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SOME COMPUTATION PROBLEMS ARISING IN FONTAINE THEORY

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Abstract. In this note we construct special types of rings $A_{\mathrm{max,n}}$ which are used in sequel work to define new types of families of continuous Fontaine sheaves. We also study the maps θ_n and q_n providing explicit description of their kernels. Finally, we implement an algorithm which leads to the computation of these kernels.

1. Introduction

Let us fix a prime integer p and a finite extension K of $\mathbb{Q}_p$ with residue field k and ring of integers $\mathcal{O}_K$ and denote by G_K the Galois group of $\overline{K}$ over K where $\overline{K}$ is a fixed algebraic closure of K. Write K_0 for the maximal unramified extension of $\mathbb{Q}_p$ in K. Also let X be a smooth, proper and connected scheme over K and denote by $X_{\overline{K}}$ the geometric generic fiber of X.

In order to decide the nature of the G_K-representation $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$, $i \geq 0$ one needs to use "comparison isomorphisms theorems" i.e. theorems comparing p-adic étale cohomology of $X_{\overline{K}}$ to other cohomology theories associated to X. For example, if X has good reduction the cohomology theory we refer to is the crystalline cohomology of the special fiber of a smooth proper model of X over $\mathcal{O}_K$. Denote this special fiber by $\overline{X}$.

The crystalline comparison conjecture was formulated by J.-M. Fontaine in [Fo1] and proved by G. Faltings in [Fa]:

Theorem 1.1. *For every $i \geq 0$ there is a canonical isomorphism of B_{cris}-modules, which respects the G_K-actions, the Frobenii and the filtrations*

$$H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^i_{\mathrm{cris}}(\overline{X}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} B_{\mathrm{cris}},$$

where B_{cris} is the crystalline period ring defined by J.-M. Fontaine in [Fo1].

In [AI] a new method of attacking comparison isomorphisms is supplied provided $K = K_0$ i.e. K is unramified over $\mathbb{Q}_p$.

One defines the Faltings's topology $\mathfrak{X}_{\overline{K}}$ on the smooth proper model of X over $\mathcal{O}_K$ (see [AI] for details).

A. Iovita and F. Andreatta are defining in [AI] new sheaves of rings $\mathbb{A}^{\nabla}_{\mathrm{cris}}$ and $\mathbb{A}_{\mathrm{cris}}$ on $\mathfrak{X}_{\overline{K}}$ and they prove the following:

Theorem 1.2. $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^i(\mathfrak{X}_{\overline{K}}, \mathbb{A}^{\nabla}_{\mathrm{cris}}) \otimes_{A_{\mathrm{cris}}} B_{\mathrm{cris}} \cong H^i_{\mathrm{cris}}(\overline{X}, K_0) \otimes_{K_0} B_{\mathrm{cris}}$.

This article deals with the construction of certain families of rings $(A_{\mathrm{max,n}})_{n\geq 1}$, $(A'_{\mathrm{max,n}})_{n\geq 1}$. The construction of these rings are carried out in section 4. We study the maps θ_n, q_n, $\bar{q}_n$ and provide explicit description of their kernels in section 3. In section 5, we discuss algorithms which allow us to compute the kernels of θ_n and q_n. The details of the computational experiments and relevant results are included in the same section. We begin in section 2 by recalling some basic facts of Fontaine theory and set out the notations that are used throughout the paper. In the appendix, the readers will find an algorithm which was used in the paper.

One uses the rings $A_{\mathrm{max,n}}$ to construct a family of sheaves of rings $(\mathbb{A}^{\nabla}_{\mathrm{max,n}})_{n\geq 1}$ on Faltings's topology $\mathfrak{X}_{\overline{K}}$ associated to X and a smooth, proper model of it and study their properties, most important the localization over small affines (see [Ga] for details). The second family of rings namely $(A'_{\mathrm{max,n}})_{n\geq 1}$ is used to define the sheaves of rings $(\mathbb{A}'^{\nabla}_{\mathrm{max,n}})_{n\geq 1}$ which are related to the first family of sheaves via the isomorphism

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$\mathbb{A}^{\nabla}_{\rm max,m}/p^n\mathbb{A}^{\nabla}_{\rm max,m} \cong \mathbb{A}'^{\nabla}_{\rm max,n}$ for $m \geq n+2$ (see [Ga], Lemma 3.2.5) and which plays a key role in proving the localization over small affines theorem (see [Ga], Theorem 3.2.7).

The rings $A_{\rm max,n}$ will also be used in sequel work to define a Riemann-Hilbert correspondence between p-adic locally constant sheaves on X and F-isocrystals on the special fiber of the fixed smooth model of X over $\mathcal{O}_K$.

The first four sections of the paper were written by the first author while the next three were the joint work of both authors.

2. Notations and Background

Let us fix as before a prime integer p, a finite extension K of $\mathbb{Q}_p$ with residue field k and an algebraic closure of K, $\overline{K}$ with residue field $\overline{k}$. Denote by G_K the Galois group of $\overline{K}$ over K, by $\mathcal{O}_K$ the ring of integers of K and by $\mathcal{O}_{\overline{K}}$ the ring of integers of $\overline{K}$. Also denote by $\mathbb{C}_K$ the completion of $\overline{K}$ for the p-adic topology. It is an algebraically closed field and it has a p-adic valuation v normalized by $v(p)=1$.

One defines the $\mathbb{F}_p$-algebra:

$$R := \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}},$$

where the inverse limit is taken with respect to Frobenius. An element $x \in R$ is then a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ satisfying $x_{n+1}^p = x_n$ for all n. R is a perfect $\mathbb{F}_p$-algebra of characteristic p and one has a bijection from $\varprojlim \mathcal{O}_{\overline{K}}$ to $\varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ which is defined by

$$(x^{(n)})_{n\geq 0} \mapsto (x^{(n)}(\mathrm{mod} p)).$$

The inverse of the map is:

$$(x_n)_{n\geq 0} \mapsto (x^{(n)})_{n\geq 0},$$

where $x^{(n)} = \lim_{m\to\infty} \widehat{x_{n+m}}^{p^m}$ for arbitrary lifts $\hat{x}_i \in \mathcal{O}_{\overline{K}}$ of $x_i \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ for all $i \geq 0$, the limit being independent of the choice of the lifts (see [Fo2], 1.2.2]).

The laws of multiplication and addition are given by the following formulae: for any $x, y \in R$ and $n \in \mathbb{N}$,

$$(xy)^{(n)} = x^{(n)}y^{(n)}$$
$$(x+y)^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$$

One gives R a valuation by defining $v_R(x) = v(x^{(0)})$ for all $x \in R$. One can prove that v_R is a valuation on R and that R is v_R-adically separated and complete with residue field $\overline{k}$ (see [BC], Lemma 4.3.3]).

Now for positive integers $n \geq 1$, let $W_n := \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ be the ring of Witt vectors of length n (on $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ valued points). We have a ring homomorphism:

$$\theta_n : \quad W_n \longrightarrow \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$$
$$(s_0, ..., s_{n-1}) \longmapsto \sum_{i=0}^{n-1} p^i \tilde{s_i}^{p^{n-1-i}}$$

where $\tilde{s_i} \in \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ are lifts of s_i. Denote by $u_n : W_{n+1} \to W_n$ the homomorphism defined by Frobenius composed with the truncation map (i.e. u_n sends $(s_0, s_1, ..., s_n)$ to $(s_0^p, s_1^p, ..., s_{n-1}^p)$). Also let $v_n : \mathcal{O}_{\overline{K}}/p^{n+1}\mathcal{O}_{\overline{K}} \to \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ be the truncation map. We have that $\theta_n \circ u_n = v_n \circ \theta_{n+1}$ for every n. Furthermore one has a G_K-equivariant morphism:

$$\theta : \quad \varprojlim_{u_n} \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \to \varprojlim_{v_n} \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} = \mathcal{O}_{\mathbb{C}_K}$$

The inverse limit of the projective system $(\mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}), u_n)_{n\in\mathbb{N}}$ is identified with the ring of Witt vectors $\mathbb{W}(R)$ which we denote by $A^+_{\rm inf}$.

3. Explicit kernel descriptions

We remark that $\mathbb{W}_n(R) \cong A^+_{\text{inf}}/p^n A^+_{\text{inf}}$ since R is perfect and since for each n the projection map

$$\pi_n : A^+_{\text{inf}} \to \mathbb{W}_n(R)$$
$$(s_0, s_1, ..., s_n, ...) \mapsto (s_0, s_1, ..., s_{n-1}).$$

has the kernel equal to:

$$\{(s_0, s_1, ...s_n, ...) \in A^+_{\text{inf}} \mid s_0 = s_1 = ... = s_{n-1} = 0\} = p^n A^+_{\text{inf}}.$$

We now describe the kernels of the maps θ and θ_n. The explicit kernel computations and related issues can be found in section 5.

Choose $\tilde{p} \in R$ such that $\tilde{p}^{(0)} = p$ (so $\tilde{p} = (p, p^{1/p}, p^{1/p^2}, ...)$), $\tilde{p}^{(n)} = p^{1/p^n}$). Then the element $\xi := [\tilde{p}] - p \in A^+_{\text{inf}}$ is a generator of $ker(\theta)$ (see [BC], Proposition 4.4.3). Also denote by $\tilde{p}_n := [p^{1/p^{n-1}}] \in W_n$ the Teichmueller lift of $p^{1/p^{n-1}} \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ and let $\xi_n := \tilde{p}_n - p \in W_n$. Remark that the sequence $\xi = \{\xi_n\}_n$ is compatible since $u_n(\xi_{n+1}) = \xi_n$ for all $n \geq 1$ and that ξ_n is a generator of $ker(\theta_n)$ because of the following proposition.

Let us first make the identification $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} = \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$.

Proposition 1. *The ideal $ker(\theta_n) \subseteq \mathbb{W}_n(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$ is the principal ideal generated by ξ_n.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{W}(R) & \xrightarrow{\pi_n} & \mathbb{W}_n(R) & \xrightarrow{q_{n,n-1}} & \mathbb{W}_n(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}) \\ \downarrow \theta & & & \swarrow \theta_n & \\ \mathcal{O}_{\mathbb{C}_K} & \longrightarrow & \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K} & & \end{array}$$

where the bottom map is the reduction modulo p^n and the map $q_{n,n-1} : \mathbb{W}_n(R) \to \mathbb{W}_n(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$ is given by

$$(s_0, s_1, ..., s_{n-1}) \overset{q_{n,n-1}}{\longmapsto} (s_0^{(n-1)}(\text{mod}p), s_1^{(n-1)}(\text{mod}p), \ldots, s_{n-1}^{(n-1)}(\text{mod}p))$$

with $(s_0, s_1, ..., s_{n-1}) \in \mathbb{W}_n(R)$. Denote by $f_n := q_{n,n-1} \circ \pi_n$ and remark that it is a surjective ring homomorphism. We first prove that the map induced by θ at the level of kernels namely $\theta_{|\ker(f_n)} : \ker(f_n) \to \ker(\text{mod}p^n)$ is surjective. For this, let $s \in \ker(\text{mod}p^n) = p^n\mathcal{O}_{\mathbb{C}_K}$ so $s = p^n \cdot t$ for some $t \in \mathcal{O}_{\mathbb{C}_K}$. Since θ is surjective, we have that $t = \theta(r)$ for some $r \in \mathbb{W}(R)$ and hence $s = p^n \cdot \theta(r) = \theta(p^n) \cdot \theta(r) = \theta(p^n \cdot r)$. Moreover, $p^n \cdot r \in p^n\mathbb{W}(R) \subset \ker(f_n)$. It follows that $\theta_{|\ker(f_n)}$ is surjective.

The inclusion $p^n\mathbb{W}(R) \subset \ker(f_n)$ follows easily: let $w := (w_0, w_1, ...) \in \mathbb{W}(R)$. We then have that $p^n \cdot w = (\underbrace{0, ..., 0}_{n}, w_0^{p^n}, w_1^{p^n}, ...) \in p^n\mathbb{W}(R)$ and consequently $f_n(p^n \cdot w) = q_{n,n-1}(\pi_n(\underbrace{0, ..., 0}_{n}, w_0^{p^n}, w_1^{p^n}, ...)) = q_{n,n-1}(0, ..., 0) = (0, ..., 0)$ hence $p^n \cdot w \in \ker(f_n)$.

We apply now the Snake Lemma in the above diagram and since $\text{coker}(\theta_{|\ker(f_n)}) = 0$ we obtain that the map induced by f_n at the level of kernels namely $\ker(\theta) \to \ker(\theta_n)$ is surjective. Consequently, since $\ker(\theta) \subseteq \mathbb{W}(R)$ is the principal ideal generated by ξ ([BC], Proposition 4.4.3), one obtains that the ideal $\ker(\theta_n) \subseteq \mathbb{W}_n(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$ is principal and generated by $f_n(\xi) = \xi_n$. □

The following two propositions are results quoted in [AI] and left as exercises. We give here the complete proof.

Proposition 2. *The kernel of the projection map*

$$\bar{q}_n : R = \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \to \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$$

on the $n+1$-th factor of the limit is generated by $\tilde{p}^{p^n}$.

Proof. For this, let $x = (x_m)_{m\geq 0} \in R$. Then $\bar{q}_n$ sends $(x_m)_{m\geq 0}$ to x_n.

Remark that since

$$v_R(x) = v(x^{(0)}) = v((x^{(n)})^{p^n}) = p^n v(x^{(n)}) \quad n \geq 0,$$

then

$$v_R(x) \geq p^n \Leftrightarrow v(x^{(n)}) \geq 1 \Leftrightarrow x^{(n)}(\mathrm{mod} p) = 0.$$

One obtains in this way a better description of $ker(\bar{q}_n)$

$$ker(\bar{q}_n) = \{x \in R / v_R(x) \geq p^n\} = \{x \in R / x^{(n)}(\mathrm{mod} p) = 0\}.$$

Now since $v_R(\tilde{p}^{p^n}) = v(p^{p^n}) = p^n$, it is true that $(\tilde{p}^{p^n}) \subseteq ker(\bar{q}_n)$. For the other inclusion, let $x \in ker(\bar{q}_n)$. Subsequently, $v(x^{(0)}) \geq p^n$ hence $x^{(0)} = p^{p^n} y^{(0)}$, for some $y^{(0)} \in \mathcal{O}_{\overline{K}}$. Since $(x^{(n)})_n$ is compatible we have that $(x^{(1)})^p = x^{(0)} = p^{p^n} y^{(0)}$ and one obtains $x^{(1)} = p^{p^{n-1}} y^{(1)}$, $y^{(1)} \in \mathcal{O}_{\overline{K}}$ and moreover $(y^{(1)})^p = y^{(0)}$ (recall that the multiplication in R (through the above mentioned bijection) is $(st)^{(n)} = (s)^{(n)}(t)^{(n)}$ and that $\mathcal{O}_{\overline{K}}$ is normal). We construct in this way a compatible sequence $y = (y^{(n)})_n \in R$ such that $x = \tilde{p}^{p^n} y$. □

The projection $\bar{q}_n$ induces a ring homomorphism:

$$q_n : \mathbb{W}_n(R) \to \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$$
$$(s_0, s_1, ..., s_{n-1}) \mapsto (s_0^{(n)}(\mathrm{mod} p), s_1^{(n)}(\mathrm{mod} p), \cdots, s_{n-1}^{(n)}(\mathrm{mod} p)).$$

Since q_n is surjective, we have the isomorphism:

$$\mathbb{W}_n(R)/ker(q_n) \cong \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) = W_n.$$

Denote by $V : \mathbb{W}_n(R) \to \mathbb{W}_{n+1}(R)$ the Verschiebung i.e.

$$V((s_0, s_1, ..., s_{n-1})) = (0, s_0, s_1, ..., s_{n-1}),\ (s_0, s_1, ..., s_{n-1}) \in \mathbb{W}_n(R).$$

The following proposition describes the kernel of the map q_n.

Proposition 3. *The kernel of the ring homomorphism q_n is the ideal generated by*

$$\{[\tilde{p}]^{p^n}, V([\tilde{p}]^{p^n}), V^2([\tilde{p}]^{p^n}), \cdots, V^{n-1}([\tilde{p}]^{p^n})\}.$$

Proof. For $n = 1$ the statement is obvious by using Proposition 2. For $n \geq 2$ we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{W}_{n-1}(R) & \xrightarrow{V\circ(*)^p} & \mathbb{W}_n(R) & \xrightarrow{pr_1\circ(*)^{1/p^n}} & \mathbb{W}_1(R) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_{n-1}} & & \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_1} & & \\
0 & \longrightarrow & \mathbb{W}_{n-1}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) & \xrightarrow{V} & \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) & \xrightarrow{pr_1} & \mathbb{W}_1(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) & \longrightarrow & 0
\end{array}$$

where by pr_1 we denote the projection map on the first component.

One can easily check the exactness of the second row so we omit it. For the first one, remark that $(V \circ (*)^p)((s_0, s_1, ..., s_{n-2})) = (0, s_0^p, s_1^p, ..., s_{n-2}^p)$, $s_i \in R$, $0 \leq i \leq n-2$, and that $(pr_1 \circ (*)^{1/p^n})((0, s_0^p, s_1^p, ..., s_{n-2}^p)) = pr_1((0, s_0^{1/p^{n-1}}, s_1^{1/p^{n-1}}, ..., s_{n-2}^{1/p^{n-1}})) = 0$.

On the other hand, $V \circ (*)^p$ is injective since Verschiebung is injective and $(*)^p$ is bijective due to the fact that R is perfect. Similarly, $pr_1 \circ (*)^{1/p^n}$ remains surjective (for $s_0 \in \mathbb{W}_1(R)$, we have that $(pr_1 \circ (*)^{1/p^n})((s_0^{p^n}, s_1, ..., s_{n-1})) = s_0$, where $(s_0^{p^n}, s_1, ..., s_{n-1}) \in \mathbb{W}_n(R)$).

Take now $(s_0, s_1, ..., s_{n-1}) \in ker(pr_1 \circ (*)^{1/p^n})$ so $s_0^{1/p^n} = 0$. Since R is perfect it follows that $s_0 = 0$ and consequently $(s_0, s_1, ..., s_{n-1}) = (V \circ (*)^p)((s_1^{1/p}, s_2^{1/p}, ..., s_{n-1}^{1/p}))$ hence $ker(pr_1 \circ (*)^{1/p^n}) \subseteq Im(V \circ (*)^p)$.

One obtains that the first row is exact. Note that the first square diagram is exact since, for a choice of $s_i \in R$, $0 \leq i \leq n-2$, we have:

$$\begin{array}{ccc}
(s_0, s_1, ..., s_{n-2}) & \xrightarrow{V\circ(*)^p} & (0, s_0^p, s_1^p, ..., s_{n-2}^p) \\
\downarrow{\scriptstyle q_{n-1}} & & \downarrow{\scriptstyle q_n} \\
(s_0^{(n-1)}(p), s_1^{(n-1)}(p), ..., s_{n-2}^{(n-1)}(p)) & \xrightarrow{V} & (0, s_0^{(n-1)}(p), s_1^{(n-1)}(p), ..., s_{n-2}^{(n-1)}(p))
\end{array}$$

Also the second square diagram commutes since, for a choice of $s_i \in R$, $0 \leq i \leq n-1$, we have:

$$\begin{array}{ccc}
(s_0, s_1, ..., s_{n-1}) & \xrightarrow{pr_1\circ(*)^{1/p^n}} & (s_0^{1/p^n}) \\
\downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_1} \\
(s_0^{(n)}(p), s_1^{(n)}(p), ..., s_{n-1}^{(n)}(p)) & \xrightarrow{pr_1} & (s_0^{(n)}(p))
\end{array}$$

One applies further the induction hypothesis at the level of kernels in the main diagram. □

4. Constructing the rings $A_{\max,n}$

Definition 1. Let A be a p-adically complete $\mathcal{O}_K$-algebra and T a variable. Define

$$A\{T\} := \varprojlim A[T]/p^n A[T].$$

Also define

$$A_{\max,n} := W_n[\delta]/(p\delta - \xi_n)$$
$$A_{\max} := \varprojlim_n A_{\max,n}.$$

Using these definitions, we then have

$$A_{\max} = A^+_{\inf}\left\{\left[\frac{\xi}{p}\right]\right\} = A^+_{\inf}\{\delta\}/(p\delta - \xi)$$
$$= \left\{\sum_{i\geq 0} a_i\delta^i \text{ such that } a_i \in A^+_{\inf} \text{ and } a_i \xrightarrow{i\to\infty} 0\right\}.$$

i.e. we recover the ring introduced by Colmez in [Col].

Let $A'_{\max,n} := W_n[\delta]/(p\delta - \xi_{n+1})$. (By ξ_{n+1} we mean here the projection on the first n components of this vector namely $pr_n(\xi_{n+1}) = \underbrace{(p^{1/p^n}, -1, 0, ..., 0)}_{n})$. Note that we also have that:

$$V^i([\tilde{p}]^{p^n}) = p^i([\tilde{p}]^{p^n})^{p^{-i}} = p^i[\tilde{p}]^{p^{n-i}} = p^i(\xi + p)^{p^{n-i}} = p^i(p(\delta+1))^{p^{n-i}}$$
$$\equiv p^{i+p^{n-i}}\delta^{p^{n-i}} \equiv 0(\text{mod} p^n A_{\max}),$$

where for the first equality one uses the Witt coordinatization $((r_0, r_1, ...) = \sum p^n[r_n^{p^{-n}}]$ (or one computes it directly)).

By using Proposition 3 one obtains that $ker(q_n) \subseteq p^n A_{\max}$. We will use this fact in the proof of the following:

Proposition 4.

$$A_{\max}/p^n A_{\max} \cong A'_{\max,n}.$$

Proof. Since $ker(q_n) \subseteq p^n A_{\max}$, we obtain that:

$$\frac{A_{\max}}{p^n A_{\max}} = A_{\max}/(p^n, ker(q_n))A_{\max} = \frac{A^+_{\inf}\{\delta\}/(p\delta-\xi)}{(p^n, ker(q_n))(A^+_{\inf}\{\delta\}/(p\delta-\xi))}$$
$$= \frac{A^+_{\inf}\{\delta\}/(p\delta-\xi)}{(p^n, ker(q_n), p\delta-\xi)A^+_{\inf}\{\delta\}/(p\delta-\xi)} \cong A^+_{\inf}[\delta]/(p^n, ker(q_n), p\delta-\xi)A^+_{\inf}[\delta]$$
$$\cong \frac{A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta]}{(p^n, ker(q_n), p\delta-\xi)A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta]} \cong \frac{(A^+_{\inf}/p^n A^+_{\inf})[\delta]}{(ker(q_n), p\delta-\xi(\mathrm{mod} p^n))(A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta])}.$$

By using now the isomorphisms of rings

$$A^+_{\inf}/p^n A^+_{\inf} \cong \mathbb{W}_n(R) \text{ and } A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta] \cong (A^+_{\inf}/p^n A^+_{\inf})[\delta]$$

one obtains that

$$A_{\max}/p^n A_{\max} \cong \mathbb{W}_n(R)[\delta]/(ker(q_n), p\delta-\xi(\mathrm{mod} p^n)).$$

Since $\mathbb{W}_n(R)/ker(q_n) \cong W_n$ and $q_n(\xi(\mathrm{mod} p^n)) = pr_n(\xi_{n+1})$, q_n induces the isomorphism

$$\mathbb{W}_n(R)[\delta]/(ker(q_n), p\delta-\xi(\mathrm{mod} p^n)) \cong W_n[\delta]/(p\delta - pr_n(\xi_{n+1})) =: A'_{\mathrm{max,n}}.$$

We obtain that $A_{\max}/p^n A_{\max} \cong A'_{\mathrm{max,n}}$. □

Remark 1. One can also prove the previous proposition by showing that there is a surjective map $A_{\max} \twoheadrightarrow A'_{\mathrm{max,n}}$ whose kernel is $p^n A_{\max}$. One can prove (see [Ga], Lemma 3.2.5) that for any positive integers $m > n$ there is an isomorphism of rings $A_{\max}/p^n A_{\max} \cong A_{\mathrm{max,m}}/p^n A_{\mathrm{max,m}}$.

Note that, via the isomorphism $A_{\max}/p^n A_{\max} \cong A'_{\mathrm{max,n}}$, we have a surjective map of rings:

$$q'_n : A_{\max}/p^n A_{\max} \to A_{\mathrm{max,n}}$$

sending $pr_n(\xi_{n+1}) \to \xi_n$, induced by Frobenius on W_n and that we also have a map:

$$u_n : A_{\mathrm{max,n+1}} \to A_{\max}/p^n A_{\max}$$

sending $\xi_{n+1} \to pr_n(\xi_{n+1})$, induced by the natural projection $W_{n+1} \to W_n$.

One further uses the rings $A_{\mathrm{max,n}}$ to construct the family of sheaves $(\mathbb{A}^{\nabla}_{\mathrm{max,n}})_{n\geq 1}$ and study their properties (see [Ga] for details).

5. Computing the kernels of θ_n and θ

As we have seen in section 3, computing of the kernels of the map θ (w.r.p θ_n) amounts to the task of computing Witt vectors of finite length in the Witt ring $A^+_{\inf}$ (w.r.p W_n). In this section, we present two approaches that will facilitate the computation task at hand.

First let us recall some basic facts about Witt vectors. The readers may consult [Se, chapter 2] for more details on the topic. Let p be a prime and n a positive integer. The n'th Witt polynomial is by definition

$$W_n(X_0, \dots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^{n-1}X_{n-1}^p + p^n X_n. \tag{1}$$

There exist polynomials S_n, P_n in $\mathbb{Z}[X_0, \dots, X_n, Y_0, \dots Y_n]$ satisfying

$$W_n(S_0, \dots, S_n) = W_n(X_0, \dots, X_n) + W_n(Y_0, \dots, Y_n) \tag{2}$$

and

$$W_n(P_0, \dots, P_n) = W_n(X_0, \dots, X_n) \cdot W_n(Y_0, \dots, Y_n). \tag{3}$$

Let A be a commutative ring. Suppose $\mathbf{a} = (a_0, a_1 \dots)$ and $\mathbf{b} = (b_0, b_1, \dots)$ are elements of $A^{\mathbb{N}}$, set

$$\mathbf{a} + \mathbf{b} = (S_0(\mathbf{a}, \mathbf{b}), S_1(\mathbf{a}, \mathbf{b}), \dots)$$
$$\mathbf{a} \cdot \mathbf{b} = (P_0(\mathbf{a}, \mathbf{b}), P_1(\mathbf{a}, \mathbf{b}), \dots).$$

p	no. terms	CPU time
11	2672	0.421s
19	22856	6.739s
23	48644	54.554s
29	121886	1102.459s
31	158812	2808.408s

TABLE 1. Polynomial S_2 calculation for various p

The laws of composition defined above make $A^{\mathbb{N}}$ into a commutative unitary ring (called the ring of Witt vectors).

Thus in order to compute $\mathbf{a}+\mathbf{b}$ (w.r.p. $\mathbf{a}\cdot\mathbf{b}$), one has to compute $S_n(\mathbf{a},\mathbf{b})$ (w.r.p. $P_n(\mathbf{a},\mathbf{b})$) for all n. In what follows, we present two approaches for computing S_n (w.r.p. P_n). For the sake of clarity, we have chosen to focus on the computation of S_n. The computation of P_n follows in a similar line. The first approach computes the polynomials S_n explicitly, the evaluation $S_n(\mathbf{a},\mathbf{b})$ is achieved by evaluating S_n at $\mathbf{a},\mathbf{b}$. The second approach is to use the recursion formula (6) (derived below) to compute $S_n(\mathbf{a},\mathbf{b})$ directly from the already computed values $S_0(\mathbf{a},\mathbf{b}), S_1(\mathbf{a},\mathbf{b}),\ldots,S_{n-1}(\mathbf{a},\mathbf{b})$.

5.1. **Polynomial Evaluation.** In the first place, we may use (2) to explicitly compute the polynomial functions $S_0,S_1,S_2,\ldots,S_n$ successively in $\mathbb{Z}[X_0,\ldots,X_n,Y_0,\ldots Y_n]$. $S_n(\mathbf{a},\mathbf{b})$ is then computed by evaluating S_n at $\mathbf{a},\mathbf{b}$.

For $n\le 2$, we have

$$S_0 = X_0+Y_0, \qquad S_1 = X_1+Y_1+\frac{1}{p}\left(X_0^p+Y_0^p-(X_0+Y_0)^p\right); \tag{4}$$

and

$$\begin{aligned} S_2 &= X_2+Y_2+\frac{1}{p}\left(X_1^p+Y_1^p-\left(X_1+Y_1+\frac{X_0^p+Y_0^p-(X_0+Y_0)^p}{p}\right)^p\right)\\ &\quad+\frac{X_0^{p^2}+Y_0^{p^2}-(X_0+Y_0)^{p^2}}{p^2} \\ &\ \vdots \end{aligned} \tag{5}$$

In expanded form, the polynomial S_1 has $p+1$ terms. As we can see, the number of terms for S_n when $n>1$ gets large very quickly. It turns out that even in the case of S_2, the computing becomes inefficient for a small p. Experimentally, we carried out the task of computing the polynomial S_2 explicitly for various small p. The calculations are done using MAPLE 12 on a Dell laptop with a Intel Duo CPU at 2.10GHz and 4 GB RAM. The table 5.1 summaries the experiment results.

We have for each p recorded the number of monomials of S_2 in expanded form and the CPU time it took to complete the calculation.

5.2. **Recursion Formula.** Alternatively, we may derive using definition (1) a recursion formula. More explicitly, we have for $n\ge 1$

$$S_n = (X_n+Y_n)+\frac{1}{p}(X_{n-1}^p+Y_{n-1}^p-S_{n-1}^p)+\cdots+\frac{1}{p^n}(X_0^{p^n}+Y_0^{p^n}-S_0^{p^n}), \tag{6}$$

p	CPU time
541	0.078s
1223	1.217s
2011	2.044s
3181	6.224s
4409	10.390s
5279	15.927s
6133	21.185s
7001	27.238s
7499	out of memory

TABLE 2. Recursive evaluation of S_2

and

$$\begin{aligned}
P_n &= \frac{1}{p^n}\Big((X_0^{p^n} + \cdots p^n X_n)(Y_0^{p^n} + \cdots + p^n Y_n) - (P_0^{p^n} + \cdots + p^{n-1}P_{n-1}^p\Big) \\
&= (X_0^{p^n}Y_n + X_1^{p^{n-1}}Y_{n-1}^p + \cdots + X_nY_0^{p^n}) \\
&+ \frac{1}{p}(X_0^{p^n}Y_{n-1}^p + \cdots X_{n-1}^pY_0^{p^n}) \\
&\ \vdots \\
&+ \frac{1}{p^n}(X_0^{p^n}Y_0^{p^n}) - \frac{1}{p^n}P_0^{p^n} - \cdots - \frac{1}{p}P_{n-1}^p \\
&+ p(X_1^{p^{n-1}}Y_n + X_2^{p^{n-2}}(Y_{n-1}^p + pY_n) + \cdots).
\end{aligned} \tag{7}$$

Example 5.1. As we have seen in Proposition 1, the kernel of θ_n is a principal ideal generated by ξ_n. Let

$$a = (p^{1/p^n}, 0, 0, \dots, 0) = (a_0, a_1, \dots, a_n)$$
$$b = (0, -1, 0, \dots, 0) = (b_0, b_1, \dots, b_n).$$

Then using (6), for every n

$$\xi_{n+1} := \tilde{p}_{n+1} - p = [p^{1/p^n}] - p = a + b = (p^{1/p^n}, -1, 0, \dots, 0).$$

The recursion formula (6) can be coded. For instance, the reader will find in the appendix a MAPLE code for the evaluation of $S_n(\mathbf{a}, \mathbf{b})$ with input Witt vectors $\mathbf{a}, \mathbf{b}$ in a field characteristic p. Observe that in order to compute $S_j = S_j(\mathbf{a}, \mathbf{b})$, one has to compute $S_0, S_1, \dots, S_{j-1}$ a priori. Therefore the complexity of the algorithm is $\mathcal{O}(n^2)$ for computing S_n. The table 5.2 summaries the experiment we carried out on the same laptop with a Intel Duo CPU at 2.10GHz and 4GB RAM. We first randomly generate Witt vectors $\mathbf{a}, \mathbf{b}$. We then evaluated $S_2(\mathbf{a}, \mathbf{b})$ for various primes p using the MAPLE code provided in the appendix.

6. APPENDIX

The following Maple code calculates S_n using formula (6).

Algorithm 1 INPUT: prime p, positive integer n and Witt vectors $\mathbf{a}, \mathbf{b}$. Output: $S_n(\mathbf{a}, \mathbf{b})$.

```
1: S := proc(a, b, p, n)
2: if n = 1 then
3:     (a_1 + b_1) mod p
4: else
5:     add(p^(n-k-1) * (a_(n-k)^(p^k) + b_(n-k)^(p^k) - S(a, b, p, n-k)^(p^k), k = 1..n-1))/p^(n-1) + (a_n + b_n) mod p
6: end if;
```

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