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ASYMPTOTICALLY OPTIMAL TESTS WHEN PARAMETERS ARE ESTIMATED

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Abstract. The main purpose of this paper is to provide an asymptotically optimal test. The proposed statistic is of Neyman-Pearson-type when the parameters are estimated with a particular kind of estimators. It is shown that the proposed estimators enables us to achieve this end. Two particular cases, $AR(1)$ and $ARCH$ models were studied and the asymptotic power function was derived.

Introduction

Local asymptotic normality LAN for the log likelihood ratio was studied for a several classes of nonlinear time series model, from a LAN the contiguity property follows, for more details the interested reader may refer to [2], [11], and [4]. Applying the contiguity property, we construct a statistic for testing a null hypothesis H_0 against the alternative hypothesis $H_1^{(n)}$, often a various classical test statistics depends on the central sequence which appears in the expression of the log likelihood ratio. In the case when the parameter of the time series model is known we obtain good properties of the test, precisely, the optimality, see for instance [8, Theorem 3].

However, in a general case, particularly in practice, the parameter is unspecified, in the expression of the estimated central sequence appears an additional term which is non degenerate asymptotically. The latter, alters the power function of the constructed test.

In order to solve this very problem, and on a basis of an estimator of the unknown parameter, we introduce and define another estimator which does not effects asymptotically the power function of the test, more precisely the additional term is absorbed. The principle of this construction is to modify one component of the

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first estimator in order to avoid the additional term, the details of this method are expanded further in the section (1).
The main purpose of this paper is to investigate the problem of testing two hypothesis corresponding to a stochastic model which is described in the following way. Let $\{(Y_i, X_i)\}$ be a sequence of stationary and ergodic random vectors with finite second moment such that for all $i \in \mathbb{Z}$, where Y_i is a univariate random variable and X_i is a d-variate random vector.
We consider the class of stochastic models

$$Y_i = T(Z_i) + V(Z_i)\,\epsilon_i, \quad i \in \mathbb{Z}, \tag{0.1}$$

where, for given non negative integers q and s, the random vectors Z_i is equal to $(Y_{i-1}, Y_{i-2}, \dots, Y_{i-s}, X_i, X_{i-1}, \dots, X_{i-q})$, the ϵ_i's are centered i.i.d. random variables with unit variance and density function $f(\cdot)$, such that for each $i \in \mathbb{Z}$, ϵ_i is independent of the filtration $\mathcal{F}_i = \sigma(Z_j, j \leq i)$, the real-valued functions $T(\cdot)$ and $V(\cdot)$ are assumed to be unknown. We consider the problem of testing whether the bivariate vector of functions $(T(\cdot), V(\cdot))$ belongs to a given class of parametric functions or not. More precisely, let

$$\mathcal{M} = \{(m(\rho, \cdot), \sigma(\theta, \cdot))\,,\ (\rho', \theta')' \in \Theta_1 \times \Theta_2\}\,,$$

$\Theta_1 \times \Theta_2 \subset \mathbb{R}^\ell \times \mathbb{R}^p$, $\mathring{\Theta}_1 \neq \emptyset$, $\mathring{\Theta}_2 \neq \emptyset$, where for all set A, $\mathring{A}$ denotes the interior of the set A and the script " $'$ " denotes the transpose, ℓ and p are two positive integers, and each one of the two functions $m(\rho, \cdot)$ and $\sigma(\theta, \cdot)$ has a known form such that $\sigma(\theta, \cdot) > 0$. For a sample of size n, we derive a test of

$$H_0 : [(T(\cdot), V(\cdot)) \in \mathcal{M}] \ \text{against}\ H_1 : [(T(\cdot), V(\cdot)) \notin \mathcal{M}]\,. \tag{0.2}$$

It is easy to see that the null hypothesis H_0 is equivalent to

$$H_0 : [(T(\cdot), V(\cdot)] \quad = \quad \Big(m(\rho_0, \cdot), \sigma(\theta_0, \cdot)\Big), \tag{0.3}$$

while the alternative hypothesis H_1 is equivalent to

$$H_1 : [(T(\cdot), V(\cdot)] \quad \neq \quad \Big(m(\rho_0, \cdot), \sigma(\theta_0, \cdot)\Big),$$

for some $(\rho_0', \theta_0')' \in \Theta_1 \times \Theta_2$.
In the sequel, our study will be focused on the following alternative hypotheses. For all integers $n \geq 1$, the alternative hypothesis $H_1^{(n)}$ is defined by the following equation

$$H_1^{(n)} : [(T(\cdot), V(\cdot)] \quad = \quad \Big(m(\rho_0, \cdot) + n^{-\frac{1}{2}} G(\cdot), \sigma(\theta_0, \cdot) + n^{-\frac{1}{2}} S(\cdot)\Big), \tag{0.4}$$

where $G(\cdot)$ and $S(\cdot)$ are two specified real functions. The situation is different in the case when the used statistic is the Neyman-Pearson test which is based on the log-likelihood ratio Λ_n defined as follows

$$\Lambda_n = \log\left(\frac{f_n}{f_{n,0}}\right) = \sum_{i=1}^{n} \log(g_{n,i}), \tag{0.5}$$

where $f_{n,0}(\cdot)$ and $f_n(\cdot)$ denote the probability densities of the random vector $(Y_1, \dots, Y_n)$ corresponding to the null hypothesis and the alternative hypothesis, respectively.

The use of the Neyman-Pearson statistics needs to resort to the following conditions:
Under the hypothesis H_0, there exists a random variable $\mathcal{W}_n$ such that

$$\mathcal{W}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and some constant $\tau > 0$ depending on the parameter $\phi_0 = (\rho_0', \theta_0')'$, such that

$$\Lambda_n = \mathcal{W}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1). \tag{0.6}$$

The equality (0.6) is a modified version of the LAN given by [8, Theorem 1]. We mention that there exist other versions of the LAN, we may refer to [7], [10] and the references therein. On the basis of the LAN, an efficient test of linearity based on Neyman-Pearson-type statistics was obtained in a class of nonlinear time series models contiguous to a first-order autoregressive process $AR(1)$ and its asymptotic power function is derived, see for instance
([8, Theorem 1 and Theorem 3]). Note that this proposed test is given by this equality:

$$T_n \quad = \quad I\left\{ \frac{\mathcal{W}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha) \right\}, \tag{0.7}$$

where $Z(\alpha)$ is the (1 -α)-quantile of a standard normal distribution $\Phi(\cdot)$.
The expression of the obtained test depends on the central sequence $\mathcal{W}_n(\phi_0)$ which itself depends on the parameter ϕ_0. In a general case the parameter ϕ_0 is unspecified, so, in order to estimate it, we need to present of a coherent methodology for the estimation of the parameters of mathematical models. An experimental data will be examined in the end.
More precisely, under some assumptions, we define and introduce an estimator preserving, asymptotically, the power on Neyman-Pearson test when we replace, in the expression of the statistics, the parameter ϕ_0 by an appropriate estimator, $\bar{\phi}_n$. Say, this estimator will be constructed on the tangent space with the direction of the partial derivatives of the central sequences in $\hat{\phi}_n$, where $\hat{\phi}_n$ is a $\sqrt{n}$-consistent estimator of ϕ_0. In the sequel, $\bar{\phi}_n$ will be called a *modified estimator* M.E..
This paper describes a method to estimate parametric models and consists of two parts essentially:
The first part corresponds to the introducing of a new estimator (Modified estimator) of the unknown parameter of the time series model.More precisely, if $\hat{\phi}_n$ is a consistency estimator of unknown parameter ϕ, and $\mathcal{W}_n$ a real random variable defined on the set $\Theta_1 \times \Theta_2$ such that the following condition is satisfied:

$$\mathcal{W}_n(\hat{\phi}_n) \quad = \quad \mathcal{W}_n(\phi_0) - D_n + o_P(1),$$

where D_n is a specified bounded random function. Then, we shall construct another estimator $\bar{\phi}_n$ of the parameter ϕ_0 which absorb the error corresponding to the function D_n.The proprieties of this estimator are expanded further in the section (1).
The second part corresponds to the applying of this new estimator (Modified estimator) in the problem of the test. In this case the random variable $\mathcal{W}_n$ is equal to the central sequence $\mathcal{V}_n$ which appears in the LAN version defined by the equality (0.6). Under some assumptions, the optimality of the constructed test is obtained

and its asymptotic power function is derived.
This paper is organized as follows:
Section (1) is devoted for the estimation. In Subsection (1.1), we describe the methodology used to construct the M.E.. In Subsection (1.2), we give the asymptotic properties of the proposed estimator.
In Section (2), we applied the modified estimator in the problem of testing. Section (3) treats specially the problem of testing in $AR(1)$ in two cases.
In Section (4), we conduct a simulation in order to evaluate the power of the proposed test in $AR(1)$ model.
All mathematical developments are relegated to Section (5).

1. Estimation

Large sample theory of estimation is developed. Attention was confined to parametric model. Other problems having a link with the considered problem depend on the used estimator. More precisely, random functions based on the unspecified parameter of the model appear. The replacing of this unknown parameter by its estimator induces an additional term. This latter is not asymptotically degenerate, the performance of the considered study problem is effected.
For instance, if we consider the problem of testing, the most of the classical statistics tests are based on the central sequence which appears in the expression of the established local asymptotic normality of the log-likelihood ratio. In this case the random function corresponds to the central sequence. By replacing the unknown parameter by its estimator, an additional term appears in the expression of the estimated central sequence. Therefore, the power of the constructed test is effected.
Our goal in this paper is to treat this problem in a general case. In order to avoid this additional term, we develop under some assumptions a method for constructing another estimator. The principle of this construction is to absorb this additional term asymptotically.
Next we discuss estimation and testing for $AR(1)$ with an extension to $ARCH$ models and note that these models lead to some interesting and particular problems.

In Subsection (1.1), we give under some assumptions, the methodology of constructing this estimator.

In Subsection (1.2), we expand further the problem of the consistency of our constructed estimator. Under some assumptions, the consistency of the modified estimator is established.
Throughout, $\hat{\phi}_n = (\hat{\rho}'_n, \hat{\theta}'_n)'$ a $\sqrt{n}$-consistent estimator of the parameter $\phi_0 = (\rho'_0, \theta'_0)'$, where

$$\begin{aligned} \hat{\rho}'_n &= \Big(\hat{\rho}_{n,1}, \ldots, \hat{\rho}_{n,\ell}\Big), \quad \hat{\theta}'_n = \Big(\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,p}\Big), \\ {\rho_0}' &= \Big(\rho_1, \ldots, \rho_\ell\Big) \quad \text{and} \quad {\theta_0}' = \Big(\theta_1, \ldots, \theta_p\Big). \end{aligned}$$

D_n and $\mathcal{W}_n$ are the additional term and a real random function respectively.

1.1. Estimation with modifying one component.

Our purpose is to construct another estimator $\bar{\phi}'_n$ of the parameter $(\rho'_0; \theta'_0)'$, such that the following fundamental equality is fulfilled

$$\mathcal{W}_n(\bar{\phi}_n) - \mathcal{W}_n(\hat{\phi}_n) = D_n, \tag{1.1}$$

where D_n is a specified bounded random function.
Our goal, is to find an estimator $\bar{\phi}_n$ satisfying (1.1) pertaining to the tangent space Γ_n, such that, for $(X', Y')' \in \mathbb{R}^\ell \times \mathbb{R}^p$, the following equation holds

$$\Gamma_n : \mathcal{W}_n((X,Y)) - \mathcal{W}_n(\hat{\phi}_n) = \partial \mathcal{W}_n'(\hat{\phi}_n).\Big((X-\hat{\rho}_n)', (Y-\hat{\theta}_n)'\Big)',$$

where

$$\partial \mathcal{W}_n(\hat{\phi}_n)' = \Big(\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_1}, \ldots, \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_\ell}, \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_1}, \ldots, \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_p}\Big),$$

and the script " $\cdot$ " denotes the inner product.
With the connection with the equality (1.1), the new estimator is then given by imposing that the value $(X', Y')'$ satisfied the following identity

$$D_n = \partial \mathcal{W}_n(\hat{\phi}_n)'.\Big((X-\hat{\rho}_n)', (Y-\hat{\theta}_n)'\Big)'. \tag{1.2}$$

Clearly, the equation (1.2) has $\ell+p$ unknown values, so it has an infinity of solutions, after modification of the j_n-th component of the first estimator $\hat{\rho}_n$, we shall propose an element in tangent space Γ_n which satisfies the equality (1.2). We obtain then a new estimator $\bar{\phi}_n' = \phi_n^{(1,j_n)'} = (\bar{\rho}_n', \hat{\theta}_n')'$ of the unknown parameter ϕ_0, where

$$\bar{\rho_n}' = \Big(\bar{\rho}_{n,1}, \ldots, \bar{\rho}_{n,\ell}\Big),$$

and such that: for s $\in \{1, \ldots, \ell\}$, $\bar{\rho}_{n,s} = \hat{\rho}_{n,s}$ if $s \neq j_n$ and $\bar{\rho}_{n,j_n} \neq \hat{\rho}_{n,j_n}$.
The use of the notation $\phi_n^{(1,j_n)}$ explains that we obtain the new estimator $\bar{\phi}_n$ of the parameter ϕ_0 when we change in the expression of the estimator $\hat{\phi}_n$ the j_n component with respect to the first estimator $\hat{\rho}_n$ corresponding to the step n of the estimation. It follows from the equality (1.1) combined with the constraint (1.2) that

$$\begin{aligned} \mathcal{W}_n(\phi_n^{(1,j_n)}) - \mathcal{W}_n(\hat{\phi}_n) &= \sum_{s=1}^{\ell} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_s}(\bar{\rho}_{n,s} - \hat{\rho}_{n,s}) + \sum_{t=1}^{p} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_t}(\bar{\theta}_{n,t} - \hat{\theta}_{n,t}), \\ &= \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}(\bar{\rho}_{n,j_n} - \hat{\rho}_{n,j_n}). \qquad (1.3) \end{aligned}$$

By imposing the following condition

$$\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \neq 0, \tag{1.4}$$

and with the use of the equality (1.2) combined with (1.4), we deduce that

$$\bar{\rho}_{n,j_n} = \frac{D_n}{\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} + \hat{\rho}_{n,j_n}. \tag{1.5}$$

In summary, we define the modified estimator by

$$\bar{\phi}_n' = \phi_n^{(1,j_n)'} = \Big(\hat{\rho}_{n,1}, \ldots, \hat{\rho}_{n,j_n-1}, \bar{\rho}_{n,j_n}, \hat{\rho}_{n,j_n+1}, \ldots, \hat{\rho}_{n,\ell}, \hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,p}\Big)'.$$

With a same reasoning as the previous case and after modifying the k_n-th component with respect to the second estimator, we shall define a new estimator

$$\bar{\phi_n}' = \phi_n^{(2,k_n)'} = (\hat{\rho}_n', \bar{\theta}_n')',$$

such that for $t \in \{1, \ldots, p\}$, $\bar{\theta}_{n,t} = \hat{\theta}_{n,t}$ if $t \neq k_n$ and $\bar{\theta}_{n,k_n} \neq \hat{\theta}_{n,k_n}$. We obtain

$$\mathcal{W}_n(\phi_n^{(2,k_n)}) - \mathcal{W}_n(\hat{\phi}_n) = \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}}(\bar{\theta}_{n,k_n} - \hat{\theta}_{n,k_n}). \tag{1.6}$$

Under the following condition

$$\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} \neq 0, \tag{1.7}$$

it follows from the equality (1.2) combined with (1.7), that

$$\bar{\theta}_{n,k_n} = \frac{D_n}{\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}}} + \hat{\theta}_{n,k_n}. \tag{1.8}$$

In summary, we obtain the modified estimator

$$\bar{\phi}_n' = \phi_n^{(2,k_n)'} = \Big(\hat{\rho}_{n,1}, \ldots, \hat{\rho}_{n,\ell}, \hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,k_n-1}, \bar{\theta}_{n,k_n}, \hat{\theta}_{n,k_n+1}, \ldots, \hat{\theta}_{n,p}\Big)'.$$

The estimator $\phi_n^{(1,j_n)}$ (respectively, $\phi_n^{(2,k_n)}$) is called a modified estimator in j_n-th component with respect to the first estimator (respectively, in k_n-th component with respect to second estimator). We denote this estimator by M.E..

Remark 1.1. For each step n of the estimation corresponding a value of the position j_n or k_n of the component where the estimator was modified.

1.2. **Consistency.** Throughout, $\hat{\phi}_n$ is a $\sqrt{n}$-consistent estimator of the unknown parameter ϕ_0. The conditions (1.4) and (1.7) are not sufficient to get the consistency of the modified estimator M.E.. In order to get its consistency, we need to resort to one of the following additional conditions :

(C.1):

$$\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \xrightarrow{P} c_1 \quad \text{as} \quad n \to \infty,$$

(C.2):

$$\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} \xrightarrow{P} c_2 \quad \text{as} \quad n \to \infty,$$

where c_1 and c_2 are two constants, such that $c_1 \neq 0$ and $c_2 \neq 0$.

Note that $\xrightarrow{P}$ denotes the convergence in probability.

Our first result concerning the consistency of the proposed estimator is summarized in the following proposition.

Proposition 1.1. *Under (1.4) and* $(C.1)$ *(or (1.7) and* $(C.2)$*), the estimator* $\phi_n^{(1,j_n)}$ *(or* $\phi_n^{(2,k_n)}$*,) is a* $\sqrt{n}$*-consistent estimator of the unknown parameter* ϕ_0.

Remark 1.2. In practice, it is not easy to verify the condition $(C.1)$ (or $(C.2)$). In the case when the unknown parameter ϕ_0 is univariate, a sufficient condition will be stated in Lemma (1.3). In this case, we need the following assumption:

(C.3): : For all real sequence $(\eta_n)_{n\geq 1}$ with values in the interval $[0, 1]$, we have:

$$\frac{1}{\sqrt{n}} \ddot{\mathcal{W}}_n(\eta_n \phi_0 + (1 - \eta_n)\hat{\phi}_n)) = O_P(1),$$

where $\ddot{\mathcal{W}}_n$ is a second derivative of $\mathcal{W}_n$.

Now, we may state the sufficient condition which implies assumptions $(C.1)$ corresponding to the case when the parameter of the time series model is univariate.

Lemma 1.3. *Let $\hat{\phi}_n$ be a $\sqrt{n}$-consistent estimator of the parameter ϕ_0. Let c_1 be a constant, such that $c_1 \neq 0$, then we have:*

(i) *Under $(C.3)$, if $\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\phi_0) \xrightarrow{P} c_1$, as $n \to \infty$, then $\forall A > 0$,*

$$P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\phi}_n) - c_1\right| > A\right) \to 0, \text{ as } n \to \infty.$$

Remark 1.4. Consequently, with the applying of the modified estimator and in the case when the error between two central sequences is bounded. Under some assumptions, it is possible to absorb this error. This result is stated and proved in the following proposition.

Proposition 1.2. *Let $\hat{\phi}'_n$ be an estimator ($\sqrt{n}$ consitency) of the unknown parameter $(\rho', \theta')'$. We assume that there exists a known bounded function D_n, such that*

$$\mathcal{W}_n(\hat{\phi}_n) = \mathcal{W}_n(\phi_0) - D_n + o_P(1). \tag{1.9}$$

Then, there exists an estimator $\bar{\phi_n}'$ of $(\rho', \theta')'$ such that

$$\mathcal{W}_n(\bar{\phi_n}) = \mathcal{W}_n(\phi_0) + o_P(1).$$

Remark 1.5. These previous results will be used in the problem of testing. If we consider the problem of testing of the null hypothesis H_0 against the alternative hypothesis H_1^n corresponding to the equalities (0.3) and (0.4), and when the local asymptotic normality of log likelihood ratio corresponding to the equality (0.6) is established. It is possible to construct an optimal test.

2. Testing in parametrical model

The literature on specification testing in parametric model is vast. The goal is to obtain a test that is consistent. This paper provides a general framework for constructing specification tests for parametric models.
Now, we are ready to apply the results obtained in the section (1) in the testing problem. More precisely, under some assumptions, it is possible to preserve asymptotically the power function of the constructed test when we replace the unknown parameter by the modified estimator. Consequently the optimality of the test is proved and the power function is derived.
Consider again the problem of testing the hypothesis H_0 and H_1^n corresponding to the equalities (0.3) and (0.4) respectively. In the sequel the functions $(\rho, \cdot) \longrightarrow m(\rho, \cdot)$ and $(\theta, \cdot) \longrightarrow \sigma(\theta, \cdot)$ are assumed to be twice differentiable on the sets Θ_1 and Θ_2 respectively.
Throughout, we assume that the function $f(\cdot)$ is positive with a third derivative, we denote by $\dot{f}(\cdot)$, $\ddot{f}(\cdot)$ and $f^{(3)}(\cdot)$ the first, the second and the third derivative respectively. For all $x \in \mathbb{R}$, let

$$M_f(x) = \frac{\dot{f}(x)}{f(x)}.$$

For the considered problem of testing, we use the classical large sample of Neyman-Pearson. On the basis on the results of the section (1), under some assumptions, we shall prove that, asymtotically the power function of the constructed test is no effected with the replacing of the unknown parameter by the modified estimator. This latter is stated in the next theorem:

2.1. **Optimality of the proposed test.** Throughout, $\bar{T}_n$ and $\bar{\tau}$ are the statistics test and the constant respectively obtained with the subsisting of the unspecified parameter ϕ_0 by its modified estimator $\bar{\phi}_n$ in the expression of the test (0.7) and the constant τ appearing in the expression of the log likelihood ratio (0.6) respectively. We assume in the problem of testing the two hypothesis H_0 against $H_1^{(n)}$ that the LAN of the the model (0.1) is established, in order to prove the optimality of the proposed test. To this end, we need the following assumption:

(E.1): There exists a $\sqrt{n}$-estimator $\hat{\phi}_n$ of the unknown parameter ϕ_0 and a random bounded function D_n, such that

$$\mathcal{V}_n(\hat{\phi}_n) \quad = \quad \mathcal{V}_n(\phi_0) - D_n + o_P(1).$$

Note in this case that, the random variable $\mathcal{V}_n$ corresponds to the central sequence $\mathcal{W}_n$ which appears in the expression of the log likelihood ratio (0.6).

It is now obvious from the previous definitions that we can state the following theorem:

Theorem 2.1. *Under LAN and the conditions (1.4) (respectively, (1.7)),* $(C.1)$ *(*$(C.2)$*, respectively) and* $(E.1)$ *the asymptotic power of* $\bar{T}_n$ *under* H_1^n *is equal to to* $1 - \Phi(Z(\alpha) - \bar{\tau}^2)$*. Furthermore,* $\bar{T}_n$ *is asymptotically optimal.*

Remark 2.2. In practice, the use of the condition $(E.1)$ requires the specification of the random variable D_n. In this way, we specify this random variable in the problem of testing corresponding to the $AR(1)$ model. This specially case is expanded further in the next section.

3. Testing in $AR(1)$ model

In this work, we treat specially the problem of testing for the $AR(1)$ model in two cases: firstly, we study in the Subsection (3.1) the case when the sequence of nonlinear model is contiguous to $AR(1)$, and secondly, we discuss in the Subsection(3.2) the extension to autoregressive conditionally heteroscedastic contiguous alternative models to $AR(1)$.
In the aim to achieve this, we need, firstly to establish the local asymptotic normality for the log likelihood ratio, secondly, to specify the random function D_n, and thirdly to construct a modified estimator and an optimal test. Therefore, we require some results and assumptions for these two problems of testing.
Throughout, the scripts "$\|\cdot\|_{\ell+p}$ " , "$\|\cdot\|_{\ell}$ " and "$\|\cdot\|_p$ " denote the euclidian norms in $\mathbb{R}^{\ell+p}$, $\mathbb{R}^{\ell}$ and $\mathbb{R}^p$ respectively.

3.1. **Nonlinear time series contiguous to $AR(1)$ processes.** Consider the s-th order (nonlinear) time series

$$Y_i = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) + \epsilon_i, \quad |\rho_0| < 1. \tag{3.1}$$

In this case and with the comparison to the equality (0.1), we have

$$Z_i = Y_i \quad , \quad T(Z_i) = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) \quad \text{and} \quad V(Z_i) = 1.$$

In the sequel, it will be assumed that the model is a stationary and ergodic time series with finite second moment.
Consider the problem of testing the null hypothesis $H_0 : \alpha = 0$ against the alternative hypothesis $H_1^{(n)} : \alpha = n^{-\frac{1}{2}}$. With the comparison to (0.3) and (0.4), we have the following equalities:

$$\begin{aligned}\Big(m(\rho_0, Y_{i-1}), \sigma(\theta_0, Y_{i-1})\Big)' &= \Big(\rho_0\, Y_{i-1}, 1\Big)', \quad \mathcal{M} = \{m(\rho, \cdot), \rho \in \Theta_1\}, \\ Z_i' &= \Big(Y_{i-1}, \ldots, Y_{i-s}\Big) \quad \text{and} \quad S(\cdot) = 0.\end{aligned}$$

Note that this problem of testing is equivalent to test the linearity of the s-th $AR(1)$ time series model ($\alpha = 0$) against the nonlinearity of the s-th $AR(1)$ time series model ($\alpha = n^{-\frac{1}{2}}$).
In order to study this problem, we require some assumptions and results. We suppose that the following conditions are satisfied:

- (A.1): There exists positive constants η and c such that for all u with $\|u\|_{\ell+p} > \eta$, $G(u) \leq c\|u\|_{\ell+p}$.
- (A.2): for a location family $\{f(\epsilon_i - c),\ -\infty < c < -\infty\}$, there exist a square integrable functions Ψ_1, Ψ_2 and a constant δ such that for all ϵ_i and $|c| < \delta$, such that :

$$\Big|\frac{d^k f(\epsilon_i - c)}{f(\epsilon_i)\, dc^k}\Big| \leq \Psi_k(\epsilon_i), \quad \text{for} \quad k = 1, 2.$$

We begin by processing the propriety of the local asymptotic normality, then we have:

3.1.1. *Local asymptotic normality.* To aim to establish the local asymptotic normality for the the local asymptotic normality LAN and according to the equality (0.5), we require that the following conditions are satisfied under H_0:

- (L.1): $\max_{1\leq i\leq n} |g_{n,i} - 1| = o_P(1)$,
- (L.2): there exists a positive constante τ^2 such that $\sum_{i=1}^n (g_{n,i} - 1)^2 = \tau^2 + o_P(1)$,
- (L.3): there exists a-$\mathcal{F}_n$ mesurable $\mathcal{V}_n$ satisfying $\sum_{i=1}^n (g_{n,i} - 1) = \mathcal{V}_n + o_P(1)$, where $\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2)$.

(L.1), (L.2) and (L.1) imply under H_0, the local asymptotic normality LAN for the log likelihood ratio corresponding to this problem of testing is established . This version of LAN is given by the following equality:

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1), \quad \text{with} \quad \mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2).$$

Fore more details, refer to ([8, Theorem 1]).
One consequence of the applying of the ([8, Theorem 1]), that, under H_0, (A.1) and (A.2) imply the local asymptotic normality LAN for the log likelihood. More precisely, we have:

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1), \quad \text{with} \quad \mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \sum_{i=1}^n M_f(\epsilon_i) G(Y(i-1)), \quad \text{and} \quad \tau^2 = \mathbf{E}(M_f^2(\epsilon_0))\mathbf{E}(G^2(Y(0))).$$

For more details, see ([8, Theorem 2]).

3.1.2. *The considering test.* According to the notation and results of the previous Subsection, under the conditions (A.1) and (A.2), the proposed test T_n is the Neyman-Pearson statistic which is given by the following equality

$$T_n \quad = \quad I\left\{\frac{\mathcal{V}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha)\right\}.$$

The asymptotic power of the test is derived and equal to $1 - \Phi(Z(\alpha) - \tau^2)$. Recall that when ρ_0 is known, we obtain an efficiency test, for more details see [8, Theorem 3].
To achieve this problem of testing, it remains to specify the random function D_n, the method is developed in the next subsection.

3.1.3. *Specification of the random variable D_n.* Our aim is to specify the form of the function D_n which is defined in (1.9).
In the sequel, the parameter ρ_0 is estimated by the $\sqrt{n}$-consistent estimator $\hat{\rho}_n$ and the residual ϵ_i is estimated by $\hat{\epsilon}_{i,n} = Y_i - Y_{i-1}\hat{\rho}_n$. We have the following statement:

Proposition 3.1. *Assume that, under H_0, the conditions (A.1) and (A.2) hold and ϵ_i's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$. We have*

$$\mathcal{V}(\hat{\rho}_n) = \mathcal{V}_n(\rho_0) - D_n + o_P(1), \tag{3.2}$$

where

$$D_n = -c_1\sqrt{n}(\hat{\rho}_n - \rho_0), \text{ and } c_1 = -\mathbb{E}\Big[Y_0 G(Y(0))\Big]. \tag{3.3}$$

3.1.4. *Modified estimator and optimal test.* Under the conditions of the Proposition (3.1), the modified estimator and an optimal test are given by the following proposition

Proposition 3.2. *The modified estimator is given by the equality*

$$\bar{\rho_n} = \frac{D_n}{\dot{\mathcal{V}}_n(\phi_n)} + \hat{\rho}_n,$$

and the statistic test is given by

$$\bar{T_n} \quad = \quad I\left\{\frac{\mathcal{V}_n(\bar{\rho_n})}{\bar{\tau}} \geq Z(\alpha)\right\}.$$

Remark 3.1.
- The use of the ergodicity of the model imposes to require the condition $\mathbb{E}\Big[Y_{-1}G(Y_0)\Big] < \infty$, therefore we choose the function $G(\cdot)$ in order to get this condition. For instance, we shall choose $G(Y(i-1)) = \frac{2a}{1+Y_{i-1}^2}$, where $a \neq 0$.
- With this choice of the function G, the condition $(A.1)$ remains satisfied, in fact, we can remark that $|G(u)| \leq 2|a|$, then for all u with $\|u\|_{\ell+p} \geq \eta$ we have $G(u) \leq 2a \times \|u\|_{\ell+p} \times \frac{1}{\|u\|_{\ell+p}} \leq \frac{2a}{\eta} \times \|u\|_{\ell+p}$, therefore, we shall choose $c = \frac{2a}{\eta}$.

3.2. **An extension to** $ARCH$ **processes.** Consider the following time series model with conditional heteroscedasticity

$$(3.4) \qquad Y_i = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) + \sqrt{1+\beta B(Y(i-1))}\,\epsilon_i, \quad i \in \mathbb{Z}.$$

We consider the problem of testing the null hypothesis H_0 against the alternative hypothesis $H_1^{(n)}$ such that

$$\begin{aligned} H_0 \quad &: \quad m(\rho, Z_i) = \rho_0 Y_{i-1} \quad \text{and} \quad \sigma(\theta_0, \cdot) = 1, \\ H_1^{(n)} \quad &: \quad m(\rho, Z_i) = \rho_0 Y_{i-1} + n^{-\frac{1}{2}} G(Y(i-1)) \text{ and } \sigma(\theta_0, Z_i) = \sqrt{1+n^{-\frac{1}{2}} B(Y(i-1))}. \end{aligned}$$

Remark that H_0, $H_1^{(n)}$ correspond to $\alpha = \beta = 0$ (linearity of (3.4)) and $\alpha = \beta = n^{-\frac{1}{2}}$ (non linearity of (3.4)) with the comparison to the equality (0.1), we have

$$Z_i = Y_i, \quad T(Z_i) = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) \text{ and } V(Z_i) = \sqrt{1+\beta B(Y(i-1))}.$$

Note that when n is large, we have

$$\sigma(\theta_0, Z_i) = \sqrt{1+n^{-\frac{1}{2}} B(Y(i-1))} \sim 1 + \frac{n^{-\frac{1}{2}}}{2} B(Y(i-1)) = 1 + n^{-\frac{1}{2}} S(Y(i-1)).$$

It is assumed that the model (3.4) is ergodic and stationary. It will be assumed that the conditions (B.1), (B.2) and (B.3) are satisfied, where

- **:** (B.1): The fourth order moment of the stationary distributions of (3.4) exists.
- **:** (B.2): There exists a positive constants η and c such that for all u with $\|u\|_{\ell+p} > \eta$, $B(u) \leq c\|u\|^2_{\ell+p}$.
- **:** (B.3): for a location family $\{b^{-1} f(\frac{\epsilon_i - a}{b}),\ -\infty < a < -\infty,\ b > 0\}$, there exists a square integrable function $\varphi(\cdot)$, and a strictly positive real ς, where $\varsigma > \max(|a|, |b-1|)$, such that,

$$\left| \frac{\partial^2 b^{-1} f\left(\frac{\epsilon_i - a}{b}\right)}{f(\epsilon_i)\, \partial a^j\, \partial b^k} \right| \leq \varphi(\epsilon_i),$$

  where j and k are two positive integers such that $j + k = 2$.

3.2.1. *Local asymptotic normality.* Under the conditions (A.1), (B.1), (B.2) and (B.3), the local asymptotic normality for the LAN corresponding to this problem of testing is established. In this case we have, under H_0:

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1),$$

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{n} M_f(\epsilon_i) G(Y(i-1)) + \sum_{i=1}^{n} (1 + \epsilon_i M_f(\epsilon_i)) B(Y(i-1)) \right\},$$

$$\text{with} \quad \mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

$$\text{and} \quad \tau^2 = I_0 \mathbf{E}\left(G(Y(0))\right)^2 + \frac{(I_2 - 1)}{4} \mathbf{E}\left(B(Y(0))\right)^2 + I_1 \mathbf{E}\left(G(Y(0)) B(Y(0))\right),$$

$$\text{where } I_j = \mathbf{E}\Big(\epsilon_0^j M_f^2(\epsilon_0)\Big) \quad \text{where} \quad j = 0, 1, 2.$$

For more details, see ([8, Theorem 4]).

3.2.2. *The considering test.* The proposed test is then given by

$$T_n = I\left\{\frac{\mathcal{V}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha)\right\}.$$

This test is asymptotically optimal with a power function equal asymptotically to $1-\Phi(Z(\alpha)-\tau^2)$? for more details refer to ([8, Theorem 3]).

3.2.3. *Specification of the random variable D_n.* By the subsisting ρ_0 by its $\sqrt{n}$-consistent estimator $\hat{\rho}_n$ in the expression of the central sequence, we shall state the following proposition:

Proposition 3.3. *Suppose that, under H_0 the conditions $(A.1)$, $(B.1)$, $(B.2)$ and $(B.3)$ hold and ϵ_i's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$. We have*

$$\mathcal{V}(\hat{\rho}_n) = \mathcal{V}_n(\rho_0) - D_n + o_P(1), \tag{3.5}$$

where

$$D_n = -c_1\sqrt{n}(\hat{\rho}_n - \rho_0), \quad \text{and} \quad c_1 = -\mathbb{E}\Big[Y_0 G(Y(0))\Big]. \tag{3.6}$$

3.2.4. *Modified estimator and optimal test.* Under the conditions of the Proposition (3.1), the modified estimator and an optimal test are given by the following proposition

Proposition 3.4. *The modified estimator is given by the equality*

$$\bar{\rho_n} = \frac{D_n}{\dot{\mathcal{V}}_n(\phi_n)} + \hat{\rho}_n,$$

and the statistic test is given by

$$\bar{T}_n = I\left\{\frac{\mathcal{V}_n(\bar{\rho_n})}{\bar{\tau}} \geq Z(\alpha)\right\}.$$

Remark 3.2. We mention that the limiting distributions appearing in Proposition (3.1) and Proposition (3.3) depend on the unknown quantity $b_n = (\hat{\rho}_n - \rho_0)$, i.e., in practice ρ_0 is not specified, in general. To circumvent this difficulty, we use the Efron's Bootstrap in order to evaluate b_n, more precisely, the interested reader may refer to the following references : [6] for the description of the Bootstrap methods, [1], [9] for the Bootstrap methods in $AR(1)$ time series models and [Fryzlewicz *et al.*(2008)] for the $ARCH$ models.

We shall now apply the results of the Section(1) and theorem (2.1) in order to conduct simulations corresponding to the representation of the derived asymptotic power function. The concerned model is the Nonlinear time series contiguous to $AR(1)$ processes with an extension to $ARCH$ processes which are detailed in the Subsections (3.1) and (3.2) respectively.

4. Simulations

In this section we consider particular classes which results already figure in the Subsections (3.1) and (3.2). We illustrate these results by doing simulations. We represent simultaneously the power functions, with the true parameter, with the consistency estimator of this parameter and with the modified estimator of this parameter respectively. This representation is given in term of the value of a which appears in the expressions of the random functions G and S. When n is large, we

compare the power functions.
The first aim of the conducted simulation is to evaluate the performance of the modified estimator. The second aim is to obtain a best power by the use of the modified estimator. The considering problem of testing concerns the linearity against a contiguous (to $AR(1)$) sequence of no alternative nonlinear models. An extension to contiguous autoregressive conditional heteroscedastic model is treated.
Simulations are carried out with comments in Subsections (4.1) and (4.2) for these problems of testing.
Throughout, we suppose that ϵ_i's are centered i.i.d. where $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, in this case, we have:
$\mathbb{E}(\epsilon_i) = 0, \quad \mathbb{E}(\epsilon_i^2) = 1, \quad \text{and } \mathbb{E}(\epsilon_i^4) = 3.$

4.1. **Simulations: Nonlinear time series contiguous to $AR(1)$ processes.** In this subsection, current simulation are carried out. All this results and representations are detailed in the subsection (3.1).
We assume, under H_0 that the conditions (A.1) and (A.2) are satisfied. We treat the case when the unknown parameter $\phi_0 = \rho_0 \in \Theta_1 \subset \mathbb{R}$, under H_0, the considering time series model can also rewritten

$$Y_i = \rho_0 Y_{i-1} + \epsilon_i \text{ where } \quad |\rho_0| < 1. \tag{4.1}$$

In the case when the parameter ρ_0 is known, the test T_n is optimal and its power is asymptotically equal to $1 - \Phi(Z(\alpha) - \tau^2)$, for more details see
[8, Theorem 3]. In a general case, when the parameter ρ_0 is unspecified, firstly, we estimate it with the least square estimators L.S.E. $\hat{\rho}_n = \frac{\sum_{i=1}^n Y_i Y_{i-1}}{\sum_{i=1}^n Y_{i-1}^2}$,
secondly, under the conditions (1.4) and $(C.1)$, the modified estimator M.E. $\bar{\rho}_n$ exists and remains $\sqrt{n}$-consistent, making use of (1.5) in connection with the Proposition (3.2) it follows:

$$\bar{\rho}_n = \frac{D_n}{\dot{\mathcal{V}}_n(\hat{\rho}_n)} + \hat{\rho}_n = \frac{-c_1(\hat{\rho}_n - \rho_0)}{\frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}}} + \hat{\rho}_n, \tag{4.2}$$

with the substitution of the parameter ρ_0 by its estimator $\bar{\rho}_n$ in (3.5), we obtain from Theorem (2.1), the following optimal statistics test

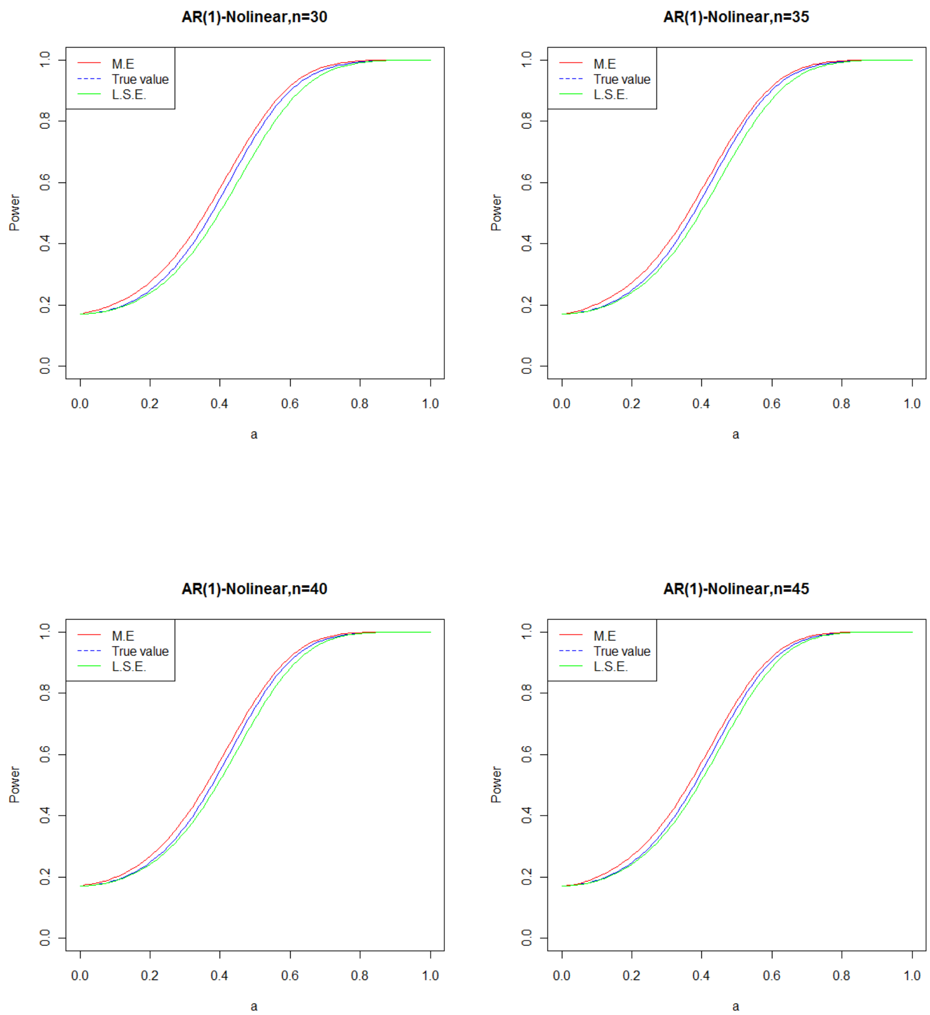
$$\bar{T}_n = \left\{ \frac{\mathcal{V}_n(\bar{\rho}_n)}{\tau(\bar{\rho}_n)} \geq Z(\alpha) \right\} \text{ where } \bar{\tau}^2 = \mathbf{E}(M_f^2(\bar{\epsilon}_{0,n}))\mathbf{E}(G^2(Y_0)),$$
$$\text{and } \bar{\epsilon}_{0,n} = Y_0 - Y_{-1}\bar{\rho}_n.$$

In this case, we have, We choose the function G like this

$$G : \Big(x_1, x_2, \cdots, x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+q}\Big) \longrightarrow \frac{5a}{1 + x_1^2} \text{ where } a \neq 0.$$

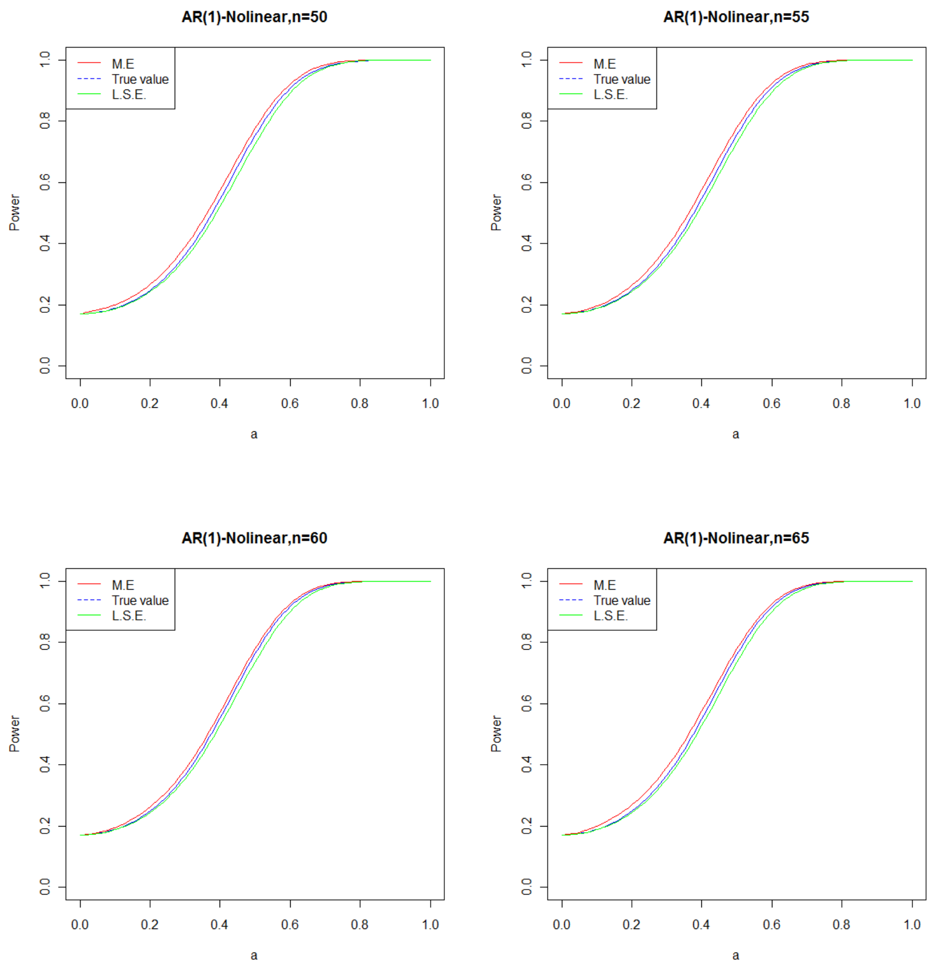
It follows from Theorem (2.1) that $\bar{T}_n$ is optimal with an asymptotic power function equal to $1 - \Phi(Z(\alpha) - \tau^2(\bar{\rho}_n))$.
In our simulations, the true value of the parameter ρ_0 is fixed at 0.1 and the sample sizes are fixed at $n = 30, 35, 40, 45, 50, 55, 60$ and 65. For a level
$\alpha = 0.05$, the power relative for each test estimated upon $m = 1000$ replicates. We represent simultaneously the power test with a true parameter ρ_0, the empirical power test which is obtained with the replacing the true value ρ_0 by its estimator M.E. $\bar{\rho}_n$ corresponding to the equality (4.2), and the empirical power test which is

obtained with the subsisting the true value ρ_0 by its least square estimator L.S.E. $\hat{\rho_n}$ (an estimator with no correction).
We observe that, the two representations with the true value and the modified estimator M.E. are close for large n and large a.



4.2. **Simulations: An extension to** $ARCH$ **processes.** These results concern the problem of testing which is described in the subsection (3.2).
In this case, we assume under H_0, that the conditions (A.1), (B.1), (B.2) and (B.3) are satisfied. On a basis of the results of the Propositions (3.3) and (3.4) and by following the same previous reasoning as the previous Subsection, it follows that:

$$\bar{T}_n \quad = \quad I\left\{\frac{\mathcal{V}_n(\bar{\rho}_n)}{\tau(\bar{\rho}_n)} \geq Z(\alpha)\right\},$$



such that

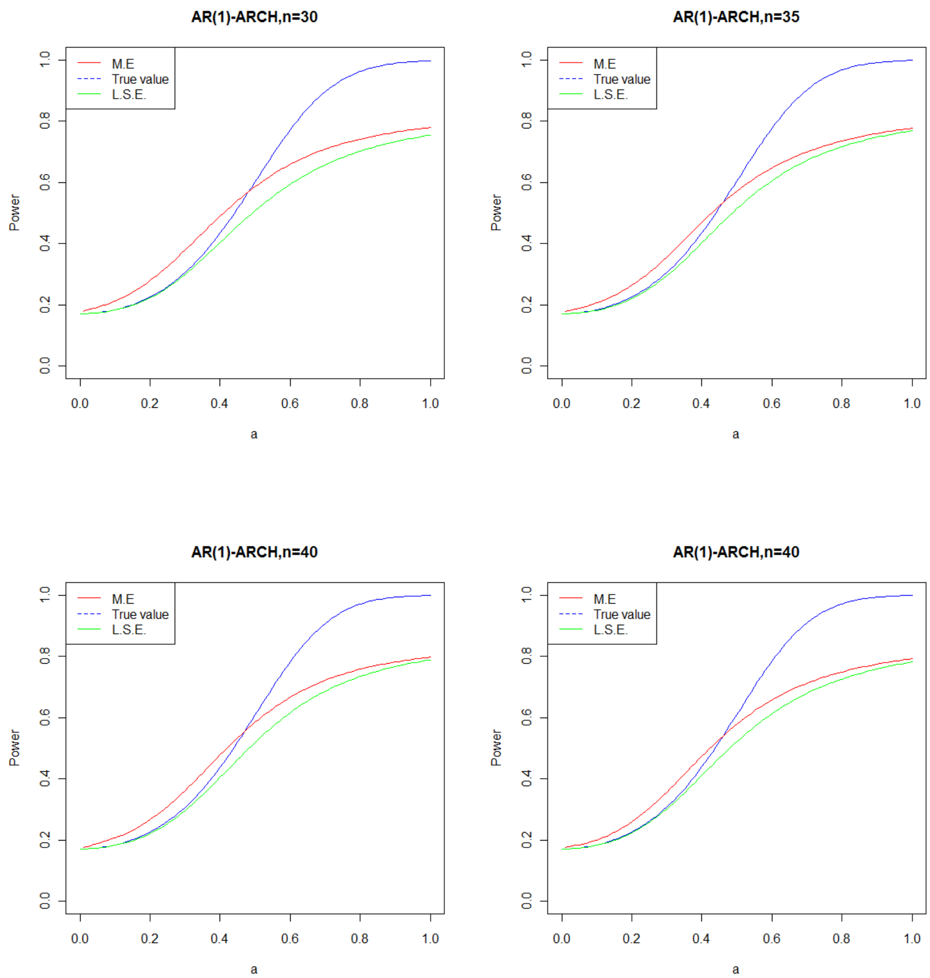
$$\begin{aligned}
\bar{\tau}^2 &= \bar{I}_{0,n}\mathbf{E}\left(G(Y(0))^2 + \frac{(\bar{I}_{2,n}-1)}{4}\mathbf{E}\left(B(Y(0)\right)^2 + \bar{I}_{1,n}\mathbf{E}\left(G(Y(0))B(Y(0)\right)\right), \\
\bar{I}_{j,n} &= \mathbf{E}\Big(\bar{\epsilon}_{0,n}^j M_f^2(\bar{\epsilon}_{0,n})\Big), \quad j=0,1,2, \text{ and } \bar{\epsilon}_{0,n} = Y_0 - Y_{-1}\bar{\rho}_n.
\end{aligned}$$

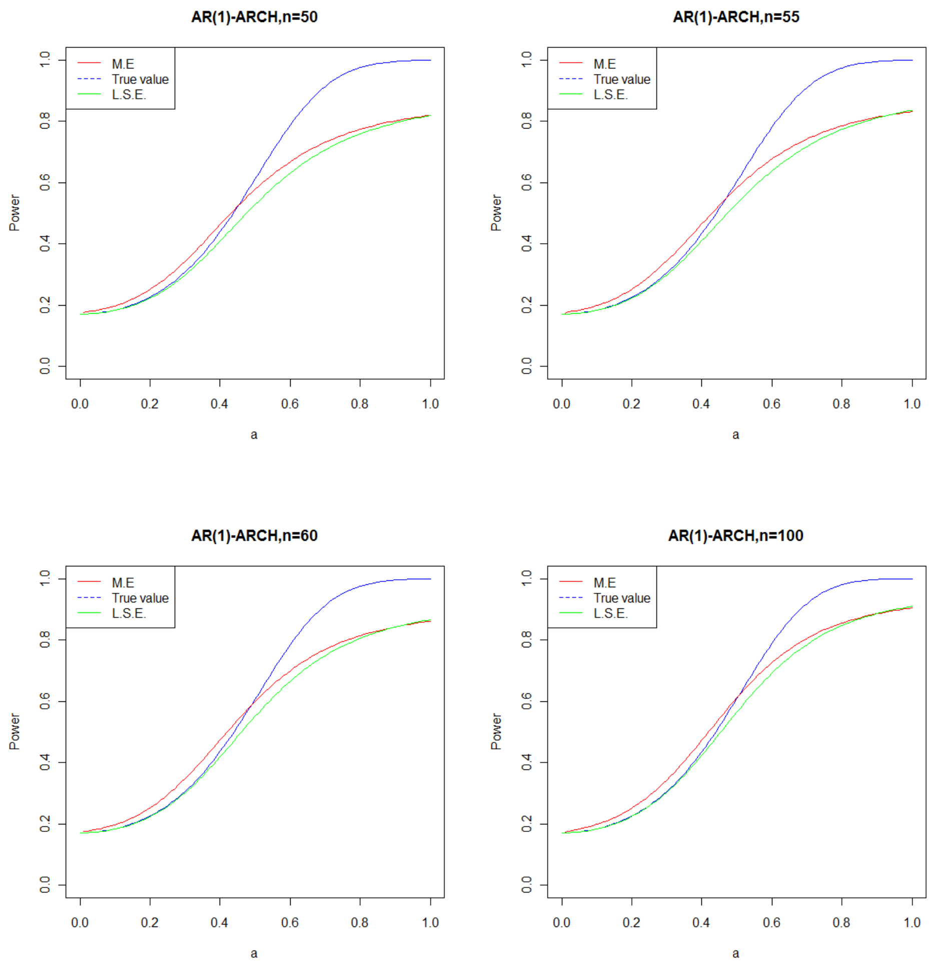
In this case, we choose the functions G and B like this

$$G = B : \Big(x_1, x_2, \cdots, x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+q}\Big) \longrightarrow \frac{3.5a}{1+x_1^2} \text{ where } a \neq 0.$$

In our simulations, the true value of the parameter ρ_0 is fixed at 0.1 and the sample sizes are fixed at $n = 30, 35, 40, 45, 50, 55, 60$ and 100. For a level $\alpha = 0.05$, the power relative for each test estimated upon $m = 1000$ replicates.

We remark that, when n and a are large, we have a similar conclusion as the previous case.


AR(1)-ARCH,n=30
M.E
True value
L.S.E.
Power
a
AR(1)-ARCH,n=35
M.E
True value
L.S.E.
Power
a
AR(1)-ARCH,n=40
M.E
True value
L.S.E.
Power
a
AR(1)-ARCH,n=40
M.E
True value
L.S.E.
Power
a



5. Proof of the results

Proof of the Proposition 1.1. Consider the following fundamental decomposition:

$$(\phi_n^{(1,j_n)})' = (\hat{\phi}_n)' + (O_{j_n})', \tag{5.1}$$

where

$$O'_{j_n} = (O_{j_n,i})'_{i\in\{1,\ldots,\ell+p\}}, \text{ such that } O_{j_n,i} = 0 \text{ when } i \neq j_n, \text{ and } O_{j_n,j_n} = \bar{\rho}_{n,j_n} - \hat{\rho}_{n,j_n}.$$

Firstly, we have $\hat{\phi}_n \xrightarrow{P} \phi_0$.
Secondly we can deduce from (1.5) that:

$$O_{j_n,j_n} = \frac{D_n}{\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} = \frac{1}{\sqrt{n}} D_n \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}. \tag{5.2}$$

Since D_n is bounded, we can remark that $\frac{1}{\sqrt{n}} D_n \xrightarrow{P} 0$.
From $(C.1)$, there exists some constant $c_1 \neq 0$, such that

$$\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \xrightarrow{P} c_1.$$

From (1.4) and since the function $x \to \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$, it follows that the random variable $\frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} \xrightarrow{P} \frac{1}{c_1}$, it results that the couple $\left(\frac{1}{\sqrt{n}} D_n; \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}\right)$ converges in probability to the couple $\left(0 \ ; \ \frac{1}{c_1}\right)$.
Since the function $(x, y) \to xy$ is continuous on $\mathbb{R} \times \mathbb{R}$, it result from (5.2), that the random variable $O_{j_n, j_n} \xrightarrow{P} \frac{0}{c_1} = 0$, therefore

$$O_{j_n}{}' = (0, \ldots 0, O_{j_n, j_n}, 0 \ldots 0)' \xrightarrow{P} (0, \ldots 0, 0, 0 \ldots 0)'. \tag{5.3}$$

Consider again the equality (5.1). Since the function $(x, y) \to x + y$ is continuous on $\mathbb{R}^{\ell+p} \times \mathbb{R}^{\ell+p}$, it results from (5.3) that $\phi_n^{(1,j_n)}$ converges in probability to ϕ_0 as $n \to \infty$.
Note that the last previous convergence in probability follows immediately with the use of the continuous mapping theorem, for more details, see for instance [3] or [12].
By following the same previous reasoning, we shall prove the consistency of the estimator $\phi_n^{(2,k_n)}$.
Note that $\phi_n^{(1,j_n)}$ is $\sqrt{n}$-consistent estimator of the parameter ϕ_0 and $\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = O_P(1)$, where $O_P(1)$ is bounded in probability in $\mathbb{R}^{\ell+p}$.
In fact, it follows from (5.1) that:

$$\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = \sqrt{n}(\hat{\phi}_n - \phi_0) + \sqrt{n} O_{j_n} = O_P(1) + \sqrt{n} O_{j_n}.$$

Since $\sqrt{n} O_{j_n, j_n} = D_n \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}$ and under $(C.1)$, it results that $\sqrt{n} O_{j_n} = O_{P_1}(1)$, where $O_{P_1}(1)$ is bounded in probability in $\mathbb{R}$.
We deduce that:

$$\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = O_P(1). \tag{5.4}$$

Note that with a similar argument and with changing $\phi_n^{(1,j_n)}$, $(C.1)$ and (1.4) by $\phi_n^{(2,k_n)}$, $(C.2)$ and (1.7) respectively, we obtain

$$\sqrt{n}(\phi_n^{(2,k_n)} - \phi_0) = O_P(1). \tag{5.5}$$

Proof of the Lemma 1.3. In this case $\phi_0 = \rho_0 \in \Theta_1 \subset \mathbb{R}$, we denote by $\hat{\rho}_n$ the $\sqrt{n}$-consistent estimator of ρ_0.
Let $A > 0$, from the triangle inequality combined with a classical inequality, we

obtain:

$$\begin{aligned} & P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-c_1\right|>A\right)\\ &= P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right|+\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)-c_1\right|>A\right)\\ &\leq P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right|>\frac{A}{2}\right)+P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)-c_1\right|>\frac{A}{2}\right).\end{aligned}$$

Firstly, we have

$$P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)-c_1\right|>\frac{A}{2}\right)\to 0 \text{ as } n\to\infty, \tag{5.6}$$

Secondly, we have

$$\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right| = \frac{1}{\sqrt{n}}\left|\ddot{\mathcal{W}}_n(\tilde{\rho}_n)\right|\left|\hat{\rho}_n-\rho_0\right| \tag{5.7}$$

$$= \frac{1}{\sqrt{n}}\left|\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\tilde{\rho}_n)\right|\left|\sqrt{n}(\hat{\rho}_n-\rho_0)\right|, \tag{5.8}$$

where $\tilde{\rho}_n$ is a point between ρ_0 and $\hat{\rho}_n$, then there exists a sequence η_n with values in the interval $[0,1]$, such that $\tilde{\rho}_n=\eta_n\rho_0+(1-\eta_n)\hat{\rho}_n$.

This implies that
$|\tilde{\rho}_n-\rho_0|\leq(1-\eta_n)|\hat{\rho}_n-\rho_0|\leq|\hat{\rho}_n-\rho_0|$.

This last inequality enables us to concluded that $\tilde{\rho}_n$ is $\sqrt{n}$-consistency estimator of ρ_0, it follows from $(C.3)$ applied on the equality (5.8) that

$$P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right|>\frac{A}{2}\right)\to 0 \text{ as } n\to 0. \tag{5.9}$$

Thus we obtain (i).

Proof of Proposition 1.2. It suffices to choose under (1.4) and $(C.1)$ the estimator $\bar{\phi_n}=\phi_n^{(1,j_n)}$, (or to choose under (1.7) and $(C.2)$ the estimator $\bar{\phi_n}=\phi_n^{(2,k_n)}$,).

Proof of Proposition 3.1. Since (A.1) and (A.2) hold, we deduce from ([8, Theorem 1]), that the local asymptotic normality LAN for the log likelihood ratio is established.

ϵ_i's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, making use of the results of [8, Theorem 2], we have

$$\mathcal{W}_n(\rho_0)=-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}M_f(\epsilon_i)G(Y(i-1)).$$

The estimated central sequence is

$$\mathcal{W}_n(\hat{\rho}_n)=-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}M_f(\hat{\epsilon}_{i,n})G(Y(i-1)).$$

By Taylor expansion with order 2, we have :

$$\mathcal{W}_n(\hat{\rho}_n)-\mathcal{W}_n(\rho_0)=\dot{\mathcal{W}}_n(\hat{\rho}_n)(\hat{\rho}_n-\rho_0)+\frac{1}{2}\ddot{\mathcal{W}}_n(\tilde{\rho_n})(\hat{\rho}_n-\rho_0)^2,$$

where $\tilde{\rho_n}$ is a point between ρ_0 and $\hat{\rho}_n$ and

$$\dot{\mathcal{W}}_n(\tilde{\rho_n}) = \frac{-1}{\sqrt{n}}\sum_{i=1}^{n} Y_{i-1}G(Y(i-1)).$$

Note that

$$R_n = \frac{1}{2}\ddot{\mathcal{W}}_n(\tilde{\rho_n})(\hat{\rho}_n-\rho_0)^2 = \frac{1}{2\sqrt{n}}\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\tilde{\rho_n})\Big(\sqrt{n}(\hat{\rho}_n-\rho_0)\Big)^2.$$

Since the estimator $\hat{\rho}_n$ is $\sqrt{n}$-consistent, it results that

$$\Big(\sqrt{n}(\hat{\rho}_n-\rho_0)\Big)^2 = O_P(1),$$

from the assumption $(C.3)$, it follows that

$$R_n = o_P(1),$$

finally we deduce that,

$$\mathcal{W}_n(\hat{\rho}_n) - \mathcal{W}_n(\rho_0) = \dot{\mathcal{W}}_n(\hat{\rho}_n)(\hat{\rho}_n-\rho_0) + o_P(1). \tag{5.10}$$

This implies that

$$\frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}} - \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}} = \frac{\ddot{\mathcal{W}}_n(\check{\rho}_n)}{\sqrt{n}}(\hat{\rho}_n-\rho_0) + o_P(1) = \frac{1}{\sqrt{n}}\frac{\ddot{\mathcal{W}}_n(\check{\rho}_n)}{\sqrt{n}}\sqrt{n}(\hat{\rho}_n-\rho_0) + o_P(1), \tag{5.11}$$

where $\check{\rho}_n$ is between $\hat{\rho}_n$ and ρ_0, and $\ddot{\mathcal{W}}_n$ is the second derivative of $\mathcal{W}_n$. From the assumption $(C.3)$, we have

$$\frac{1}{\sqrt{n}}\frac{\ddot{\mathcal{W}}_n(\check{\rho}_n)}{\sqrt{n}} = o_P(1),$$

since the estimator $\hat{\rho}_n$ is $\sqrt{n}$-consistent, it result that

$$\frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}} - \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}} = o_P(1),$$

this implies that

$$\frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}} = \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}} + o_P(1). \tag{5.12}$$

With the use of (5.12), the equality (5.10) can also rewritten

$$\begin{aligned}\mathcal{W}_n(\hat{\rho}_n) - \mathcal{W}_n(\rho_0) &= \frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}}\sqrt{n}(\hat{\rho}_n-\rho_0) + o_P(1),\\ &= \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}}\sqrt{n}(\hat{\rho}_n-\rho_0) + o_P(1). \qquad (5.13)\end{aligned}$$

It follows from the assumption $(C.1)$ combined with the ergodicity and the stationarity of the model that, the random variable $\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)$ converges in probability to the constant c_1, as $n \to +\infty$, where

$$c_1 = -\mathbb{E}\Big[Y_0G(Y(0))\Big],$$

therefore there exists a random variable X_n, $X_n \xrightarrow{P} 0$ such that

$$\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0) = c_1 + X_n.$$

We deduce from the equality (5.13) and the $\sqrt{n}$-consistence of the estimator $\hat{\rho_n}$, that

$$\mathcal{W}_n(\hat{\rho}_n) - \mathcal{W}_n(\rho_0) = c_1\sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1) = -D_n + o_P(1),$$

where $D_n = -c_1\sqrt{n}(\hat{\rho}_n - \rho_0)$.
Recall that the second derivative $\ddot{\mathcal{W}}_n$ is equal to 0, this implies that the assumption $(C.3)$ is satisfied.

Proof of Proposition 3.2. This proposition is one consequence of the results of Subsection (1.1). More precisely the direct application of the definition (1.5) (or (1.8)).

Proof of Proposition 3.3. The assumption $(C.1)$ remains satisfied. From ([8, Theorem 4]), assumptions $(A.1)$, $(B.1)$, $(B.2)$ and $(B.3)$ imply the local asymptotic normality LAN for the log likelihood ratio. The proof is similar as the proof of Proposition (3.1), in this case, for all $\rho \in \Theta_1$, we have

$$\ddot{\mathcal{W}}_n(\rho) = \frac{-1}{\sqrt{n}} \sum_{i=1}^{n} Y_{i-1}^2 B(Y(i-1)) 2\dot{M}_f(\rho).$$

By a simple calculus and since the the function f is the density of the standard normal distribution, it is easy to prove that the quantity $2\dot{M}_f(\rho)$ is bounded, therefore, there exists a positive constant w such that $2\dot{M}_f(\rho) \leq w$, then

$$|\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\rho)| \leq w\frac{1}{n}\sum_{i=1}^{n} Y_{i-1}^2 |B(Y(i-1))|.$$

With the choice $B(Y(i-1)) = \frac{2a}{1+Y_{i-1}^2}$ with $a \neq 0$, it results that

$$|\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\rho)| \leq 2w|a|\frac{1}{n}\sum_{i=1}^{n} Y_{i-1}^2.$$

By the use of the ergodicity of the model and since the model is with finite second moments, it follows that the random variable $\frac{1}{n}\sum_{i=1}^{n} Y_{i-1}^2 \xrightarrow{a.s} k$, where k is some constant, this implies that the condition $(C.3)$ is straightforward.

Proof of Proposition 3.4. The proof is similar as the proof of the Proposition (3.2).

Proof of the Theorem 2.1. Since it assumed that local asymptotic normality LAN for the log likelihood ratio is established, then we have

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1).$$

In this case the random variable $\mathcal{W}_n$ is equal to $\mathcal{V}_n$.

From the conditions (1.4) ((1.7), respectively), $(C.1)$ ($(C.2)$, respectively), it results the existence and the $\sqrt{n}$-consistency of the modified estimator $\bar{\phi}_n$ corresponding to the equation (1.5) ((1.8), respectively).
The combinaison of the condition (E_1) and the Proposition (1.2) enable us to get under H_0 the following equality

$$\mathcal{V}_n(\bar{\phi}_n) \quad = \quad \mathcal{V}_n(\phi_0) + o_P(1).$$

This last equation implies that with $o_P(1)$, the estimated central and the central sequences are equivalent, in the expression of the test (3.2), the replacing of the central sequence by the estimated central sequence has no effect.
LAN implies the contiguity of the two hypothesis (see, [5, Corrolary 4.3]), by Le Cam third lemma's (see for instance, [7, Theorem 2]), under $H_1^{(n)}$, we have

$$\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(\tau^2, \tau^2).$$

It follows from the convergence in probability of the estimator $\bar{\phi}_n$ to ϕ_0, the continuity of the function $\tau : \cdot \longrightarrow \tau(\cdot)$ and the application of the continuous mapping theorem see, for instance ([12]) or [3], that asymptotically, the power of the test is not effected when we replace the unspecified parameter ϕ_0 by it's estimator, $\bar{\phi}_n$, hence the optimality of the test.

The power function of the test is asymptotically equal to $1 - \Phi(Z(\alpha) - \tau^2(\bar{\phi}_n))$. The proof is similar as [8, Theorem 3].

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References

[1] Bertail, P. (1994). Un test bootstrap dans un modèle AR(1). *Ann. Économ. Statist.*, (36), 57–79.

[2] Bickel, P. J. (1982). On adaptive estimation. *Ann. Statist.*, **10**(3), 647–671.

[3] Billingsley, P. (1968). *Convergence of probability measures.* John Wiley & Sons Inc., New York.

[4] Cassart, D., Hallin, M., and Paindaveine, D. (2008). Optimal detection of Fechner-asymmetry. *J. Statist. Plann. Inference*, **138**(8), 2499–2525.

[5] Droesbeke, J.-J. and Fine, J. (1996). *Inférence non paramétrique. Les statistiques de rangs.* Editions de l'Université de Bruxelles.

[6] Efron, B. (1979). Bootstrap methods: another look at the jackknife. *Ann. Statist.*, **7**(1), 1–26.

[Fryzlewicz *et al.*(2008)] Fryzlewicz, P., Sapatinas, T., and Subba Rao, S. (2008). Normalized least-squares estimation in time-varying ARCH models. *Ann. Statist.*, **36**(2), 742–786.

[7] Hall, W. and Mathiason, D. J. (1990). On large-sample estimation and testing in parametric models. *Int. Stat. Rev.*, **58**(1), 77–97.

[8] Hwang, S. Y. and Basawa, I. V. (2001). Nonlinear time series contiguous to AR(1) processes and a related efficient test for linearity. *Statist. Probab. Lett.*, **52**(4), 381–390.

[9] Kvam, P. H. and Vidakovic, B. (2007). *Nonparametric statistics with applications to science and engineering.* Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ.

[10] Le Cam, L. (1960). Locally asymptotically normal families of distributions. Certain approximations to families of distributions and their use in the theory of estimation and testing hypotheses. *Univ. California Publ. Statist.*, **3**, 37–98.

[11] Swensen, A. R. (1985). The asymptotic distribution of the likelihood ratio for autoregressive time series with a regression trend. *J. Multivariate Anal.*, **16**(1), 54–70.

[12] van der Vaart, A. W. (1998). *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics.* Cambridge University Press, Cambridge.

