ALBANIAN JOURNAL
OF MATHEMATICS
Volume 5, Number 2, Pages 95–102
ISSN 1930-1235: (2011)

ON F-SUPPLEMENTED MODULES

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Abstract. Let R be a ring and M a right R–module. In this paper we prove that if M is weakly F-supplemented, then every factor module and every F-coclosed submodule of M is again weakly F-supplemented. In [5], it is shown that $Rad(M)$ has finite uniform dimension iff M does not contain an infinite direct sum of nonzero small submodules. Here we replace F-small submodules instead of small submodules (which is a weaker condition) and obtain the same result; i.e, we show that if M does not contain an infinite direct sum of F-small submodules, then $Rad(M)$ has finite uniform dimension.

1. Introduction

Throughout this article, R denotes an associative ring with identity, and modules are unitary right R-modules.

We write $N \leq M$ to denote that N is a submodule of the module M while $N \subseteq^{\oplus} M$ means that N is a direct summand of M. A submodule L of M is called *small* in M (denoted by $L \ll M$) if, for every proper submodule K of M, $L + K \neq M$. A module M is called *hollow* if every proper submodule of M is small in M.

We denote the ring of all endomorphisms of M by $End(M)$ and the *Jacobson radical* of M by $Rad(M)$ and the Jacobson radical of the ring R by $J(R)$.

A module M is called *lifting* (or said to *satisfy condition* D_1) if for every submodule N of M, M has a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll B$.

For two submodules N and K of a module M, N is called a *supplement* of K in M if N is minimal with respect to the property $M = K + N$, equivalently $M = K + N$ and $N \cap K \ll N$. Also N is called a *weak supplement* of K in M if, $M = N + K$

1991 *Mathematics Subject Classification.* Primary 16D70, 16D80; Secondary 16D90.
Key words and phrases. F-supplemented module, F-lifting module, F-coclosed submodule.

and $N \cap K \ll M$. A module M is called *supplemented* if every submodule of M has a supplement in M. M is called *amply supplemented* if whenever $M = A + B$ for submodules A, B of M, then A has a supplement in M contained in B. Also M is called *weakly supplemented* if any submodule of M has a weak supplement in M.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. The submodule A of M is called *coclosed* in M if A has no proper *cosmall* submodule. Also B is called a *coclosure* of A in M if B is a cosmall submodule of A and B is coclosed in M.

Supplemented and lifting modules and some generalizations of these kinds of modules are studied by many authors, see for example [16, 8, 13, 14]. We refer for other basic notions to [6, 17].

2. F-small submodules

The class of small modules and some other classe of modules relative to small modules (for example semiperfect modules, supplemented modules and ...) are studied by many authors. For example, see [10, 4, 11, 12, 1, 9, 15, 7]. In this section we define the F–small class and then in section 3, we investigate the class of F–supplemented modules. Let M be a module and $K \leq M$, then K is called an *F-small* submodule of M, denoted by $K \ll_F M$, if K is a finitely generated small submodule of M. It is clear that any F-small submodule of M is small in M, and in noetherian modules small submodules and F-small submodules coincide.

Lemma 2.1. *Let M be a module and $N \leq M$ such that $N \leq Rad(M)$ and N is finitely generated. Then $N \ll_F M$.*

Proof. It is clear by the proof of [3, Proposition 9.13]. □

It is an immediate conclusion of Lemma 2.1 that the sum of all F-small submodules of the module M is equal to $Rad(M)$.

The proof of following three statements are straightforward and are omitted.

Proposition 2.2. *Let M be a module and A, B submodules of M. If $A \ll_F M$ and $B \ll_F M$, then $A + B \ll_F M$. The converse is true if both A and B are finitely generated. Especially for submodules $A_1, A_2, ..., A_n$ of M, $\bigoplus_{i=1}^{n} A_i \ll_F M$ if and only if $A_i \ll_F M$ (i = 1, 2, ..., n).*

Proposition 2.3. *Let M be a module and $K \leq N \leq M$. If $K \ll_F M$ and $N/K \ll_F M/K$, then $N \ll_F M$. Moreover $N \ll_F M$ implies that $N/K \ll_F M/K$.*

Proposition 2.4. *Let M be a module and $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, sucht that $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_F M_1 \oplus M_2$ iff $K_1 \ll_F M_1$ and $K_2 \ll_F M_2$.*

Example 2.5. Let M denote the $\mathbb{Z}$-module $\bigoplus_{i=1}^{\infty} \mathbb{Z}/4\mathbb{Z}$. Consider the submodule $N = \bigoplus_{i=1}^{\infty} 2\mathbb{Z}/4\mathbb{Z}$ of M. Then N is a small submodule of M but not F-small.

3. F-Supplemented Modules

Let M be a module and N, K submodules of M. Then N is called an *$F-$supplement* of K in M if, $M = N + K$ and $N \cap K \ll_F N$. Similarly N is called a *weak $F-$supplement* of K in M if $M = N+K$ and $N \cap K \ll_F M$. The submodule N of M is called an F-supplement (weak F-supplement, resp.) submodule, if there exists a submodule K of M such that N is an F-supplement (weak F-supplement, resp.) of K in M.

The module M is called *F-supplemented* if every submodule of M has an F-supplement in M. M is called *weakly F-supplemented* if every submodule of M has a weak F-supplement in M and M is called *amply F-supplemented* if whenever $M = A+B$ for submodules A, B of M, then A has an F-supplement in M contained in B.

For two submodules $K \leq N \leq M$ of M, we say that K is an F-*cosmall* submodule of N in M, if $N/K \ll_F M/K$. The submodule N of M is called F-*coclosed* in M if N has no proper F-cosmall submodule, equivalently $N/K \ll_F M/K$ implies $N = K$ for any submodule K of N.

For two submodules N, K of M, we say K is an F-*coclosure* of N in M, if $N/K \ll_F M/K$ (K is an F-cosmall submodule of N in M) and K is F-coclosed in M.

The module M is called *F-lifting* if for any submodule N of M there exists a direct summand A of M such that $A \leq N$ and $N/A \ll_F M/A$. By the definition of F-lifting, we deduce that a module M is F-lifting iff for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_F M_2$.

In this section we show that in an F-lifting module every F-coclosed submodule is a direct summand. Especially we show that a module M is F-lifting iff M is amply F-supplemented and any F-coclosed submodule of M is a direct summand of M.

Lemma 3.1.

(i) Let M be a module and $A \leq N \leq M$. If N is an F-coclosed submodule of M and $A \ll_F M$, then $A \ll_F N$.

(ii) In any weakly F-supplemented module, every F-coclosed submodule is an F-supplement submodule.

Proof. (i) It is enough to show that $A \ll N$. Suppose that $A + L = N$ for $L \leq N$. Now We prove that $N/L \ll M/L$. Let $N/L + K/L = M/L$ for $L \leq K \leq M$. So $M = N + K = A + L + K$. Since $A \ll M$, we get $M = L + K = K$. Hence $N/L \ll M/L$. Note that $N/L \cong A/A \cap L$ is finitely generated and so $N/L \ll_F M/L$. Therefore $N = L$, i.e. $A \ll N$.

(ii) Suppose that M is a weakly F-supplemented module and $N \leq M$ is an F-coclosed submodule of M. There exists a submodule A of M such that $N + A = M$ and $A \cap N \ll_F M$. By (1), $A \cap N \ll_F N$ and so N is an F-supplement submodule of M. □

Theorem 3.2. *Let M be a module. Then the following are equivalent:*

(i) M is F-lifting;

(ii) Every submodule A of M can be written as $A = N \oplus F$ with $N \subseteq^{\oplus} M$ and $F \ll_F M$;

(iii) M is amply F-supplemented and every F-coclosed submodule of M is a direct summand of M.

Proof. (i) $\Longrightarrow$ (ii) Let $A \leq M$. Since M is F-lifting, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll_F M_2 \leq M$. By modularity $A = M_1 \oplus A \cap M_2$.

(i) $\Longrightarrow$ (iii) Let $M = X + Y$. By (2) we may assume that $Y \subseteq^{\oplus} M$. Again by (2), $X \cap Y = Y_1 \oplus F$ for $Y_1 \subseteq^{\oplus} M$ and $F \ll_F M$. By Lemma 3.1, $F \ll_F Y$. Clearly $Y_1 \subseteq^{\oplus} Y$ and so write $Y = Y_1 \oplus Y_2$. Let $\pi : Y_1 \oplus Y_2 \longrightarrow Y_2$ denote the projection map. We have $X \cap Y = Y_1 \oplus X \cap Y \cap Y_2$ and also $X \cap Y_2 = X \cap Y \cap Y_2 = \pi(X \cap Y) = \pi(Y_1 + F) = \pi(F)$. Therefore $X \cap Y_2 \ll_F Y_2$ by Proposition 2.3. Finally we obtain $M = X + Y = X + Y_1 + Y_2 = X + Y_2$. So Y_2 is an F-supplement of X contained in Y.

Now suppose that H is an F-coclosed submodule of M. Then $H = A \oplus F$ with $F \ll_F M$. Thus $H/A \ll_F M/A$ and so $H = A$ a direct summand of M.

(iii) $\Longrightarrow$ (i) Let $X \leq M$. Then by (3), X has an F-supplement Y and Y has an F-supplement M_1 such that $M_1 \leq X$ and $M_1 \subseteq^{\oplus} M$. Write $M = M_1 \oplus M_2$. Then $X = M_1 \oplus X \cap M_2$. Furthermore $M = M_1 + Y$ and so $X = M_1 + X \cap Y$. Let $\pi : M_1 \oplus M_2 \longrightarrow M_2$ be the projection map. Then $X \cap M_2 = \pi(X) = \pi(X \cap Y)$. Since $X \cap Y \ll_F M$, $X \cap M_2 \ll_F M$. Therefore M is F-lifting.

□

Example 3.3. Let $M = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ as $\mathbb{Z}$-module. Then M is F-lifting and so is F-supplemented by Theorem 3.4. Let $N = p\mathbb{Z}/p^2\mathbb{Z} \oplus p\mathbb{Z}/p^3\mathbb{Z}$. Then N is a submodule of M which is not a direct summand of M, so N is not F-coclosed in M. Now consider the submodule $K = \mathbb{Z}/p^2\mathbb{Z}$ of M. Clearly K is not F-small in M. Also if $K/L \ll_F M/L$ for $L \leq K$, then $K = L$ and so K is an F-coclosed submodule of M.

Let M be a module, then M is called *F-hollow* if every proper submodule of M is F-small in M. By the last Theorem we conclude the following Corollary;

Corollary 3.4. *An indecomposable module M is F-lifting iff it is F-hollow.*

Remark 3.5. Let M be a module and N an F-hollow submodule of M. If N is not F-small in M then N is not small in M and so $N + K = M$ for a proper submodule K of M. Since N is F-hollow, we have $N \cap K \ll_F N$ and so N is an F-supplement submodule of M. Moreover if M is F-lifting, then N is a direct summand of M by Theorem 3.4.

Theorem 3.6. *Let M be a module. Then the following hold*

(i) If M is amply F-supplemented, then for every submodule N of M that is not small in M, there is an F-supplement submodule L of M such that $L \leq N$ and $N/L \ll_F M/L$.

(ii) If A is an F-coclosed submodule of M and $B \leq A$, then A/B is F-coclosed in M/B.

(iii) If L is a supplement submodule of M and $K \leq L$, then K is F-coclosed in L iff K is F-coclosed in M.

Proof. (i) Since N is not small in M we get a proper submodule K of M such that $N + K = M$. Let X be an F-supplement of N contained in K and L be an F-supplement of X contained in N. Since $N \cap X \ll_F X$, we have $(N \cap X)/(L \cap X) \ll_F X/(L \cap X)$. Furthermore $(N \cap X)/(L \cap X) \cong N/L$ and $X/(L \cap X) \cong (X + L)/L = M/L$. So $N/L \ll_F M/L$.

(ii) Let $(A/B)/(C/B) \ll_F (M/B)/(C/B)$ for $B \leq C \leq A$. Then $A/C \ll_F M/C$ and so $A = C$.

(iii) Let L be a supplement submodule of M. If K is F-coclosed in M then obviously K is F-coclosed in L.

For converse assume that K is F-coclosed in L. Let H be a submodule of K such that $K/H \ll_F M/H$. It is clear that L/H is a supplement submodule of M/H. So by Lemma 3.1, $K/H \ll_F L/H$. Hence $K = H$, as K is F-coclosed in L. Therefore K is F-coclosed in M. □

Proposition 3.7. *Every direct summand of an amply F-supplemented module is again amply F-supplemented.*

Proof. Let M be any amply F-supplemented module and $K \subseteq^{\oplus} M$. Write $M = K \oplus K'$. Suppose that $K = C + D$, then $M = C + (D \oplus K')$. So there exists $P \leq C$ such that $M = P + (D \oplus K')$ and $P \cap (D \oplus K') \ll_F P$. Thus $K = K \cap M = P + D$ and $P \cap D = P \cap (D \oplus K') \ll_F P$; i.e., K is amply F-supplemented. □

Suppose that M is a module and $N \leq M$. Then N is said to have *ample F-supplement* in M if N has an F-supplement contained in L, whenever $M = N + L$ for submodule L of M.

Proposition 3.8. *Let M be a module and U_1, U_2 submodules of M such that $M = U_1 + U_2$. If U_1 and U_2 have ample F-supplement in M, then so does $U_1 \cap U_2$.*

Proof. Let $V \leq M$ and $U_1 \cap U_2 + V = M$. Then $U_1 = U_1 \cap U_2 + V \cap U_1$ and $U_2 = U_1 \cap U_2 + V \cap U_2$. So $M = U_1 + V \cap U_2$ and $M = U_2 + V \cap U_1$. Therefore there exist $V_2' \leq V \cap U_2$ and $V_1' \leq V \cap U_1$ such that $U_1 + V_2' = M$ and $U_1 \cap V_2' \ll_F V_2'$, and $U_2 + V_1' = M$ and $U_2 \cap V_1' \ll_F V_1'$. Thus $V_1' + V_2' \leq V$ and $U_1 = U_1 \cap U_2 + V_1'$ and $U_2 = U_1 \cap U_2 + V_2'$. So $(U_1 \cap U_2) + (V_1' + V_2') = M$ and $(U_1 \cap U_2) \cap (V_1' + V_2') = (U_2 \cap V_1') + (U_1 \cap V_2') \ll_F V_1' + V_2'$, that completes the proof. □

Proposition 3.9. *Let M be a module and $U \leq M$. Then the following are equivalent*

(i) There is a decomposition $M = X \oplus X'$ with $X \leq U$ and $X' \cap U \ll_F X'$;
(ii) There is an idempotent $e \in End(M)$ such that $(M)e \leq U$ and $(U)(1-e) \ll_F (M)(1-e)$;
(iii) There is a direct summand X of M such that $X \leq U$ and $U/X \ll_F M/X$;
(iv) U has an F-supplement V in M such that $U \cap V$ is a direct summand of U.

Proof. (i) $\Longrightarrow$ (ii) For $M = X \oplus X'$, there exists an idempotent $e \in End(M)$ such that $(M)e = X$ and $(M)(1-e) = X'$. Since $X \leq U$, we conclude $(U)(1-e) \leq U \cap (M)(1-e) \ll_F (M)(1-e)$.

(ii) $\Longrightarrow$ (iii) Take $X = (M)e$. Then $M = X \oplus (M)(1-e)$ and $U/X \ll_F M/X$.

(iii) $\Longrightarrow$ (i) Write $M = X \oplus X'$. So $U = X \oplus (X' \cap U)$ by modularity. Also we have $X' \cap U \cong U/X \ll_) FM/X \cong X'$. Thus $X' \cap U \ll_F X'$.

(i) $\Longrightarrow$ (iv) By (1), X' is an F-supplement of U in M and also $U = X \oplus (X' \cap U)$.

(iv) $\Longrightarrow$ (i) Let V be an F-supplement of U in M such that $U = X \oplus (V \cap U)$ for some $X \leq U$. Then $M = U + V = X + (V \cap U) + V = X + V$ and $X \cap V = (X \cap U) \cap V = X \cap (U \cap V) = 0$, i.e., X is a direct summand of M. □

Example 3.10. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}/24\mathbb{Z}$ and $U = 4\mathbb{Z}/24\mathbb{Z} \leq M$. Let $X = 8\mathbb{Z}/24\mathbb{Z}$ and $X' = 3\mathbb{Z}/24\mathbb{Z}$ be submodules of M. We have $M = X \oplus X'$. Since $X \leq U$, by Proposition 3.11 (4), there exists an F-supplement V of U in M such that $U \cap V \subseteq^{\oplus} U$. If we get $V = X'$ in this example, then obviously V is an F-supplement of U in M and $U \cap V = 12\mathbb{Z}/24\mathbb{Z}$ is a direct summand of U; as desired.

Proposition 3.11. *Let M be module such that every submodule of M is F-supplemented. Then, M is amply F-supplemented.*

Proof. Suppose that $M = X+Y$ for some submodules X, Y of M. Then there exists a submodule A of X such that $(X \cap Y) + A = X$ and $(X \cap Y) \cap A = Y \cap A \ll_F A$. Thus $M = X+Y = (X \cap Y)+A+Y = A+Y$, and so M is amply F-supplemented. □

Corollary 3.12. *Let R be a ring. Then every R–module is amply F-supplemented if and only if every R–module is F-supplemented.*

Let M be a module. Then M is called $\pi - projective$ if for two submodules X, Y of M, there exists $f \in End(M)$ with $Im(f) \leq X$ and $Im(1-f) \leq Y$.

Proposition 3.13. *Let M be a π–projective module. If M is F-supplemented, then M is amply F-supplemented.*

Proof. Suppose that $M = A + B$ for $A, B \leq M$. Then there is an endomorphism e of M such that $(M)e \leq A$ and $(M)(1-e) \leq B$. Let C be an F-supplement of A in M. Therefore we have $M = (M)e + (M)(1-e) = (M)e + (A+C)(1-e) \leq A + (C)(1-e)$. So $M = A + (C)(1-e)$, where $(C)(1-e) \leq B$. Now we have $A \cap (C)(1-e) = (A \cap C)(1-e) \ll_F (C)(1-e)$; as $A \cap C \ll_F C$. Thus $(C)(1-e)$ is an F-supplement of A in M contained in B; i.e, M is amply F-supplemented. □

Proposition 3.14. *Let M be a weakly F-supplemented module. Then*

(i) Every F-coclosed submodule of M is weakly F-supplemented.
(ii) Every factor module of M is weakly F-supplemented.

Proof. (i) Let K be an F-coclosed submodule of M and $N \leq K$. Since M is weakly F-supplemented, there exists $L \leq M$ such that $M = N + L$ and $N \cap L \ll_F M$. Thus $K = N + (K \cap L)$. Also $N \cap (K \cap L) = N \cap L \ll_F K$ by Lemma 3.1.

(ii) Let N be a submodule of M and $L/N \ll_F M/N$. Since M is weakly F-supplemented, there exists $K \leq M$ such that $M = K + L$ and $K \cap L \ll_F M$. So $M/N = L/N + (K+N)/N$. Let $\pi : M \longrightarrow M/N$ denote the natural epimorphism. Then $L/N \cap (K+N)/N = (N + L \cap K)/N = \pi(L \cap K) \ll_F M/N$ by Proposition 2.3. Therefore M/N is weakly F-supplemented. □

Lemma 3.15. *Let M be a module and $B \leq C \leq M$. Moreover suppose that $M = A + B$. Then $C/B \cong (A \cap C)/(A \cap B)$.*

Proof. Let $B' = A \cap C$, then $B' \cap B = A \cap B$. Now $B'/(B' \cap B) \cong (B + B')/B'$, so that $B + B' = A \cap C + B = C$. Hence $(A \cap C)/(A \cap B) \cong C/B$. □

Lemma 3.16. *Let M be a module such that $M = A + B = (A \cap B) + C$ for submodules A, B, C of M. Then $M = (B \cap C) + A = (A \cap C) + B$.*

Proof. See [5, Lemma 1.2]. □

Lemma 3.17. *Let M be a module such that $M = A+B$, for $A, B \leq M$. If $B \leq C$ and $C/B \ll_F M/B$, then $(A \cap C)/(A \cap B) \ll_F M/(A \cap B)$.*

Proof. By [5, Lemma 1.3], $(A \cap C)/(A \cap B) \ll M/(A \cap B)$ and by Lemma 3.15, $(A \cap B)/(A \cap C)$ is finitely generated. So $(A \cap C)/(A \cap B) \ll_F M/(A \cap B)$. □

Proposition 3.18. *Let M be a module and $B \leq C \leq M$. If C/B is an F-supplement submodule of M/B and B is an F-supplement submodule of M. Then C is an F-supplement submodule of M.*

In particular if M is weakly F-supplemented, then we can replace F-supplement by F-coclosed.

Proof. Let $M/B = C/B + C'/B$ and $C/B \cap C'/B \ll_F C/B$. Also suppose that $M = B + B'$ and $B \cap B' \ll_F B$, for $B \leq C' \leq M$ and $B' \leq M$. Since $B \leq C$ and also $B \leq C'$, we have $M = (C \cap C') + B'$. Also $M = C + C'$. These implies $M = C + (B' \cap C')$ by Lemma 3.16. Therefore it remains we show that $C \cap C' \cap B' \ll_F C$. For this, since $C = C \cap (B + B') = B + (C \cap B')$ and $(C \cap C')/B \ll_F C/B$, we obtain $(C \cap C' \cap B')/(B \cap B') \ll_F C/(B \cap B')$ by Lemma 3.17. Moreover $B \cap B' \ll_F C$. Now by Proposition 2.2, $C \cap C' \cap B' \ll_F C$. The last statement follows immediately from Lemma 3.1. □

Proposition 3.19. *Homomorphic images of amply F-supplemented modules are amply F-supplemented.*

Proof. Suppose that M is an amply F-supplemented module and $f : M \longrightarrow N$ is an epimorphism where N is an arbitrary module. Assume $N = N_1 + N_2$ for two submodules N_1, N_2 of N. Then $M = f^{-1}(N) = f^{-1}(N_1) + f^{-1}(N_2)$. So there exists $X \leq f^{-1}(N_2) \leq M$ such that $M = f^{-1}(N_1) + X$ and $X \cap f^{-1}(N_1) \ll_F X$. Thus $N = N_1 + f(X)$ and $N_1 \cap f(X) = f(f^{-1}(N_1) \cap X) \ll_F f(X)$ and also $f(X) \leq N_2$. Therefore N is amply F-supplemented. □

Let M be a module. Then M is said to have *finite uniform (Goldie) dimension* if, M does not contain an infinite set of independent submodules.

If $Sup\{k \in \mathbb{N} | M$ contains k independent submodules$\} = n$, then M is said to have uniform dimension n and denoted by $u.dim(M) = n$. In this case M contains n independent uniform submodules $N_1, N_2, ..., N_n$ with $\bigoplus_{i=1}^{n} N_i \trianglelefteq_e M$. So there exists an essential monomorphism from a direct sum of n uniform modules to M. If $M = 0$ then we denote $u.dim(M) = 0$, else $u.dim(M) \geq 1$.

It is clear that if M has finite uniform dimension and $M = \bigoplus_{i=1}^{n} N_i$, then $u.dim(M) = \sum_{i=1}^{n} u.dim(N_i)$.

Suppose that N is a submodule of M and M has finite uniform dimension, then $u.dim(N) \leq u.dim(M)$.

Proposition 3.20. *Let M be a module. Then the following statements are equivalent:*

(i) $Rad(M)$ has finite uniform dimension;

(ii) Every F-small submodule of M has finite uniform dimension and there exists a positive integer n such that $u.dim(N) \leq n$ for every F-small submodule N of M;

(iii) M deos not contain an infinite direct sum of nonzero F-small Submodules.

Proof. $1 \Longrightarrow 2$: This is clear.

$2 \Longrightarrow 3$: Suppose that $N_1 \oplus N_2 \oplus ...$ is an infinite direct sum of non-zero F-small submodules of M. Then $N_1 \oplus N_2 \oplus ... \oplus N_{n+1}$ is F-small in M, and also

$u.dim(N_1 \oplus N_2 \oplus ... \oplus N_{n+1}) \geq n+1$, that is a contradiction by hypothesis. So M does not contain an infinite direct sum of non-zero F-small submodules.

$3 \Longrightarrow 1$: Let $K_1 \oplus K_2 \oplus ...$ be an infinite direct sum of non-zero submodules of $Rad(M)$ and $0 \neq x_i \in K_i$ for each $i \geq 1$. Then $x_i R \ll_F M$ by Lemma 2.1 and so $x_1 R \oplus x_2 R \oplus ...$ is an infinite direct sum of non-zero F-small submodules of M, that is a contradiction. So $Rad(M)$ has finite uniform dimension. □

Example 3.21. Suppose that M is a module with $Rad(M) = M$. Then by Proposition 3.20, M has finite uniform dimension iff M does not contain an infinite direct sum of non-zero F-small submodules. Abelean groups as $\mathbb{Z}$-modules have no maximal submodule, so such modules have finite uniform dimension iff they do not contain an infinite direct sum of non-zero F-small submodules. Especially $\mathbb{Q}_{\mathbb{Z}}$ has finite uniform dimension.

Acknowledgement: This research is partially supported by the "Research Center on Algebraic Hyperstructure and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran.

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