mathbb{R}^{k+1}$ with equation $z_0 + \cdots + z_k = mP(m)$. In particular, the trivial character, which we denote $\mathbf{0}_m$, is represented by the point on this hyperplane with all coordinates equal. That is,

$$\mathbf{0}_m = \left(\frac{mP(m)}{k+1},\dots,\frac{mP(m)}{k+1}\right)$$

for outer states. Similarly, when we are working with inner states, a formula for the trivial character is

$$\mathbf{0}_m = \left(\frac{m\binom{k+m}{k} - mP(m)}{k+1},\dots,\frac{m\binom{k+m}{k} - mP(m)}{k+1}\right).$$

The connection between Hilbert semistability and state polytopes is given by the Hilbert-Mumford Numerical Criterion applied to Hilbert points and can be phrased as follows:

Proposition 3.12 ([3, Theorem 4.1], [17, Criterion 3.4])**.** *Let $X \subset \mathbb{P}^k$ have ideal $I \subset \mathbb{K}[x_0,\dots,x_k]$, and let T be the maximal torus scaling these variables. The m^{th} inner (respectively, outer) Hilbert point of X is T-semistable if and only if the trivial character lies in the m^{th} inner (respectively, outer) state polytope of I.*

Observe that the proposition only gives Hilbert semistability with respect to T. However, under certain additional hypotheses, T semistability establishes $\mathrm{SL}(k+1)$ semistability.

Proposition 3.13. *Consider $X \subset \mathbb{P}^k$. Let $G \subseteq \mathrm{Stab}_{\mathrm{SL}(k+1)}(X)$ be a linearly reductive group. We say that $X \subset \mathbb{P}^k$ is* multiplicity free *with respect to G if no irreducible G-submodule has multiplicity greater than 1 in the representation of $G \to \mathrm{SL}(k+1)$.*

Suppose that X is multiplicity free. Choose coordinates $x_0, \ldots, x_k$ on $\mathbb{P}^k$ that are adapted to the decomposition of $\mathbb{K}^{k+1}$ into irreducible G-submodules. Let T be the maximal torus scaling these variables.

Let $[X]_m$ be the m^{th} Hilbert point of $X \subset \mathbb{P}^k$. Then $[X]_m$ is T-semistable if and only if $[X]_m$ is $\mathrm{SL}(k+1)$-semistable.

Proof. This is proved in [17, Proposition 4.7] using Kempf's instability results [13]. (Morrison and Swinarski state the result for finite groups G, but their proof applies verbatim in the case of an arbitrary linearly reductive G.) When $G = \mathbb{G}_m$, as often is the case, the claim also follows directly from Luna's criteria for orbit closedness [Cor. 2 and Rem. 1][14]. □

Our strategy is now clear. To prove Hilbert stability of the rational normal curve for some finite degree m, we want to show that the trivial character is in the m^{th} state polytope of the rational normal curve. For this, it is enough to exhibit two initial ideals (vertices of the inner state polytope) such that the trivial character lies between them. Not surprisingly, our two choices are the lex and grevlex initial ideals. We leave the calculations to the reader.

Proposition 3.14.

(1) *The m^{th} outer state of the lex initial ideal is*

$$\left(\frac{1}{2}m^2 + \frac{1}{2}m,\ m^2,\ \ldots,\ m^2,\ \frac{1}{2}m^2 + \frac{1}{2}m\right).$$

(2) *The m^{th} outer state of the grevlex initial ideal is*

$$\left(\frac{k}{2}m^2 - \frac{k-2}{2}m,\ m,\ \ldots,\ m,\ \frac{k}{2}m^2 - \frac{k-2}{2}m\right).$$

(3) *For any $m \geq 2$, the m^{th} Hilbert point of the rational normal curve of degree k is $\mathrm{SL}(k+1)$-semistable.*

Remark 3.15. *A more conceptual proof of Proposition 3.14 part (3) follows from the fact that a rational normal curve is a homogeneous variety embedded by a complete linear system* [13, Corollary 5.1].

4. Equations and syzygies of balanced ribbons

We now apply to ribbons the techniques illustrated in the previous section for rational normal curves. In this section, we describe equations and syzygies of canonically embedded balanced ribbons, which we now define.

Definition 4.1. *Let $g = 2k+1$ be an odd integer with $k \geq 1$. The* balanced ribbon of genus g *is the nonreduced curve C obtained as follows: Let $U := \mathrm{Spec}\, \mathbb{K}[u, e]/(e^2)$,*

$V := \operatorname{Spec} \mathbb{K}[v,f]/(f^2)$, and glue $U \smallsetminus \{0\}$ and $V \smallsetminus \{0\}$ via the isomorphism

$$u \mapsto v^{-1} - v^{-k-2} f,$$
$$e \mapsto v^{-g-1} f.$$

In [1, Lemma 3.1], Alper, Fedorchuk, and Smyth describe a basis of differentials on the balanced ribbon. Their result in our notation is as follows:

Proposition 4.2. *A basis of $\mathrm{H}^0(C, \omega_C)$ is given by differentials of the form $f(t,e)\frac{dt \wedge de}{e^2}$, where $f(t,e)$ ranges over the following functions:*

$$\begin{array}{ll} t^i & i = 0, 1, \ldots, k, \\ t^{2k-j} + (k-j)t^{k-j-1}e, & j = k-1, k-2, \ldots, 0. \end{array}$$

This leads to a parametrization of the canonically embedded balanced ribbon of genus g. Namely, the ribbon C is the closure of the map $\operatorname{Spec} \mathbb{K}[t,e]/(e^2) \to \mathbb{P}^{g-1}$ given by

$$t \mapsto [1 : t : t^2 : \cdots : t^k : t^{k+1} + e : t^{k+2} + 2te : \cdots : t^{2k} + kt^{k-1}e].$$

Definition 4.3. *Let $S = \mathbb{K}[t, e, x_0, \ldots, x_{2k}]$. The* elimination ideal *I_E of the canonically embedded balanced ribbon is the ideal generated by the following equations:*

$$\begin{array}{ll} t^i x_0 - x_i & i = 0, 1, \ldots, k, \\ (t^{2k-j} + (k-j)t^{k-j-1}e)x_0 - x_{2k-j}, & j = k-1, k-2, \ldots, 0, \\ e^2. & \end{array}$$

Equations for the canonically embedded balanced ribbon can be obtained from the parametric description above by eliminating the variables t and e from I_E.

Theorem 4.4. *The following $\binom{g-2}{2} + g$ quadrics and g cubics form a Gröbner basis with respect to the Bayer-Stillman elimination term order for the elimination ideal I_E of the balanced ribbon of genus g:*

(1) *The 2×2 minors of the catalecticant matrix*

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{k-1} \\ x_1 & x_2 & x_3 & \cdots & x_k \end{bmatrix}$$

(2) *The 2×2 minors of the catalecticant matrix*

$$\begin{bmatrix} x_{2k} & x_{2k-1} & x_{2k-2} & \cdots & x_{k+1} \\ x_{2k-1} & x_{2k-2} & x_{2k-3} & \cdots & x_k \end{bmatrix}$$

(3) *For each pair i, j with $0 \le i \le k-2$ and $0 \le j \le k-2$ the following trinomial quadric:*

$$x_{i+2}x_{2k-j-2} - 2x_{i+1}x_{2k-j-1} + x_i x_{2k-j}.$$

(4) $\{tx_i - x_{i+1} \mid i = 0, \ldots, k-1\}$
(5) $\{ex_i + tx_{k+i} - x_{k+i+1} \mid i = 0, \ldots, k-1\}$
(6) $\{e(tx_{k+i} - x_{k+i+1}) \mid i = 0, \ldots, k-1\}$
(7) $\{t^2 x_{k+i} - 2tx_{k+i+1} + x_{k+i+2} \mid i = 0, \ldots, k-2\}$
(8) *One additional quadric:* e^2
(9) *The cubic* $ex_k x_{2k-1} + tx_{2k-1}x_{2k} - x_{2k}^2$
(10) *The cubic* $t^2 x_{2k-1} + ex_k - tx_{2k}$

Proof of Theorem 4.4. First, we show that polynomials listed in the statement of Theorem 4.4 are in the ideal I_E. This is straightforward, so we give just one example. We verify that a quadric from the third group is in I_E:

$$\begin{aligned}(t^{i+2}x_0 - x_{i+2})((t^{2k-j-2} + (k-j-2)t^{k-j-3}e)x_0 - x_{2k-j-2})\\ -2(t^{i+1}x_0 - x_{i+1})((t^{2k-j-1} + (k-j-1)t^{k-j-2}e)x_0 - x_{2k-j-1})\\ +(t^i x_0 - x_i)((t^{2k-j} + (k-j)t^{k-j-1}e)x_0 - x_{2k-j})\\ = x_{i+2}x_{2k-j-2} - 2x_{i+1}x_{2k-j-1} + x_i x_{2k-j}.\end{aligned} \tag{4.5}$$

Next, we show that the polynomials listed generate I_E. For this, observe that

$$t^i x_0 - x_i = \sum_{j=0}^{i-1} t^{i-j-1}(tx_j - x_{j+1}) \tag{4.6}$$

and

$$\begin{aligned}t^{2k-j}x_0 + (k-j)t^{k-j-1}ex_0 - x_{2k-j} =\\ (k-j)t^{k-j-1}(ex_0 + tx_k - x_{k+1}) + \sum_{p=0}^{k-1} t^{2k-j-p-1}(tx_p - x_{p+1})\\ - \sum_{p=0}^{k-j-2}(k-j-1-p)t^{k-j-p-2}(t^2 x_{k+p} - 2tx_{k+p+1} + x_{k+p+2}).\end{aligned} \tag{4.7}$$

It remains to show that the polynomials listed in the statement of Theorem 4.4 form a Gröbner basis of I_E. For this, we use Buchberger's Algorithm.

There are 10 different types of generators, and hence 55 types of S-pairs. However, the generator types 8, 9, and 10 contain only one polynomial each, so we do not need to consider S-pairs of types $(8,8)$, $(9,9)$, or $(10,10)$. This leaves 52 types of S-pairs that we must reduce to zero. Of these pairs, 18 are coprime. We outline the calculations needed for the first three of the remaining 34 cases below.

Type (1,1). Consider two polynomials from the first group. Let $f = \underline{x_{a+1}x_b} - x_a x_{b+1}$, $g = \underline{x_{c+1}x_d} - x_c x_{d+1}$ with $a < b$ and $c < d$. We have done this calculation before in the context of the rational normal curve.

Type (1,2). Consider a polynomial from the first group and a quadric from the second group. The leading terms will be coprime. The only variable that can occur in both quadrics is x_k, and it never occurs in the leading term of the quadric from the first group.

Type (1,3). Consider a polynomial from the first group and a quadric from the third group. Let $f = \underline{x_{a+1}x_b} - x_a x_{b+1}$, $g = \underline{x_{i+2}x_{2k-j-2}} - 2x_{i+1}x_{2k-j-1} + x_i x_{2k-j}$. The leading terms of f and g are coprime unless $i+2 = a+1$ or $i+2 = b$.

Suppose $i+2 = a+1$. The S-pair reduction is

$$\begin{aligned}
&x_{2k-j-2}(\underline{x_{i+2}x_b} - x_{i+1}x_{b+1}) - x_b(\underline{x_{i+2}x_{2k-j-2}} - 2x_{i+1}x_{2k-j-1} + x_i x_{2k-j}) \\
&= -\underline{x_{b+1}x_{i+1}x_{2k-j-2}} + 2x_b x_{i+1}x_{2k-j-1} - x_b x_i x_{2k-j} \\
&\quad + x_{b+1}(\underline{x_{i+1}x_{2k-j-2}} - 2x_i x_{2k-j-1} + x_{i-1}x_{2k-j}) \\
&= 2\underline{x_b x_{i+1}x_{2k-j-1}} - x_b x_i x_{2k-j} - 2x_{b+1}x_i x_{2k-j-1} + x_{b+1}x_{i-1}x_{2k-j} \\
&\quad - 2x_{2k-j-1}(\underline{x_b x_{i+1}} - x_{b+1}x_i) \\
&= -\underline{x_b x_i x_{2k-j}} + x_{b+1}x_{i-1}x_{2k-j} \\
&\quad + x_{2k-j}(\underline{x_b x_i} - x_{i-1}x_{b+1}) \\
&= 0.
\end{aligned}$$

The proof when $i+2 = b$ is similar.

All 52 cases are typed up in an appendix to this paper available at the third author's website. □

Corollary 4.8. *The quadrics of the first three types shown above form a Gröbner basis with respect to the grevlex term order for the ideal of the balanced ribbon.*

Definition 4.9. *For each triple (i,j,ℓ) with $0 \le i \le k-3$, $0 \le j \le k-2$, and $0 \le \ell \le k-1$, we define*

$$\begin{aligned}
S_{i,j,l} := {} & x_{\ell+1}(x_{i+2}x_{2k-j-2} - 2x_{i+1}x_{2k-j-1} + x_i x_{2k-j}) \\
& - x_\ell(x_{i+3}x_{2k-j-2} - 2x_{i+2}x_{2k-j-1} + x_{i+1}x_{2k-j}) \\
& + x_{2k-j-2}(x_{i+3}x_\ell - x_{i+2}x_{\ell+1}) \\
& - 2x_{2k-j-1}(x_{i+2}x_\ell - x_{i+1}x_{\ell+1}) \\
& + x_{2k-j}(x_{i+1}x_\ell - x_i x_{\ell+1}).
\end{aligned} \tag{4.10}$$

Corollary 4.11. *A basis for the module of linear syzygies between quadrics of the canonically embedded balanced ribbon of genus g is given by the syzygies $S'_{p,q,r}$, $S''_{p,q,r}$, and $S_{i,j,l}$ defined above and their images under the involution $x_i \leftrightarrow x_{2k-i}$.*

Proof. This follows from the calculations in the proof of Theorem 4.4. (Not all of the calculations are shown here, but they are all shown in the appendix.) □

5. Applications

5.1. **Betti numbers of ribbons.** We may combine the Gröbner basis calculation of the previous section with Fong's theorem to obtain amusing new proofs of weak versions of two classical theorems on canonical curves. The history of algebraic geometry in the twentieth century most certainly did not proceed via calculations on a single nonreduced curve in each odd genus!

Proposition 5.1 (Weak version of Petri's Theorem)**.** *The ideal of a general smooth canonical curve of odd genus is generated by quadrics.*

Proof. Since the ideal of the balanced ribbon is generated by quadrics, it has $\beta_{1,1+j} = 0$ for all $j \ge 2$. Since graded Betti numbers are upper semicontinuous in flat families, and ribbons smooth to canonical curves by Fong's theorem, this implies the desired result. □

We reprove a weak version of a theorem due to Vishik and Finkelberg [20]; Polishchuk [19]; and Pareschi and Purnaprajna [18].

Proposition 5.2. *The ideal of a very general smooth canonical curve of odd genus is Koszul.*

Proof. Let $S = \mathbb{K}[x_0, \ldots, x_k]$. If I has a quadratic Gröbner basis for some term order, then S/I is Koszul (see for instance [5, Theorem 6.7]). Since the balanced ribbon has a quadratic Gröbner basis, it is Koszul. Koszulity is not an open condition, but it is defined by the vanishing of *countably* many Ext groups. Since ribbons smooth to canonical curves by Fong's theorem, this implies the desired result. □

To execute Bayer and Eisenbud's original plan of using ribbons to give a new proof of Generic Green's Conjecture (Voisin's Theorem), one would need to show that the Betti table of the balanced ribbon is pure. Unfortunately, the next proposition shows that the analogue for ribbons of the proof of Proposition 3.7 fails.

Proposition 5.3. *The genus* 7 *balanced ribbon has* $50,913$ *monomial initial ideals with* $31,881$ *unique saturations. None of these monomial initial ideals has a pure Betti table; in particular, each of these monomial initial ideals has* $\beta_{3,4} > 0$.

In summary, to get a new proof of Generic Green's Conjecture (Voisin's Theorem) via ribbons, the Betti numbers of the balanced ribbon would need to be computed some other way.

5.2. **Finite Hilbert stability of ribbons.** In [1], Alper, Fedorchuk, and Smyth show that the m^{th} Hilbert point of a general bicanonical or canonical curve is semistable for any $m \geq 2$. They split the proof into four separate cases, treating odd genus and even genus separately, and canonical and bicanonical curves separately. We give a Gröbner interpretation of their proof for one of these cases: the case of odd genus canonical curves.

Balanced ribbons are used to establish semistability of odd genus canonical curves in [1, Section 4.1]. There the cases $m = 2$ and $m \geq 3$ are analyzed separately; for convenience, we will focus on the case $m = 2$ below. Alper, Fedorchuk, and Smyth's approach is to produce two points in the state polytope such that the trivial character lies between them. Specifically, in [1, (4.1) and (4.2)], they construct two monomial bases $\mathcal{B}^+$ and $\mathcal{B}^-$ of $\mathrm{H}^0(C, \omega_C^2)$ such that the outer state of $\mathcal{B}^+$ overrepresents the coordinates x_0, x_k, x_{2k} relative to the other coordinates, and the outer state of $\mathcal{B}^-$ underrepresents the coordinates x_0, x_k, x_{2k} relative to the other coordinates. Namely, these monomial bases are

$$\mathcal{B}^+ = \left\{ \{x_0 x_i\}_{i=0}^{2k},\ \{x_k x_i\}_{i=1}^{2k},\ \{x_{2k} x_i\}_{i=1}^{k-1},\ \{x_{2k} x_i\}_{i=k+1}^{2k} \right\} \tag{5.4}$$

and

$$\mathcal{B}^- = \left\{ \begin{array}{ll} \{x_i^2\}_{i=0}^{2k}, & \{x_i x_{i+1}\}_{i=0}^{2k-1}, \\ & \\ \{x_i x_{k+i}\}_{i=1}^{k-1}, & \{x_i x_{k+i+1}\}_{i=0}^{k-1} \end{array} \right\}. \tag{5.5}$$

In [1, Lemma 3.6], Alper, Fedorchuk, and Smyth describe arbitrary monomial bases of $\mathrm{H}^0(C, \omega_C^m)$, thus obtaining a complete description of the m^{th} outer state polytope of I_C.

It is natural to wonder if the outer states of $\mathcal{B}^+$ and $\mathcal{B}^-$ are the outer states of initial ideals of I_C. To this end, we have the following result for $\mathcal{B}^+$:

Proposition 5.6. *For* $m = 2$, $\mathcal{B}^+$ *is the complement of the set of degree two generators in the initial ideal of* I_C *arising from the term order given by grevlex with the variables ordered* $x_0, x_k, x_{2k}, x_1, \ldots, \widehat{x_k}, \ldots, x_{2k-1}$.

Proof. Follows immediately from definitions and (5.4). □

The set $\mathcal{B}^-$ also has a Gröbner interpretation, but it is more subtle. First, we give the following easy lemma:

Lemma 5.7. *Let T be a torus and V be a T-representation. Suppose*

$$Q = [V \to W \to 0] \in \mathrm{Grass}(p, V)$$

is a point which is invariant under a linear subgroup $G \subset T$. Let

$$W = \bigoplus_{\chi \in S} W_\chi$$

be the weight space decomposition, where S is a finite set of distinct characters of G. Set

$$Q_\chi := [V \to W_\chi \to 0] \in \mathrm{Grass}(\dim(W_\chi), V).$$

Then

$$\mathrm{State}(W) = \sum_{\chi \in S} \mathrm{State}(W_\chi), \tag{5.8}$$

where the operation on the right is Minkowski sum of polytopes.

Proof. Let $v_1, \ldots, v_n$ be a basis of V diagonalizing the T-action. A state of Q corresponds to a T-basis $\{v_{i_1}, \ldots, v_{i_p}\}$ of W; see Remark 3.10. Evidently, every T-basis of W is obtained as the concatenation of T-bases of the summands W_χ. Hence a T-state of W is a sum of T-states of W_χ, and, conversely, a sum of T-states of W_χ is a state of W. It follows that $\mathrm{State}(W)$ is the Minkowski sum of $\mathrm{State}(W_\chi)$, as desired. □

Proposition 5.9.

(1) *For genus 7, there exists no term order for which $\mathcal{B}^-$ is the complement of the set of degree two generators of the initial ideal of I_C.*[1]

(2) *The ideal I_C is bigraded, where the first grading is by degree and the second grading is by the weights of the $\mathbb{G}_m$-action. Let $I_C = \bigoplus I_p$ be its decomposition into $\mathbb{G}_m$-weight spaces. There exists a term order $\leq_p$ on each I_p such that $\mathcal{B}^-$ is the complement of the union of the degree two generators of the initial ideals $\mathrm{in}_{\leq_p} I_p$. Specifically:*

   (a) *If $p \leq k+2$ or $p \geq 3k-2$, let $\leq_p$ be the lexicographic term order with the variables $x_0, \ldots, x_{2k}$ in the usual order.*

   (b) *If $k+3 \leq p \leq 3k-3$, set $q = \lfloor \frac{p-k}{2} \rfloor$ and let $\leq_p$ be the lexicographic term order with the variables ordered $x_0, \ldots, \widehat{x_q}, \ldots, x_k, x_q, x_{k+1}, \ldots, x_{2k}$.*

Proof. For the first part, we can use `gfan` [12] to compute all 50,913 initial ideals for this example, and none of them gives $\mathcal{B}^-$.

For the second part, we can compute the initial ideals for each I_p with the given term orders. Observe that for $p \leq k+1$, $I_p(2)$ only contains binomials, and for $p = k+2$, $I_p(2)$ contains exactly one trinomial, and the initial ideals with respect to the lex term order are easily computed in these cases. For $k+3 \leq p \leq 2k$, the variable x_q cannot appear in any binomial in $I_p(2)$, so these leading monomials are also easily computed. The trinomials in $I_p(2)$ are indexed by $i = 0, \ldots, p-k-2$ and it is easy to compute the leading monomials under the term orders described. □

[1] Presumably the same result is true for all $g \geq 7$.

References

[1] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth, *Finite Hilbert stability of (bi)canonical curves*, Invent. Math. **191** (2013), no. 3, 671–718, DOI 10.1007/s00222-012-0403-6. MR3020172 ↑56, 58, 64, 67

[2] Dave Bayer and David Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 719–756, DOI 10.2307/2154871. MR1273472 (95g:14032) ↑55, 56

[3] David Bayer and Ian Morrison, *Standard bases and geometric invariant theory. I. Initial ideals and state polytopes*, J. Symbolic Comput. **6** (1988), no. 2-3, 209–217, DOI 10.1016/S0747-7171(88)80043-9. Computational aspects of commutative algebra. MR988413 (90e:13001) ↑61, 62

[4] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR1322960 (97a:13001) ↑60

[5] Viviana Ene and Jürgen Herzog, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics, vol. 130, American Mathematical Society, Providence, RI, 2012. MR2850142 ↑67

[6] Maksym Fedorchuk and David Jensen, *Stability of 2nd Hilbert points of canonical curves*, Int. Math. Res. Not. IMRN **2013** (2013), no. 22, 5270–5287. MR3129099 ↑56

[7] Lung-Ying Fong, *Rational ribbons and deformation of hyperelliptic curves*, J. Algebraic Geom. **2** (1993), no. 2, 295–307. MR1203687 (94c:14020) ↑55

[8] David Gieseker, *Geometric invariant theory and applications to moduli problems*, Invariant theory. Proceedings of the 1st 1982 Session of the Centro Internazionale Matematico Estivo (CIME), Montecatini, June 10–18, 1982, 1983, pp. v+159. ↑57

[9] ———, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 69, Published for the Tata Institute of Fundamental Research, Bombay, 1982. ↑57

[10] Ian P. Goulden and David M. Jackson, *Combinatorial enumeration*, Dover Publications, Inc., Mineola, NY, 2004. With a foreword by Gian-Carlo Rota; Reprint of the 1983 original. MR2079788 (2005b:05001) ↑60

[11] Mark L. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), no. 1, 125–171. ↑57

[12] Anders Jensen, `gfan`*: a software package for computing Gröbner fans and tropical varieties* (2011). Version 0.5. ↑68

[13] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316. ↑63

[14] Domingo Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238. ↑63

[15] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR2110098 (2006d:13001) ↑60

[16] Ian Morrison, *GIT constructions of moduli spaces of stable curves and maps*, Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces, 2009, pp. 315–369. ↑56, 58

[17] Ian Morrison and David Swinarski, *Gröbner techniques for low-degree Hilbert stability*, Exp. Math. **20** (2011), no. 1, 34–56, DOI 10.1080/10586458.2011.544577. MR2802723 (2012g:14083) ↑61, 62, 63

[18] Giuseppe Pareschi and B. P. Purnaprajna, *Canonical ring of a curve is Koszul: a simple proof*, Illinois J. Math. **41** (1997), no. 2, 266–271. MR1441677 (98c:14027) ↑66

[19] Alexander Polishchuk, *On the Koszul property of the homogeneous coordinate ring of a curve*, J. Algebra **178** (1995), no. 1, 122–135, DOI 10.1006/jabr.1995.1342. MR1358259 (96j:14019) ↑66

[20] Alexander Vishik and Michael Finkelberg, *The coordinate ring of general curve of genus $g \geq 5$ is Koszul*, J. Algebra **162** (1993), no. 2, 535–539, DOI 10.1006/jabr.1993.1269. MR1254790 (94m:14036) ↑66

[21] Claire Voisin, *Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface*, J. Eur. Math. Soc. (JEMS) **4** (2002), no. 4, 363–404, DOI 10.1007/s100970200042. ↑57

[22] ———, *Green's canonical syzygy conjecture for generic curves of odd genus*, Compos. Math. **141** (2005), no. 5, 1163–1190, DOI 10.1112/S0010437X05001387. MR2157134 (2006c:14053) ↑57

