SPECTRAL DECOMPOSITION OF A 4TH-ORDER COVARIANCE DOUGLAS TENSOR IN *P-FINSLER MANIFOLDS

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ABSTRACT. In this note we wish to implement some very technical consequences due to professor(s) P.J. Basser and Sinisa Pajevic [5] to discuss the spectral decomposition of 4th-order Douglas tensor in *P-Finsler manifold. Actually, throughout the article, we have just reviewed the results of [5] for a particular 4th-order covariance Douglas tensor, most often enunciated in Finslerian geometry. Moreover, spectral decomposition techniques have been studied for isotropic Douglas tensor.

1. INTRODUCTION

A Finsler metric of a manifold or vector bundle is defined as a smooth assignment for each base point a norm on each fibre space, and thus the class of Finsler metrics contains Riemannian metrics as a special sub-class. For this reason, Finsler geometry is usually treated as a generalization of Riemannian geometry. In fact, there are many contributions to Finsler geometry which contain Riemannian geometry as a special case (see e.g., [4], [22], and references therein).

On the other hand, we can treat Finsler geometry as a special case of Riemannian geometry in the sense that Finsler geometry may be developed as differential geometry of fibred manifolds (e.g., [1]). In fact, if a Finsler metric in the usual sense is given on a vector bundle, then it induces a Riemannian inner product on the vertical subbundle of the total space, and thus, Finsler geometry is translated to the geometry of this Riemannian vector bundle. It is natural to question why we need Finsler geometry at all. To answer this question, we have few aspects of complex Finsler geometry to some subjects which are impossible to study via Hermitian geometry.

Let F be a Finsler metric on a holomorphic vector bundle $\pi : E \rightarrow M$ over a complex manifold M. The geometry of a Finsler bundle (E, F) is the study of

the vertical bundle $V_E = \ker_\pi$ with a Hermitian metric g_{V_E} induced from the given
Finsler metric. The main tool of the investigation in Finsler geometry is the Finsler
connection. The connection is a unique one on the Hermitian bundle (V_E, g_{V_E}) sat-
isfying some geometric conditions. Although it is natural to investigate (V_E, g_{V_E}) by
using the Hermitian connection (V_E, g_{V_E}), it is convenient to use Finsler connection
for investigating Finsler metrics. For example, the flatness of Hermitian connection
(V_E, g_{V_E}) implies that the Finsler metric F is reduced to a flat Hermitian metric.
However, if the Finsler connection is flat, then the metric F belongs to an important
class, the so-called locally Minkowski metrics (we simply call these special metrics
at Finsler metrics). If the Finsler connection is induced from a connection on E,
then the metric F belongs to another important class, the so-called Berwald metrics
(sometimes a Berwald metric is said to be modeled on a Minkowski space). In this
sense, the big difference between Hermitian geometry and Finsler geometry is the
connection used for the investigation of the bundle (V_E, g_{V_E}).

1.1. **Preliminaries** [6]**.** We consider an n-dimensional Finsler space

$$F^n = (M^n, L(x, y))$$

on a connected differentiable manifold M^n of dimension n. The fundamental func-
tion $L(x, y)$, a real valued function on the tangent bundle TM^n, is usually supposed
to satisfy certain conditions from the geometrical standpoint, but only the homo-
geneity and the regularity are mainly important for our further considerations.

(1) $L(x, y)$ be positively homogeneous in y^i of degree one:
   $L(x, py) = pL(x, y)$, for any $x \in M^n, y \in TM^n$ and $\forall p > 0$.
(2) $L(x, y)$ be regular:
   $g_{ij} = \dot{\partial}_i \dot{\partial}_j F$ has non-zero $g = \det g_{ij}$,
   where $F = L^2/2$ and $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

Let (g^{ij}) be the inverse of the metric (g_{ij}). We construct the following:

$$2\gamma^i_{jk} = g^{ir}(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{kj}),$$

$$2G^i(x, y) = g^{ij}\{(\dot{\partial}_j \partial_r F)y^r - \partial_j F\},$$

where $\partial_j = \frac{\partial}{\partial x^j}$. Then we have $\gamma^i_{jk}(x, y)y^j y^k = 2G^i(x, y)$.
We, now, consider a geodesic curve $C : x^i = x^i(t), (t_0 \leq t \leq t_1)$. on M^n whose
arc length is defined by the integral $s = \int_{t_0}^{t_1} L(x, \dot{x})dt, \dot{x} \equiv \frac{dx^i}{dt}$. Then the extreme
of this integral, called the geodesic, is given by the Euler differential equations
$d(\dot{\partial}L)/dt - \partial_i L = 0$, which are written in the form:

(1.1) $\dot{x}^i[\ddot{x}^j + 2G(x, \dot{x})] - \dot{x}^j[\ddot{x}^i + 2G(x, \dot{x})] = 0.$

The system of differential equations given by (1.1) can also be written as;

(1.2) $\frac{d^2 x^i}{dt^2} = -2G^i(x, y), \ y^i = \frac{dx^i}{dt},$

where

$$(1.3) \qquad G^i = \frac{1}{4} g^{ir} \left[y^s \left(\frac{\partial L^2_{(r)}}{\partial x^s} \right) - \frac{\partial L^2}{\partial x^r} \right],$$

and $g_{ij} = \frac{1}{2} L^2_{(i)(j)}$, $(i) = \dot{\partial} = \frac{\partial}{\partial y^i}$ and $(g^{ij}) = (g_{ij}^{-1})$.

Moreover, in order to introduce the geometrical quantities in F^n, we shall be concerned with a Finsler connection $F\Gamma = (F^i_{jk}(x,y), N^i_j(x,y), V^i_{jk}(x,y))$ on F^n. For a tensor field $F\Gamma$ gives rise to the h and v- covariant differentiations, we treat a tensor field $X^i(x,y)$ of type $(1,0)$ for brevity. Then we obtain two tensor fields as follows:

$$(1.4) \qquad \nabla^h_j X^i = \delta_j X^i + X^r F^i_{rj}(x,y)$$

$$(1.5) \qquad \nabla^v_j X^i = \dot{\partial}_j X^i + X^r V^i_{rj}(x,y),$$

where $\delta_j = \partial_j - N^r_j(x,y)\dot{\partial}_r$. The h and v-covariant derivatives $\nabla^h X$ and $\nabla^v X$ are tensor field of $(1,1)$-type.

The Berwarld connection coefficients $B\Gamma = (G^i_{ij}(x,y)), G^i_j(x,y), 0)$ can be derived from the function G^i as follows:

$$(1.6) \qquad G^i_j = G^i_{(j)} \equiv \dot{\partial}_j G^i, \quad G^i_{jk} = G^i_{j(k)} \equiv \dot{\partial} G^i_j.$$

The Berwald covariant derivative with respect to the Berwald connection can be written as;

$$(1.7) \qquad T^i_{j;k} = \frac{\partial T^i_j}{\partial x^k} - T^i_{j(r)} G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

With the help of equation (1.7), we can obtain the commutation formulae, called Ricci identities:

$$(1.8) \quad X^i_{;j;k} - X^i_{;k;j} = X^r H^i_{rjk} - X^i_{;r} R^r_{jk}, \quad \dot{\partial}_k(X^i_{;j}) - (\dot{\partial}_k X^i)_{;j} = X^r G^i_{rjk}.$$

1.1.1. *Douglas Space, Douglas Tensor, Randers metric and *P Finsler Space.* In this subsection, we delineate a short introduction to the recent theory of Finsler manifolds.

We initiate with the equation (1.1) of geodesic of two dimensional Finsler space F^2. If we represent (x^1, x^2) by (x,y), assume x as the parameter t and use the mathematical terms $y' = \frac{dy}{dx}, y'' = \frac{dy'}{dx}$, then (1.1) $\forall i = 1, j = 2$ for F^2 can be written in the form:

$$(1.9) \qquad y'' = f(x,y,y') = X_3 y'^3 + X_2 y'^2 + X_1 y' + X_0,$$

where $X_3 = G^1_{22}, X_2 = 2G^1_{12} - G^2_{22}, X_1 = G^1_{11} - 2G^2_{12}, X_0 = -G^2_{11}$ and $G^i_{jk} = G^i_{jk}(x,y,1,y')$ [23].

In case, if we are particularly concerned with a Riemannian space of dimension 2, then $G^i_{jk} = \gamma^i_{jk}$ are the usual Christoffel symbols, and hence $X's$ of (1.9) do not contain y' by definition. Consequently, $f(x,y,y')$ of those spaces is a polynomial

in y' of degree at most three. Thus such a special property of $f(x, y, y')$ is equivalent to the fact that the expression $\dot{x}^1 G^2(x, \dot{x}) - \dot{x}^2 G^1(x, \dot{x})$ of equation (1.1) is a homogeneous polynomial in $\dot{x}^1, \dot{x}^2$ of degree three.

Now, we can extend the above fact to have the following definitions:

Definition 1.1. A Finsler space F^n is said to be of Douglas type or known as a Douglas space, if $D^{ij}(x, y) = G^i(x, y)y^j - G^j(x, y)y^i$ are homogeneous polynomials in y^i of degree three.

Proposition 1.1. *A Berwald space is said to be of Douglas type, if $G^i(x, y)$ of equation (1.1) are of the form $G^i_{jk}(x)y^j y^k / 2$.*

Theorem 1.2. *A Finsler space F^2 of dimension 2 is said to be Douglas type, if and only if, in every local coordinate system (x, y) the differential equation $y'\prime = f(x, y, y')$ of geodesic is such that $f(x, y, y')$ is a polynomial of degree at most three.*

Now, let us consider the two Finsler spaces $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ defined over a common underlying manifold M^n. A diffeomorphism $F^n \to \bar{F}^n$ is called geodesic if it maps an arbitrary geodesic of F^n to a geodesic of $\bar{F}^n$. In this case the change $L \to \bar{L}$ of the metric is called projective. It is also well known that the mapping $F^n \to \bar{F}^n$ is geodesic if and only if $\exists$ a scalar field $p(x, y)$ satisfying the following equation:

$$(1.10) \qquad \bar{G}^i = G^i + p(x, y)y^i, \quad p \neq 0.$$

The projective factor $p(x, y)$ is positive homogeneous function of degree one in y. From equation (1.10), we obtain the following equations [6]:

$$(1.11) \qquad \bar{G}^i_j = G^i_j + p\delta^i_j + p_j y^i, \quad p_j = p_{(j)}$$

$$(1.12) \qquad \bar{G}^i_{jk} = G^i_{jk} + p_j \delta^i_k + p_k \delta^i_j + p_{jk} y^i, \quad p_{jk} = p_{j(k)},$$

$$(1.13) \qquad \bar{G}^i_{jkl} = G^i_{jkl} + p_{jk}\delta^i_l + p_{jl}\delta^i_k + p_{kl}\delta^i_j + p_{jkl}y^i, \quad p_{jkl} = p_{jk(l)}.$$

If we substitute $p_{ij} = (\bar{G}_{ij} - G_{ij})/(n+1)$ and $p_{ijk} = (\bar{G}_{ij(k)} - G_{ij(k)})/(n+1)$ into equation (1.13), we obtain the so called Douglas tensor which is invariant under geodesic mappings, i.e.,

$$(1.14) \qquad D^i_{jkl} = \frac{1}{(n+1)} \left[G^i_{jkl} - (y^i G_{jk(l)} + \delta^i_j G_{kl} + \delta^i_k G_{jl} + \delta^i_l G_{jk}) \right],$$

which is invariant under geodesic mapping, that is,

$$(1.15) \qquad D^i_{jkl} = \bar{D}^i_{jkl}.$$

We now consider the following notations and theorems for the Finsler space.

Definition 1.2. [2] In an n-dimensional differentiable space M^n, a Finsler metric $L(x, y) = \alpha(x, y) + \beta(x, y)$ is called Randeres metric, where $\alpha(x, y) = \sqrt{(a_{ij}(x)y^i y^j)}$ is a Riemannian metric in M^n and $\beta(x, y) = b_i(x)y^i$ is a differential 1-form in M^n. The Finsler space $F^n = (M^n, L)$ with Randers metric is called Randers space.

Definition 1.3. [2] The Finsler metric $L = \alpha^2/\beta$ is called Kropina metric and the Finsler space (M^n, L) equipped with Kropina metric is called Kropina space.

Definition 1.4. [2],[23] A Finsler space of dimension $n > 2$ is called C-reducible, if the tensor $C_{ijk} = \frac{1}{2}g_{ij,(k)}$ can be written in the form:

$$(1.16) \qquad C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{ik}C_j + H_{jk}C_i,)$$

where $h_{ij} = g_{ij} - l_i l_j$ is the angular metric ensor and $l_i = L_{(i)}$.

Theorem 1.3. [23]*A Finsler space $F^n, n \geq 3$ is said to be C-reducible if and only if the metric is a Randers metric of a Kropian metric.*

Definition 1.5. [17, 18] A Finsler space F^n is called $^*P$-Finsler space, if the tensor $P_{ijk} = \frac{1}{2}g_{ij;k}$ can be written in the form:

$$(1.17) \qquad\qquad P_{ijk} = \lambda(x,y)C_{ijk}.$$

Theorem 1.4. [17] *For $n > 3$ in a C-reducible $^*P$-Finsler space, $\lambda(x,y) = k(x)L(x,y)$ holds and $k(x)$ is only the function of position.*

1.1.2. *Spectral Decomposition of a 4th order covariance tensor.* Various techniques to characterize the variability of scalar and vector valued random variables have been evolved by many researchers. Specially, the Principal component analysis (PCA) for analyzing sample covariance matrices has been originally proposed by [29] and developed by [15]. Concerning to the same issue, many other methods, such as factor analysis [30] and independent component analysis (ICA)[24, 7, 16, 8] have been well established. However, yet now statistical framework for the variability of a tensor-valued random variables could be found. To overcome from the above problem, [5] have proposed a framework to delineate the covariance structure of random 2nd-order tensor variables. Further, expressions for the sample mean and covariance tensor associated with a 2nd-order tensor random variable have been derived and is shown that the covariance tensor is a 4th-order tensor, which can be decomposed as a linear combination of eigenvalues and the outer product of their corresponding eigentensors. Moreover, [5] have also proposed a new avenue to visualize angular or orientational feature of the 4th-order covariance tensor using the spectral decomposition framework.

1.2. **Methodology and Theoretical Background of Spectral Analysis.** Here we discuss a brief digest over spectral analysis suggested by [5] and many others.

[5] mentioned that in order to perform spectral analysis on vector valued data, we first need to generate a sample covariance matrix S, and then expend it as a linear combination of eigenvalues and the outer product of their corresponding eigenvectors (for instance see [13]). It is also mentioned by [5] that however, there are many data types, such as 2nd-order and higher order tensors, for which the present approach is not suitable. Even, [9, 28] have shown that it is always possible

to express there higher order tensors as a linear combination of vectors, but recombining the tensor elements in this way may result vagueness (viz. failing to recognize diagonal and off-diagonal tensor elements). Moreover, it is also uneasy to perform affine transformations such as rotation, dilation or shear etc. in the condition when one have a higher order tensor as a linear combination of vectors. Representation of higher order tensor in vectorial form may also destroy the intrinsic geometric structure of original tensor data.

Keeping these issues in mind, [10] have proposed a normal distribution for 2nd and higher order tensor data, which generalizes the normal multivariate distribution. In order to preserve the original form of given tensor in this new distribution, [10] has replaced the familiar mean vector μ in the multivariate distribution, with a 3-dimensional 2nd-order mean tensor $\bar{D}$ and replaced the covariance matrix S, in the multivariate normal distribution, with a 3-dimensional 4th-order tensor Σ.

The normal distribution for 2nd-order tensor random variables. The exponent of a multi-varite normal probability density function $p(x)$ contains the quadratic form $(x-\mu)^T M^{-1}(x-\mu)$ of an n-dimensional normal random variable x, its man vector μ and the inverse of an $n \times n$ covariance matrix M is give as [25, 3]:
(1.18)
$$p(x) = \sqrt{\left[\frac{|M^{-1}|}{(2\pi)^n}\right]} e^{-(1/2)(x-\mu)^T M^{-1}(x-\mu)} = \sqrt{\left[\frac{|M^{-1}|}{(2\pi)^n}\right]} e^{-(1/2)(x_i-\mu_i)^T M_{ij}^{-1}(x_j-\mu_j)},$$

where the Einstein summation convention have been used in the right most expression. It is also discussed by [25, 3] that the exponent $(x_i - \mu_i)^T M_{ij}^{-1}(x_j - \mu_j)$ is a scalar contraction of two n-dimensional second order covariance tensor, which in this context is a covariance matrix S, but it can be transformed as an n-dimensional 2nd-order tensor. Moreover, the interpretation of the random vector and covariance matrix as a tensor of 1st and 2nd-order respectively have been enhanced to have multi-variate normal distribution as a tensor variate normal distribution for a 2nd-order random tensor D,

$$(1.19) \quad p(D) = \sqrt{\left[\frac{|\Sigma^{-1}|}{8\pi^6}\right]} e^{-(1/2)(D-\bar{D}):\Sigma^{-1}:(D-\bar{D})}$$

$$= \sqrt{\left[\frac{|\Sigma^{-1}|}{8\pi^6}\right]} e^{-(1/2)(D_{ij}-\bar{D}_{ij}):\Sigma_{ijmn}^{-1}:(D_{mn}-\bar{D}_{mn})}.$$

In the above expression, $\bar{D}$ is the mean tensor and $(D_{ij}-\bar{D}_{ij}) : \Sigma_{ijmn}^{-1} : (D_{mn}-\bar{D}_{mn})$ is a scalar contraction of the inverse of 3-dimensional 4th-order covariance tensor Σ_{ijmn} and two 3-dimensional 2nd-order tensors $(D_{ij} - \bar{D}_{ij})$ and $(D_{mn} - \bar{D}_{mn})$ [10]. Here in equation (1.19), the resulting exponent is a linear combination of quadratic functions formed from the products of elements $D, (D_{ij}-\bar{D}_{ij}), (D_{mn}-\bar{D}_{mn})$ weighted by the suitable coefficients, Σ_{ijmn}^{-1}. Also, in equation (1.19), the tensor "double dot product" operation as defined in [26] has been employed. In the expression $D : \Sigma^{-1} : D = D_{ij}\Sigma_{ijkl}^{-1}D_{kl}$, sums are taken for all indices i, j, k and l.

Moreover, the 4th-order covariance tensor Σ and its inverse Σ^{-1} are related to the symmetric 4th-order identity tensor Y as below:

$$(1.20) \qquad \Sigma_{ijkl}\Sigma_{klmn}^{-1} = \Sigma_{ijkl}^{-1}\Sigma_{klmn} = Y_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}).$$

2. Spectral decomposition of a 4th order covariance Douglas tensor in *P-Finsler Manifolds

In order to discuss the spectral decomposition of a 4th-order covariance Douglas tensor, let us make the following useful assumptions to setup the relevant mathematical analysis.

Suppose F^n be a *P-Finsler space satisfying the condition (1.17). Also, let $x_i \in M^n$ be the n-dimensional normal random variable in *P-Finsler manifold and μ be the familiar mean vector, then for the hv-Ricci tensor (1.6), the probability density function $p(x)$ will have the form:

$$(2.1) \qquad p(x) = \sqrt{\left[\frac{|G^{-1}|}{(2\pi)^n}\right]} e^{-(1/2)(x_i - \mu_i)^T G_{ij}^{-1}(x_j - \mu_j)},$$

where G_{ij} is an n-dimensional 2nd-order covariance hv-Ricci tensor and is given as $G_{ij} = G_{i(j)} = \dot{\partial}_j G_i$.

Furthermore, if we consider the normal tensor variate probability function for G_{ij} as $P(G)$, then for the following 4th-order covariance Douglas tensor,

$$(2.2) \quad D_{ijkl} \;=\; \frac{1}{(n+1)}[G_{ijkl} \;-\; (y_i G_{jk(l)} \;+\; g_{ij}G_{kl} \;+\; g_{ik}G_{jl} \;+\; g_{il}G_{jk})],$$

the $P(G)$ will be given by

$$(2.3) \qquad P(G) = \sqrt{\left[\frac{|D^{-1}|}{8\pi^6}\right]} e^{-(1/2)(G_{ij} - \bar{G}_{ij}):D_{ijkl}^{-1}:(G_{kl} - \bar{G}_{kl})}.$$

In the above expression $\bar{G}$ in the equation (2.3) is the mean hv-Ricci tensor of the hv-Ricci tensor G. Further, the 4th-order n-dimensional covariance Douglas tensor D_{ijkl} and its inverse D_{ijkl}^{-1} satisfy the identity:

$$(2.4) \qquad D_{ijmn}D_{klmn}^{-1} = D_{ijmn}^{-1}D_{klmn} = Y_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{jk}).$$

The 4th-order covariance Douglas tensor D_{ijkl} may satisfy the following symmetry properties:

Since the hv-Ricci tensor G_{ij} is symmetric 2nd-order tensor, then D_{ijkl} given by (2.2) being the composition of G_{ij} and its partial derivatives should inherit symmetries such that its elements must remain same by exchanging particular pairs of indices.

For feasibility, if one sets the mean hv-Ricci tensor $\bar{G}_{ij}$ to zero. Then, since the product of two components of 2nd-order tensor commute in the scalar contraction (viz. $G_{ij}.D_{ijkl}.D_{kl}$), i.e., $G_{ij}G_{mn} = G_{mn}G_{ij}$, the corresponding coefficients of D_{ijkl}

must be indistinguishable and hence must be same, i.e., $D_{ijkl} = D_{klij}$.

From what has been discussed, it follows that symmetry of hv-Ricci tensor implies that $D_{ijkl} = D_{jikl}$ and $D_{ijkl} = D_{ijlk}$. Also, these symmetries of covariance Douglas tensor can reduce the possible number of independent components. In case of 3-dimension the symmetry can reduce the components of D_{ijkl} from 81 (i.e, 3^4) to 21 [19].

In concern with the symmetry properties, [5] has mentioned that actually the 21 independent components are also required to specify each element of the symmetric covariance matrix. Moreover, In case of three dimensional *P Finsler space, the 4th-order covariance Douglas tensor D_{ijkl} can be transformed to a 6-dimensional 2nd-order tensor which is a symmetric 6×6 covariance matrix having the same 21 independent components [28, 14, 32]. To perform such a transformation, we first write the scalar contraction $G_{ij}D_{ijkl}^{-1}G_{kl}$ in the form $\tilde{G}_r S_{rt} \tilde{G}_t$, where the hv-Ricci tensor G_{ij} is written as a 6-dimensional column vector $\tilde{G}$ and is expressed as $\tilde{G} = (G_{xx}, G_{yy}, G_{zz}, \sqrt{2}G_{xy}, \sqrt{2}G_{xz}, \sqrt{2}G_{yz})$. Here the factor $\sqrt{2}$ premultiplied with the off-diagonal elements of G_{ij} emphasize that the operation of matrix multiplication between $\tilde{G}$ and the 6-dimensional 2nd-order tensor S is isomorphic to the operation of tensor double product second order hv-Ricci tensor G_{ij} and 3-dimensional 4th-order covariance Douglas tensor D_{ijkl} [11, 12].

In order to have a conversion between 6-dimensional 2nd-order tensor and the 3-dimensional 4th-order covariance Douglas tensor, we have the following tensor representation as discussed by [5];

(2.5)

$$
S = \begin{pmatrix}
D_{xxxx} & D_{xxyy} & D_{xxzz} & \sqrt{2}D_{xxxy} & \sqrt{2}D_{xxxz} & \sqrt{2}D_{xxyz} \\
D_{xxyy} & D_{yyyy} & D_{yyzz} & \sqrt{2}D_{yyxy} & \sqrt{2}D_{yyxz} & \sqrt{2}D_{yyyz} \\
D_{xxzz} & D_{yyzz} & D_{zzzz} & \sqrt{2}D_{zzxy} & \sqrt{2}D_{zzxz} & \sqrt{2}D_{zzyz} \\
\sqrt{2}D_{xxxy} & \sqrt{2}D_{yyxy} & \sqrt{2}D_{zzxy} & 2D_{xyxy} & 2D_{xyxz} & 2D_{xyyz} \\
\sqrt{2}D_{xxxz} & \sqrt{2}D_{yyxz} & \sqrt{2}D_{zzxz} & 2D_{xyxz} & 2D_{xzxz} & 2D_{xzyz} \\
\sqrt{2}D_{xxyz} & \sqrt{2}D_{yyyz} & \sqrt{2}D_{zzyz} & 2D_{xyyz} & 2D_{xzyz} & 2D_{yzyz}
\end{pmatrix}.
$$

Again, according to [27, 31], the premultiplied factors 2 and $\sqrt{2}$ for different 3×3 blocks of the covariance 6×6 matrix S emphasize that this object transforms as a 6-dimensional second order tensor and the mapping between D_{ijkl} and S including corresponding multiplication operation is an isomorphism. This fact can be expressed by saying that with the column operation :, the set of 4th-order covariance Douglas tensor is isomorphic to the set of 2nd-order covariance tensors and matrix multiplication operation. Thus once we have the tensor representation of type (2.5), we can easily obtain S^{-1} from (2.5).

Let us now discuss about the latent-roots and latent-tensors of the 4th-order covariance Douglas tensor as below:

2.1. **Latent-roots and Latent-vectors of D_{ijkl}.** As it is evident from the foregoing discussion that we can represent the 2nd-order tensor in terms of a covariance matrix M and hence the latent-roots and the corresponding latent-vectors can be determined from the matrix M. Likewise, one can determine the eigenvalues (denoted by σ^2) and 2nd-order eigentensors E of a 4th-order covariance Douglas tensor [26, 20, 21]. The Fundamental expression is given as [9]:

$$(2.6) \qquad\qquad\qquad D : E = \sigma^2 E,$$

where the tensor double dot product " : " has been employed to signify the tensor product operation.

Basically, if the two tensors, say U and V are of same order, then the tensor dot product for them will be given by,

$$(2.7) \qquad\qquad U : V = \text{ Trace } (UV^T) = U_{ij}V_{kj}\delta_{ik} = U_{ij}V_{ij}.$$

Now, rearranging the terms for (2.6), we have

$$(2.8) \qquad\qquad\qquad \left(D - \sigma^2 Y\right) : E = 0,$$

where Y is the 4th-order identity tensor defined by (2.4).

Now, likewise square matrices, the equation (2.8) has a non-trivial solution if and only if the Characteristic equation given by,

$$(2.9) \qquad\qquad\qquad |D - \sigma^2 Y| = 0.$$

One can now perform the spectral decomposition, sometimes called eigentensor decomposition by developing the connection between the 4th-order covariance Douglas tensor D and the 6×6 matrix S as in equation (2.5)[9]. We now proceed to find the eigenvalues and eigenvectors of S with the fact that eigenvalues of D and S are same. for this purpose, we fabricate the 2nd-order eigentensor E of the 4th-order Douglas tensor D by considering the 6×1 eigenvectors of S using the following expression:

$$(2.10) \qquad\qquad E^i = \begin{pmatrix} \epsilon^i_{xx} & \frac{1}{\sqrt{2}}\epsilon^i_{xy} & \frac{1}{\sqrt{2}}\epsilon^i_{xz} \\ \frac{1}{\sqrt{2}}\epsilon^i_{xy} & \epsilon^i_{yy} & \frac{1}{\sqrt{2}}\epsilon^i_{yz} \\ \frac{1}{\sqrt{2}}\epsilon^i_{xz} & \frac{1}{\sqrt{2}}\epsilon^i_{yz} & \epsilon^i_{zz} \end{pmatrix},$$

where $\epsilon^i = (\epsilon^i_{xx}, \epsilon^i_{yy}, \epsilon^i_{zz}, \epsilon^i_{xy}, \epsilon^i_{yz}, \epsilon^i_{zx})^T$ is the ith normalized eigenvector of S. Here the six 3×3 eigentensors represented by E^i are symmetric and mutually orthogonal and satisfy the following identity:

$$(2.11) \qquad\qquad\qquad E^i : E^j = \delta^{3d}_{ij}.$$

here in equation (2.11), the symbol δ^{3d}_{ij} is the familiar Kronecker delta which sometimes known as 3-dimensional 2nd-order identity tensor. Moreover the superscript $3d$ placed on the Kronecker tensor is simply used to represent its dimensionality.

Also, the expression (2.11) is equivalent to the orthonormality condition for the six 6×1 eigenvectors of the corresponding covariance matrix S:

$$\epsilon^i . \epsilon^j = \delta_{ij}^{6d}, \tag{2.12}$$

where once again the symbol δ_{ij}^{6d} stands for the Kronecker tensor or the 6-dimensional 2nd-order identity tensor.

We, now, go through the spectral decomposition of the 4th-order covariance positive definite symmetric Douglas tensor D. This Douglas tensor can be decomposed into a linear combination of six positive definite latent-vectors denoted by σ_k^2 multiplied by the outer product of their corresponding six 2nd-order eigentensors, denoted by $E^k \otimes E^k$, that is to say,

$$D_{ijmn} = \sigma_k E_{ij}^k E_{mn}^k \sigma_k \text{ or } D = \sigma_k E^k \otimes E^k \sigma_k. \tag{2.13}$$

The above equation is usually refer to as spectral decomposition of a 4th-order covariance Douglas tensor. The Douglas tensor D being positive definite may possess six positive real latent-roots (even though some of them may be repeated)and the six relevant real valued 2nd-order latent-tensors. The above expression can precisely help in finding the inverse of the covariance Douglas tensor appeared in equation (2.3) as follows:

$$D_{ijmn}^{-1} = \sigma^{-1} E_{ij}^k E_{mn}^k \sigma_k^{-1} \text{ or } D^{-1} = \sigma_k^{-1} E^k \otimes E^k \sigma^{-1}. \tag{2.14}$$

For the detail theory of eigentensor decomposition,the readers should refer to [31]. Furthermore, the eigentensor decomposition can provide a lucid expression for the determinant of the 4th-order Douglas tensor $|D|$, i.e.,

$$|D| = \prod_{k=1}^{6} \sigma_k^2 \text{ and } |D^{-1}| = \prod_{k=1}^{6} \sigma_k^{-2}, \tag{2.15}$$

which can be applied in determining the normalization constant in equation (2.3). This factor can also delineate the multiplicative factor of $2^{3/2}$ between the normalization constants in equations (2.1) and (2.3). It is evident that this factor is simply the ratio of determinants of hv-ricci tensor $|G|$ and in equation (2.1) and $|D| = |S|$ in equation (2.3). Actually this factor is the Jacobian of transformation between G and D.

Let us now discuss a special case in which the 4th-order covariance Douglas tensor D_{ijkl} has the isotropic nature.

2.2. Spectrally decomposed Douglas tensor bearing isotropic nature. Here
we describe the spectral decomposition of the 4th-order covariance Douglas tensor bearing isotropic nature. The detail study regarding this issue has been already done by [5, 10].Here, We precisely utilize the results of [5, 10] to meet our purpose. According to [5], isotropy of any tensorial quantity means the quantity has no orientation dependence. Informally, we can say that the tensorial quantity has an isotropic nature, if it is invariant from the aspect of its natural behavior, i.e, the nature which it bears must be invariant under any transformation like rotation,

reflection, inversion etc. Now, as in our case, since the hv-Ricci tensor is symmetric, then the isotropy of Douglas tensor will have the form [19, 32, 31]:

$$(2.16) \qquad D_{ijkl}^{\text{iso}} = \frac{\lambda_\alpha}{3}(\delta_{ij}\delta_{kl}) + \lambda_\beta \left(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl} \right).$$

Here, in the above expression, λ_α and λ_β are constants.

Now, according to [5], the spectrally decomposed 4th-order covariance isotropic Douglas tensor D_{ijkl}^{iso} will have the following latent-roots:

$$(2.17) \qquad \sigma_1^2 = \lambda_\alpha; \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = \sigma_6^2 = \lambda_\beta$$

and the relevant normalized latent-tensors are given as;

$$(2.18) \qquad E^1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E^2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(2.19) \qquad E^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad E^4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(2.20) \qquad E^5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E^6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In Finslerian geometry, these latent-tensors are able to describe the characteristics and geometric significance of the 4th-order covariance isotropic Douglas tensor. Now, with the 6th-order characteristic equation (2.9), there associate six coefficients each of which are scalar invariants of Douglas tensor D_{ijkl}. In fact, these coefficients and their functions are independent from the effect of change of coordinate system. Here we mention the six scalar invariants I_1, I_2, I_3, I_4, I_5 and I_6 which are already determined by [5].

These scalar invariants can be obtained by expanding equation (2.9)as below:

$$(2.21) \qquad (\psi - \sigma_1^2)(\psi - \sigma_2^2)(\psi - \sigma_3^2)(\psi - \sigma_4^2)(\psi - \sigma_5^2)(\psi - \sigma_6^2) = 0,$$

where $\psi = \sigma^2$ is used for convenience. On collecting the like powers of σ and writing these coefficients in the form of latent-roots of Douglas tensor [5], we have

$$(2.22) \qquad \psi^6 - I_1\psi^5 + I_2\psi^4 - I_3\psi^3 + I_4\psi^2 - I_5\psi + I_6 = 0.$$

Here, in the above expression, the following have been used [5]:

$$(2.23) \qquad I_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2,$$

which stands for the trace of covariance matrix S.

$$(2.24) \quad I_2 = \sigma_3^2\sigma_4^2 + \sigma_3^2\sigma_5^2 + \sigma_4^2\sigma_5^2 + \sigma_3^2\sigma_6^2 + \sigma_4^2\sigma_6^2 + \sigma_5^2\sigma_6^2 + \sigma_2^2\sigma_3^2 + \sigma_2^2\sigma_4^2 +$$
$$+ \sigma_2^2\sigma_5^2 + \sigma_2^2\sigma_6^2 + \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_4^2 + \sigma_1^2\sigma_5^2 + \sigma_1^2\sigma_6^2,$$

$$(2.25) \quad I_3 = \sigma_3^2\sigma_4^2\sigma_5^2 + \sigma_3^2\sigma_4^2\sigma_6^2 + \sigma_3^2\sigma_5^2\sigma_6^2 + \sigma_5^2\sigma_5^2\sigma_6^2 + \sigma_2^2[\sigma_5^2\sigma_6^2 + \sigma_4^2(\sigma_5^2 + \sigma_6^2) +$$
$$+ \sigma_3^2(\sigma_4^2 + \sigma_5^2 + \sigma_6^2)] + \sigma_1^2[\sigma_4^2\sigma_5^2 + \sigma_4^2\sigma_6^2 + \sigma_5^2\sigma_6^2 + \sigma_3^2(\sigma_4^2 + \sigma_5^2 + \sigma_6^2) + \sigma_2^2(\sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2)],$$

$$(2.26) \quad I_4 = \sigma_3^2\sigma_4^2\sigma_6^2\sigma_6^2 + \sigma_2^2[\sigma_4^2\sigma_5^2\sigma_6^2 + \sigma_3^2(\sigma_5^2\sigma_6^2 + \sigma_4^2(\sigma_5^2 + \sigma_6^2))] + \sigma_1^2[\sigma_3^2\sigma_4^2\sigma_5^2 +$$
$$+ \sigma_3^2\sigma_4^2\sigma_6^2 + \sigma_3^2\sigma_5^2\sigma_6^2 + \sigma_4^2\sigma_5^2\sigma_6^2 + \sigma_2^2(\sigma_5^2\sigma_6^2 + \sigma_4^2(\sigma_6^2 + \sigma_6^2) + \sigma_3^2(\sigma_4^2 + \sigma_5^2 + \sigma_6^2))],$$

$$(2.27) \quad I_5 = \sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2\sigma_6^2 + \sigma_1^2[\sigma_3^2\sigma_4^2\sigma_5^2\sigma_6^2 + \sigma_2^2(\sigma_4^2\sigma_5^2\sigma_6^2 +$$
$$+ \sigma_3^2(\sigma_5^2\sigma_6^2 + \sigma_4^2(\sigma_5^2 + \sigma_6^2)))],$$

$$(2.28) \qquad\qquad\qquad I_6 = \sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2\sigma_6^2.$$

[5] mentioned that the invariant I_1 is the trace of covariance second order tensor S and I_6 is its determinant. Rest of the invariants stand for discrete combinations of eigenvalues of D_{ijkl}, which can be used to pursue distinguish features of Douglas space.

Concluding Remarks. The present paper is just a particularization of the fabulous article of [5]. The statistical methods have been employed to 4th-order covariance Douglas tensor in *P-Finsler manifold. Just the techniques provided by [5] have been applied and reviewed in the Finsler geometry. In a nutshell, in place of any general 4th-order covariance tensor, a particular Douglas tensor has been employed and then the spectral decomposition has been reviewed.

REFERENCES

[1] Aikou, T.: *Some remarks on conformal equivalence of complex Finsler structures, in finslerian geometry: A meeting of minds*, Edited by P. Antonelli, Kluwer, Amsterdam (2000).

[2] Antonelli, P.L., Ingargen, R.-S. and Matsumoto, M.: *The theory of sprays and FInsler spaces with applications in Physics and Biology*, Kulwer Acad. Publ., Dordrecht, Boston, London, (1993).

[3] Anderson, T.W.: *An Introduction to Multivariate Statistics*, second ed., Wiley, New York, p. 675, (1984).

[4] Bao, D., Chern, S.-S and Shen, Z.: *An introduction to Riemannian-Finsler geometry* , Graduate Texts in Math. 200, Springer, New York (2000).

[5] Basser, P.J. and Pajevic, S.: *Spectral decomposition of a 4th-prder covariance tensor: Application to diffusion tensor MRI*, Signal Processing (Elsevier), 87, 220-236, (2007).

[6] Bácsó, S. and Papp, I.: **P-Finsler spaces with vanishing Douglas tensor*, Acta Academiae Paedagogicae Agriensis, Sectio Mathematicae, 25, 91-95, (1998).

[7] Bell, A.J. and Sejnowski, T.J.: *The independent components of natural scenes are edge filters*, Vision Res., 37 (23), 33273338, (1997).

[8] Bell, A. and Sejnowski, T.: *An information-maximization approach to blind separation and blind deconvolution*, Neural Comput, 7 (6), 11291159, (1995).

[9] Betten, J. and Helisch, W.: *Irreduzible Invarianten eines Tensors vierter Stufe*, (ZAMM) Z. Angew. Math. Mech. 72 (1), 45-57, (1992).

[10] Basser, P.J. and Pajevic: *A normal distribution for tensorvalued random variables: applications to diffusion tensor MRI*, IEEE Trans. Med. Imaging 22 (7), 785794, (2003).

[11] Cowin, S.C. and Mehrabadi, M.M.: *On the identification of material symmetry for anisotropic elastic-materials*, Q. J. Mech. Appl. Math., 40, 4514, (1987).

[12] Dellinger, J., Vasicek, D. and Sondergeld, C.: *Kelvin notation for stabilizing elastic-constant inversion*, Rev. LInst. Francais Pétrole, 53 (5), (1998).

[13] Fukunaga, K.: *Introduction to statistical pattern recognition*, in: H.G. Booker, N. DeClaris (Eds.), Electrical Sciences, Academic Press Inc., New York, (1972).

[14] Green, A.E. and Zerna, W.: *Theoretical Elasticity*, second ed., Clarendon Press, Oxford, p. 457, (1954).

[15] Hotelling, H.: *Analysis of a complex of statistical variables into principal components*, J. Educ. Psychol., 24, 417-444 and 498-520, (1933).

[16] Herault, J. and Jutten, C.: *Blind separation of sources, part 1: an adaptive algorithm based on neuromimetic architecture*, Signal Process, 24, 110, (1991).

[17] Izumi, H.: *On *P-Finsler spaces, I., II.,*, Memoirs of the defense academy, Japan, 16, 133-138, (1976), 17, 1-9, (1977).

[18] _____: *On *P-Finsler spaces of scalar curvature*, Tensor, N.S., 38, 220-222, (1982).

[19] Jeffreys, H.: *Cartesian Tensors*, Cambridge University Press, Cambridge, p. 93, (1931).

[20] (Lord Kelvin) Thomas, W.: *Elements of a mathematical theory of elasticity*, Philos. Trans. R. Soc., 166, 481, (1856).

[21] (lord Kelvin) Thomas, W.: *Elasticity, Encyclopedia Britannica, vol. 7, ninth ed.*, Adam and Charles Black, London, Edinburgh, pp. 796825, (1878).

[22] Mathsumoto, M.: *foundations of Finsler geometry and special Finsler spaces*, Kaisesha Press, Otsu, Japan (1986).

[23] Matsumoto, M. and Hojo, S.: *A conclusive theorem on C-reducible Finsler spaces*, Tensor, N.S., 32, 225-230, (1978).

[24] Makeig, S., Jung, T.P., Bell, A.J., Ghahremani, D. and Sejnowski, T.J.: *Blind separation of auditory event-related brain responses into independent components*, Proc. Natl. Acad. Sci. USA, 94 (20), 10979-10984, (1977).

[25] Mardia, K.V., Kent, J.T. and Bibby, J.M.: *Multivariate Analysis*, Academic Press, New York, p. 519, (1979).

[26] Morse, P.M. and Feschbach, H.: *Methods of Theoretical Physics*, vol. 1, McGraw-Hill, New York, p. 997, (1953).

[27] Mehrabadi, M.M. and Cowin, S.C.: *Eigentensors of linear anisotropic elastic-materials*, Q. J. Mech. Appl. Math., 43, 1541, (1990).

[28] Onat, E.T., Boelher, J.P. and Kirillov, J.P.: *On the polynomial invariants of the elasticity tensor*, J. Elasticity, 97110, (1994).

[29] Pearson, K.: *On lines and planes of closest fit to systems of points in space*, Philos. Mag., 6 (2), 559-572, (1901).

[30] Spearman, C.: *The proof and measurement of association between two things*, Am. J. Psychol., 15, 72 and 202, (1933).

[31] Tarantola, A.: *Elements for Physics: Quantities, Qualities, and Intrinsic Theories*, Springer, Berlin, p. 279, (2005).

[32] Zheng, Q.S.: *A note on representation for isotropic functions of 4th-order tensors in 2-dimensional space*, Z. Angew. Math. Mech., 74 (8), 357359, (1994).

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