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ON A GENERALIZATION OF ATOMIC DECOMPOSITIONS

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Abstract. We generalize atomic decomposition for Banach spaces and called it T-atomic decomposition. A necessary condition for T-atomic decomposition is given. A characterization for a triangular atomic decomposition is also given. Finally, as an application of triangular atomic decompositions, we prove that if a Banach space has a triangular atomic decomposition, then it also has an approximative atomic decomposition, an atomic decomposition and a fusion Banach frame.

1. Introduction

Coifman and Weiss [3] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchenig [5] extended the notion of atomic decomposition to Banach spaces. Frazier and Jawerth [6] had constructed wavelet atomic decompositions for Besov spaces which they called as ϕ-transform. Feichtinger [4] constructed Gabor atomic decompositions for the modulation spaces which are Banach spaces similar in many respects to Besov spaces, defined by smoothness and decay conditions. Atomic decompositions have played a key role in the development of wavelet theory and Gabor theory. Atomic decompositions and Banach frames were further studied in [1, 2, 8].

Motivated by Kozolov [10], we generalize atomic decompositions for Banach spaces. Infact, we introduce the notion of T-atomic decomposition for Banach spaces. Also, a necessary condition for T-atomic decomposition has been obtained. Further, a characterization for triangular atomic decomposition and a characterization for Banach frames have been obtained. Finally, as an application of triangular atomic decompositions, it has been proved that if a Banach space E has a triangular atomic decomposition, then E also has an approximative atomic decomposition, an atomic decomposition and a fusion Banach frame.

2. Preliminaries

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the dual space of E, $L(E)$ the space of all linear operator on E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$.

A sequence $\{x_n\}$ in E is said to be *complete* if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be *total* over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. A sequence of projections $\{v_n\}$ on E is *total* on E if $\{x \in E : v_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition 2.1 ([5]). Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$. Then, $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* for E with respect to E_d, if

(i) $\{f_n(x)\} \in E_d$, $x \in E$

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(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E$

(iii) $x = \sum_{i=1}^{\infty} f_i(x)x_i, \quad x \in E.$

The constants A and B, respectively, are called lower and upper atomic bounds of the atomic decomposition $(\{f_n\}, \{x_n\})$.

Definition 2.2 ([7])**.** Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. Then, $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d, if

(i) $\{f_n(x)\} \in E_d$, $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that

$S(\{f_n(x)\}) = x, \quad x \in E.$

The constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the reconstruction operator (or, the pre-frame operator). The inequality (2.1) is called the Banach frame inequality.

A generalization of the concept of Banach frame namely, fusion Banach frame was introduced and studied in [9] and defined as follows:

Definition 2.3. Let E be a Banach space. Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. Let $\mathcal{A}$ be a Banach space associated with E and $S : \mathcal{A} \to E$ be an operator. Then, $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (or, *fusion Banach frame*) for E with respect to $\mathcal{A}$, if

(i) $\{v_n(x)\} \in \mathcal{A}$, $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{A}} \leq B\|x\|_E, \quad x \in E$

(iii) S is a bounded linear operator such that

$S(\{v_n(x)\}) = x, \quad x \in E.$

The constants A and B, respectively, are called lower and upper frame bounds of the frame of subspaces $(\{G_n, v_n\}, S)$.

The following results are referred in this paper and are listed in the form of lemmas:

Lemma 2.4 ([12])**.** *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Lemma 2.5 ([9])**.** *Let $\{G_n\}$ be a sequence of non-trival subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over E, then $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$.*

3. Main Results

We begin with the following definition of T-atomic decomposition

Definition 3.1. Let E be a Banach space, E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$ and $T = (t_{nm})$ be a matrix of scalars such that

$$\sum_{j=1}^{\infty} |t_{nj}| \leq M < \infty, \qquad n = 1, 2, 3, \ldots \tag{3.1}$$

$$\lim_{n\to\infty} t_{nj} = 0, \qquad j = 1, 2, 3, \ldots \tag{3.2}$$

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} = 1. \tag{3.3}$$

Let $\{x_n\}$ be a sequence in E and $\{f_n\}$ be a sequence in E^*. Then, $(T, \{f_n\}, \{x_n\})$ is called a *T-atomic decomposition* for E with respect to E_d, if

(i) $\{f_n(x)\} \in E_d$, $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E$

(iii) $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{\infty} t_{nj} \left(\sum\limits_{i=1}^{j} f_i(x)x_i \right) = x$, $x \in E$.

In case, T is a triangular matrix, then $(T, \{f_n\}, \{x_n\})$ is said to be a *triangular atomic decomposition* for E with respect to E_d.

Regarding the existence of T-atomic decomposition, let E be a Banach space, $(\{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ be an atomic decomposition for E with respect to an associated Banach space E_d and $T = (t_{nm})$ be a matrix such that $t_{nn} = 1$, $n \in \mathbb{N}$ and $t_{nm} = 0$, $m \neq n$. Then $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d.

Also, one may observe that if E is a Banach space and $(T, \{f_n\}, \{x_n\})$ $(T = (t_{nm}), \{f_n\} \subset E^*, \{x_n\} \subset E)$ is a T-atomic decomposition for E with respect to E_d, then $\{x_n\}$ is complete in E and for each $n \in \mathbb{N}$, $\sigma_n : E \to E$ defined by

$$\sigma_n(x) = \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x)x_i \right), \quad x \in E$$

is well defined bounded linear operator such that $\sup\limits_{1\leq n<\infty} \|\sigma_n\| < \infty$.

Conversely, we have the following example

Example 3.2. Let $E = c_0$, the space of all sequences convergent to 0 in $\mathbb{K}$. Let $T = (t_{nm})$ be a matrix such that $t_{nn} = 1$, $n \in \mathbb{N}$ and $t_{nm} = 0$, $n \neq m$. Let $\{e_n\}$ be the sequence of unit vectors in E and $\{f_n\}$ be a sequence in E^* defined by

$$f_n = (0, 0, \ldots, \underset{\substack{\uparrow \\ n^{\text{th}} \text{ position}}}{(-1)^n}, 0, 0, \ldots), \quad n \in \mathbb{N}.$$

Then, $\{e_n\}$ is complete in E and for each $n \in \mathbb{N}$, $\sigma_n : E \to E$ defined by

$$\sigma_n(x) = \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x)e_i \right), \quad x \in E$$

is well defined bounded linear operator such that $\sup\limits_{1\leq n<\infty} \|\sigma_n\| < \infty$. But $\lim\limits_{n\to\infty} \sigma_n(x) \neq x$, for some $x \in E$. Infact, if we take $x = (1, 0, 0, \ldots) \in E$ then $\lim\limits_{n\to\infty} \sigma_n(x) \neq x$. Hence, $(T, \{f_n\}, \{e_n\})$ is not a T-atomic decomposition for E with respect to any associated Banach space E_d.

In the next result, we prove that for any matrix $T = (t_{nm})$ satisfying (3.1)-(3.3), every atomic decomposition for E is also a T-atomic decomposition for E.

Theorem 3.3. *Let $(\{f_n\}, \{x_n\})$ be an atomic decomposition for a Banach space E with respect to E_d. Then, for any matrix $T = (t_{nm})$ satisfying* (3.1)-(3.3), *$(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d.*

Proof. Let c_E be the Banach space of all convergent sequences of elements of E with the norm $\|\{z_k\}\|_{c_E} = \sup_{1\leq k<\infty} \|z_k\|_E$. For each $n\in\mathbb{N}$, define $u_n : c_E \to E$ by

$$u_n(\{z_k\}) = \sum_{j=1}^{\infty} t_{nj} z_j, \quad \{z_k\} \in c_E.$$

Then, each u_n is well defined on c_E and

$$\begin{aligned} \|u_n\| &= \sup_{\{z_k\}\in c_E} \|u_n(\{z_k\})\| \\ &= \sum_{j=1}^{\infty} |t_{nj}| \leq M, \quad n \in \mathbb{N}. \end{aligned}$$

Now, for any $\{x_1, x_2, \ldots, x_m, 0, 0, \ldots\} \in c_E$, we have

$$\lim_{n\to\infty} u_n(\{x_1, x_2, \ldots x_m, 0, 0, \ldots\}) = 0$$

and, for any $\{x, x, x, \ldots\} \in c_E$, we have

$$\lim_{n\to\infty} u_n(x, x, x, \ldots) = x, \quad x \in E.$$

Since, the set of all the elements of the form $\{x_1, x_2, \ldots, x_m, 0, 0, \ldots\}$ and $\{x, x, x, \ldots\}$, where $x_1, x_2, \ldots, x_m \in E$, $1 \leq m < \infty$ and $x \in E$ is complete in c_E, we have

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} z_j = \lim_{n\to\infty} u_n(\{z_k\}) = \lim_{k\to\infty} z_k.$$

Define, $S_n(x) = \sum_{i=1}^{n} f_i(x) x_i$, $n \in \mathbb{N}$ and $x \in E$. Then $\lim_{n\to\infty} S_n(x) = x$, $x \in E$. Therefore, $\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} s_j(x) = x$, $x \in E$.

Hence, $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d. □

The converse of Theorem 3.3 may not be true as shown by the following example

Example 3.4. Let $E = \ell^2$. Define $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ by

$$\begin{aligned} x_n &= e_n - e_{n+1} \\ f_n(x) &= \langle e_1 + e_2 + \ldots + e_n, x\rangle, \quad x \in E, \ n = 1, 2, \ldots . \end{aligned}$$

Then, $(\{f_n\}, \{x_n\})$ is not an atomic decomposition for E with respect to any associated Banach space E_d. But, by Lemma 2.4, there exist an associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ and a matrix $T = (t_{nm})$ given by $t_{nm} = \frac{1}{n}$, $m = 1, 2, \ldots, n$, $t_{nm} = 0$ for $m > n$ $(n = 1, 2, \ldots)$ such that $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_{d_0}.

Indeed,

$$\begin{aligned}\sigma_n(x) &= \sum_{i=1}^{n} \frac{n-i+1}{n} f_i(x) x_i \\ &= \sum_{i=1}^{n} \frac{n-i+1}{n} \Big\langle \sum_{j=1}^{i} e_j, x \Big\rangle (e_i - e_{i+1}) \\ &= \langle e_1, x\rangle e_1 + \sum_{i=2}^{n} \left[\frac{n-i+1}{n} \Big\langle \sum_{j=1}^{i} e_j, x \Big\rangle e_i - \frac{n-i+2}{n} \Big\langle \sum_{j=1}^{i-1} e_j, x \Big\rangle e_i \right] \\ &\quad - \frac{1}{n} \Big\langle \sum_{j=1}^{n} e_j, x \Big\rangle e_{n+1}\end{aligned}$$

$$= \sum_{i=1}^{n} \frac{n-i+1}{n} \langle e_i, x\rangle e_i - \frac{1}{n} \sum_{i=2}^{n+1} \Big\langle \sum_{j=1}^{i-1} e_j, x \Big\rangle e_i, \quad x \in E, n = 1, 2, 3, \dots .$$

Since, $\lim_{n\to\infty} \sum_{i=1}^{n} \langle e_i, x\rangle e_i = x$, $x \in E$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n-i+1}{n} \langle e_i, x\rangle e_i = x, \quad x \in E.$$

For each $n \in \mathbb{N}$, define $v_n : E \to E$ by

$$v_n(x) = \frac{1}{n} \sum_{i=1}^{n} \Big\langle \sum_{j=1}^{i} e_j, x \Big\rangle e_{i+1}, \quad x \in E, n = 1, 2, \dots .$$

Then, each v_n is well defined bounded linear operator on E. Also, for each $n, k = 1, 2, 3 \dots$, we have

$$\|v_n(e_k)\|^2 = \frac{1}{n^2} \sum_{i=1}^{n} \left| \Big\langle \sum_{j=1}^{i} e_j, e_k \Big\rangle \right|^2 = \frac{n-k+1}{n^2}.$$

Therefore, $\lim_{n\to\infty} \|v_n(e_k)\|^2 = 0$. Hence, $\lim_{n\to\infty} v_n(x) = 0$, $x \in \operatorname{span}\{x_i\}_{i=1}^{\infty}$. Also, since

$$\begin{aligned}\|v_n(x)\|^2 &= \frac{1}{n^2} \sum_{i=1}^{n} \left| \Big\langle \sum_{j=1}^{i} e_j, x \Big\rangle \right|^2 \\ &\leq \frac{1}{n^2} \sum_{i=1}^{n} \left\| \sum_{j=1}^{i} e_j \right\|^2 \|x\|^2 \\ &= \frac{n(n+1)}{2n^2} \|x\|^2 \\ &\leq \|x\|^2, \qquad x \in E,\ n = 1, 2, 3, \dots ,\end{aligned}$$

we have, $\sup_{1\leq n<\infty} \|v_n\| < \infty$. Hence, $\lim_{n\to\infty} \sigma_n(x) = x$, $x \in E$.

Next, we give a necessary condition for a T-atomic decomposition in a Banach space.

Theorem 3.5. *Let E be a Banach space and $T = (t_{nm})$ be a matrix satisfying (3.1)-(3.3). If $(T, \{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ is a T-atomic decomposition for E with respect to E_d. Then for each $n, m \in \mathbb{N}$, there exists a linear operator $v_{nm} \in L(E)$ such that*

$$\lim_{n\to\infty} \lim_{m\to\infty} v_{nm}(x) = x, \quad x \in E.$$

Proof. For each $n, m = 1, 2, 3 \ldots$, define

$$v_{nm}(x) = \sum_{j=1}^{m} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right), \quad x \in E.$$

Then, $v_{nm} \in L(E)$. Also

$$\lim_{m\to\infty} v_{nm}(x) = \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right), \quad x \in E.$$

Since, $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E, therefore

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right) = x, \quad x \in E. \qquad \square$$

Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{x_n\}$ be any sequence in E and $\{f_n\}$ be any sequence in E^*. For each $n \in \mathbb{N}$, define

$$\begin{aligned} \sigma_n(x) &= \sum_{j=1}^{n} t_{nj} \sum_{i=1}^{j} f_i(x) x_i, \quad x \in E,\ n = 1, 2, 3, \ldots, \\ E_0^{(T)} &= \{x \in E : \lim_{n\to\infty} \sigma_n(x) = x\} \text{ and} \\ E_1^{(T)} &= \{x \in E : \lim_{n\to\infty} \sigma_n(x) \text{ exists}\}. \end{aligned}$$

The following result characterizes triangular atomic decompositions in terms of $\{\sigma_n\}$ and $E_0^{(T)}$

Theorem 3.6. *Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{f_n\} \subset E^*$ and $\{x_n\} \subset E$. Then there exists an associated Banach space E_{d_0} such that $(T, \{f_n\}, \{x_n\})$ is a triangular atomic decomposition for E with respect to E_{d_0} if and only if $\{\sigma_n\}$ is total on E and $E_0^{(T)} = E$.*

Proof. Assume that $E_0^{(T)} = E$ and $\{\sigma_n\}$ is total on E. Let $x \in E$ such that $f_n(x) = 0$ for all $n \in \mathbb{N}$. Then $\sigma_n(x) = 0$, $n \in \mathbb{N}$. So totality of $\{\sigma_n\}$ yields $x = 0$. Therefore, by Lemma 2.4, there exists an associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$. Also, by hypothesis, we have $\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right) = x$, $x \in E$. Hence, $(T, \{f_n\}, \{x_n\})$ is a triangular atomic decomposition for E with respect to E_{d_0}.

The converse part is straight forward. $\square$

We conclude this section with the following characterization of Banach frames in terms of $E_0^{(T)}$ and $E_1^{(T)}$

Theorem 3.7. *Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$, $i, j \in \mathbb{N}$. Then there exist an associated Banach space E_d and a bounded linear operator*

$S : E_d \to E$ such that $(\{f_n\}, S)$ is Banach frame for E with respect to E_d if and only if $E_0^{(T)} = E_1^{(T)}$.

Proof. Let $(\{f_n\}, S)$ be a Banach frame for E. Then

$$\begin{aligned} f_m(\sigma_n(x)) &= f_m\bigg(\sum_{j=1}^{\infty} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg)\bigg) \\ &= \bigg(\sum_{j=m}^{\infty} t_{nj}\bigg) f_m(x)\,, \quad n, m = 1, 2, 3\ldots \text{ and } \ x \in E. \end{aligned}$$

Let $x \in E_1^{(T)}$. Then

$$\begin{aligned} f_m(x - \lim_{n\to\infty} \sigma_n(x)) &= f_m(x) - \lim_{n\to\infty}\bigg(\sum_{j=m}^{\infty} t_{nj}\bigg) f_m(x) \\ &= f_m(x)\bigg[1 - \lim_{n\to\infty}\sum_{j=1}^{\infty} t_{nj} + \lim_{n\to\infty}\sum_{j=1}^{m-1} t_{nj}\bigg] = 0 \end{aligned}$$

Therefore, by the frame inequality for the Banach frame $(\{f_n\}, S)$, we have $x \in E_0^{(T)}$.

Conversely, let $x \in E$ be such that $f_n(x) = 0$, $n = 1, 2, 3 \ldots$. Then $\sigma_n(x) = 0$ for all $n \in \mathbb{N}$. Since, $E_0^{(T)} = E_1^{(T)}$, we have $x = 0$. Therefore, by Lemma 2.4, there exist associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ and a bounded linear operator $S : E_{d_0} \to E$ defined by $S(\{f_n(x)\}) = x$, $x \in E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_{d_0}. □

4. Applications

In this section, we give some applications of triangular atomic decompositions. First, we give the definition of approximative atomic decomposition introduced in [11].

Let E be a Banach space and let E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \subset E^*$, where $\{m_n\}$ is an increasing sequence of positive integers. Then, $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{x_n\})$ is called an approximative atomic decomposition for E with respect to E_d, if

(i) $\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \in E_d$, $x \in E$

(ii) there exist constants A and B with $0 < A \le B < \infty$ such that
$A\|x\|_E \le \|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}\|_{E_d} \le B\|x\|_E, \quad x \in E$

(iii) $x = \lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x)x_i$, $x \in E$.

In the following result, we prove that if a Banach space has a triangular atomic decomposition, then it also has an approximative atomic decomposition

Theorem 4.1. *If a Banach space has a triangular atomic decomposition then it also has an approximative atomic decomposition.*

Proof. Let E be a Banach space having a triangular atomic decomposition $(T, \{f_n\}, \{x_n\})(T = (t_{nm}), \{f_n\} \subset E^*, \{x_n\} \subset E)$ with respect to E_d. Since, T is a triangular matrix, for each $n, m \in \mathbb{N}$, $m \ge n$,

$$\sum_{j=1}^{m} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg) = \sum_{j=1}^{n} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg), \quad x \in E.$$

For each $n \in \mathbb{N}$, define $\sigma_n : E \to E$ by

$$\sigma_n(x) = \sum_{j=1}^{n} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg), \quad x \in E.$$

Then, each σ_n is well defined finite rank linear operator on E. Since, for each $n \in \mathbb{N}, \sigma_n(E)$ is finite dimensional. So, there exist a sequence $\{y_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E^* such that

$$\sigma_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x)y_{n,i}, \quad x \in E,\ n \in \mathbb{N},$$

where $\{m_n\}$ is an increasing sequence of positive integers with $m_0 = 0$. Define, $\{z_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots m_n \\ n\in\mathbb{N}}} \subset E^*$ by

$$\begin{aligned} z_i &= y_{n,i}, \quad i = m_{n-1}+1, \ldots, \quad m_n, \\ h_{n,i} &= \begin{cases} 0, & \text{if } \ i = 1, 2, \ldots, m_{n-1} \\ g_{n,i}, & \text{if } \ i = m_{n-1}+1, \ldots m_n, \end{cases} \qquad n \in \mathbb{N}\,. \end{aligned}$$

Then, for each $x \in E$,

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x)z_i = \lim_{n\to\infty} \sigma_n(x) = x\,.$$

Let $x \in E$ be such that $h_{n,i}(x) = 0$, for all $i = 1, 2, \ldots, m_n,\ n \in \mathbb{N}$. Then $x = 0$. Therefore, by Lemma 2.4, there exists an associated Banach space $E_{d_0} = \{\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} : x \in E\}$ with the norm given by $\|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ such that, $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{z_n\})$ is an approximative atomic decomposition for E with respect to E_{d_0}. □

Corollary 4.2. If a Banach space E has a triangular atomic decomposition, then it also has an atomic decomposition.

Proof. Follows in view of Theorem 4.1. □

Finally, we prove that, if for a suitably chosen triangular matrix T satisfying (3.1)-(3.3), E has a triangular atomic decomposition, then it also has a fusion Banach frame.

Theorem 4.3. *Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix such that $t_{nm} \neq 0$, $n \geq m$. If $(T, \{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ is a triangular atomic decomposition for E, then E has a fusion Banach frame.*

Proof. By Theorem 4.1, E has approximative atomic decomposition. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \subset E^*$ be sequences such that $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to E_d, where $\{m_n\}$ is an increasing sequence of positive integers. For each $n \in \mathbb{N}$, define $u_n : E \to E$ by

$$u_n(x) = \sum_{i=1}^{m_n} h_{n,i}(x)x_i, \quad x \in E.$$

Then, each u_n is a well defined continuous linear operator on E with $\dim u_n(E) < \infty$ and $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Define $G_n = u_n(E)$, $n \in \mathbb{N}$. Then, each G_n is finite dimensional. Therefore, there exist a sequence $\{y_{n,i}\}_{i=1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=1}^{m_n}$ in E^* such that

$$u_n(x) = \sum_{i=1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E \quad \text{and } n \in \mathbb{N}.$$

Now, for each $n \in \mathbb{N}$, define $v_n : E \to E$ by

$$v_n(x) = \sum_{i=1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E.$$

Then, each v_n is a projection on G_n such that $\{x \in E : v_n(x) = 0, n \in \mathbb{N}\}=\{0\}$. Therefore, by Lemma 2.5, there exist an associated Banach space $\mathcal{A} = \{v_n(x) : x \in E\}$ with the norm given by $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$ and a bounded linear operator $S : \mathcal{A} \to E$ given by $S(\{v_n(x)\}) = x$, $x \in E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. □

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