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SOLVABILITY OF EXTENDED GENERAL MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we consider and study a new class of mixed variational inequality, which is called the extended general mixed variational inequality. We use the auxiliary principle technique to study the existence of a solution of the extended general mixed variational inequality. Several special cases are also discussed.

1. Introduction

Variational inequalities, which were introduced in 1960's, are being used as a powerful tool to study a wide class of problems, which arise in various branches of mathematical, financial, regional and engineering sinces, see[1-27] and the references therein. Using the technique of Noor [16-21] and Noor et al [22], one cam show that the minimum of the sum of differentiable hg-convex function and a nondifferentiable hg-convex functions can be charactrized by a class of variational inequality. Motivated by this result, we introduce a new class of mixed variational inequalities, which is calle *extended general mixed variational inequality* involving four different operators. It is known that it is very difficult to find the projection of the operator except in very special cases. To overcome this drawback, on uses the auxiliary principle technique. This technique is mainly due to Glowinski, Lions and Tremolieres [4]. This technique is more flexible and has been used to develop several numerical methods for solving the variational inequalities and the equilibrium problems. In this paper, we again use the auxiliary principle technique to study the existence of a solution of the extended general mixed variational inequalities. Since the extended general variational inequalities include various classes of variational inequalities and complementarity problems as special cases, results proved in this paper continue to hold for these problems. Results proved in this paper may be viewed as important and significant improvement of the previously known results. It is interesting to explore the applications of these extended general variational inequalities in mathematical and engineering sciences with new and novel aspects. This may lead to new research in this field.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively. Let K be a nonempty closed and convex set in H. Let $\varphi : H \longrightarrow R \cup \{\infty\}$ be a continuous function.

For given nonlinear operators $T, g, h : H \to H$, consider the problem of finding $u \in H, h(u) \in K$ such that

$$\langle Tu, g(v) - h(u) \rangle + \varphi(g(v)) - \varphi(h(u)) \geq 0, \qquad \forall v \in H : g(v) \in K. \tag{2.1}$$

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Inequality of type (2.1) is called the *extended general mixed variational inequality involving four operators*.

We now show that the minimum of the sum of differentiable nonconvex function and a class of differentiable nonconvex functions and nondifferentiable nonconvex function on the hg-convex set K in H can be characterized by extended general variational inequality (2.1). For this purpose, we recall the following well known concepts, see [2, 16-20].

Definition 2.1. Let K be any set in H. The set K is said to be hg-convex, if there exist functions $g, h : H \longrightarrow H$ such that

$$h(u) + t(g(v) - h(u)) \in K, \quad \forall u, v \in H : h(u), g(v) \in K, \quad t \in [0, 1].$$

Note that every convex set is hg-convex, but the converse is not true, see[2]. If $g = h$, then the hg-convex set K is called the g-convex set, which was introduced by Youness [26]. See also Cristescu and Lupsa [2] for its various extensions and generalization.

Definition 2.2. The function $F : K \longrightarrow H$ is said to be hg-convex on the hg-convex set K, if there exist two functions h, g such that

$$\begin{aligned} F(h(u) + t(g(v) - h(u))) \quad &\leq \quad (1-t)F(h(u)) + tF(g(v)). \\ &\forall u, v \in H : h(u), g(v) \in K, \quad t \in [0, 1]. \end{aligned}$$

Clearly every convex function is hg-convex, but the converse is not true. For $g = h$, Definition 2.2 is due to Youness [26].

It is known [16-19] that the minimum of a differentiable hg-convex function on a hg-convex set K in H can be characterized by the extended general variational inequality (2.1). One can prove a similar result for the sum of nonconvex functions on the hg-convex set.

Lemma 2.3. *Let $F : K \longrightarrow H$ be a differentiable hg-convex function on the hg-convex set K. Then $u \in H : h(u) \in K$ is the minimum of the functional $I[v]$ defind by (2.) on the hg-convex set K, if and only if, $u \in H : h(u) \in K$ satisfies the inequality*

(2.2) $$\langle F'(h(u)), g(v) - h(u) \rangle + \varphi(g(v)) - \varphi(h(u)) \geq 0, \quad \forall v \in H : g(v) \in K,$$

where $F'(u)$ is the differential of F at $u \in K$.

Lemma 2.3 implies that hg-convex programming problem can be studied via the extended general mixed variational inequality (2.1) with $Tu = F'(h(u))$.

We now list some special cases of the extended general variational inequalities.

I. If $g = h$, then Problem(2.1) is equivalent to finding $u \in H : g(u) \in K$ such that

(2.3) $$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \forall v \in H : g(v) \in K,$$

which is known as general mixed variational inequality, introduced and studied by Noor [8]. It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via general variational inequalities.

II. For $g \equiv I$, the identity operator, the extended general variational inequality (2.1) collapses to: find $u \in H : h(u) \in K$ such that

(2.4) $$\langle Tu, v - h(u) \rangle + \varphi(v) - \varphi(g(u)) \geq 0, \quad \forall v \in K,$$

which is also called the general mixed variational inequality, see Noor et al [22].

III. For $h = I$, the identity operator, the extended general variational inequality (2.1) is equivalent to finding $u \in KI$ such that

(2.5) $$\langle Tu, g(v) - u \rangle + \varphi(g(u)) - \varphi(u) \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is also called the general mixed variational inequality involving two nonlinear operators which was introduced and studied by Noor [18-20].

We would like to emphasize the fact that general variational inequalities (2.4), (2.5) and (2.6) are quite different from each other and have different applications.

VI. For $g = h = I$, the identity operator, the extended general variational inequality (2.1) is equivalent to finding $u \in K$ such that

(2.6) $$\langle Tu, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in K,$$

which is known as the classical mixed variational inequality. We would like to remark that, if $\varphi(.) =$, then the extended general variational inequality () and its variant forms are exactly the same as considered by Noor [5-21] and Stampacchia [27]. For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulations of variational inequalities, see [1-27] and the references therein. From the above discussion, it is clear that the extended general mixed variational inequalities (2.1) is most general and includes several previously known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences.

We also need the following concepts and results.

Definition 2.4. For all $u, v \in H$, an operator $T : H \to H$ is said to be:

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||.$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

Remark 2.5. It follows from the strongly monotonicity of the operator T, that

$$\alpha\|u - v\|^2 \leq \langle Tu - Tv, u - v\rangle \leq \|Tu - Tv\|\|u - v\|, \quad \forall u, v \in H,$$

which implies that

$$\|Tu - Tv\| \geq \alpha\|u - v\|, \quad \forall u, v \in H.$$

This observation enables us to define the following concept.

Definition 2.6. The operator T is said to firmly expanding if

$$\|Tu - Tv\| \geq \|u - v\|, \quad \forall u, v \in H.$$

3. Main Results

In this Section, we use the auxiliary principle technique of Glowinski, Lions and Tremolieres [4] to study the existence of a solution of the extended general mixed variational inequality (2.1).

Theorem 3.1. *Let T be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$. Let g be a strongly monotone and Lipschitz continuous operator with constants $\sigma > 0$ and $\delta > 0$ respectively. If the operator h is firmly expanding and there exists a constant $\rho > 0$ such that*

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{3.1}$$

where

$$\theta = k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \tag{3.2}$$

$$k = \sqrt{1 - 2\sigma + \delta^2}. \tag{3.3}$$

then the extended general mixed variational inequality (2.1) has a unique solution.

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in H : g(u) \in K$ satisfying the extended general mixed variational inequality (2.1), we consider the problem of finding a solution $w \in H : h(w) \in K$ such that

$$\langle \rho Tu + h(w) - g(u), g(v) - h(w)\rangle + \rho\varphi(g(v)) - \rho\varphi(h(u)) \geq 0, \quad \forall v \in H : g(v) \in K, \tag{3.4}$$

where $\rho > 0$ is a constant.

The inequality of type (3.4) is called the auxiliary extended general mixed variational inequality associated with the problem (2.1). It is clear that the relation (3.4) defines a mapping $u \longrightarrow w$. It is enough to show that the mapping $u \longrightarrow w$ defined by the relation (3.4) has a unique fixed point belonging to H satisfying the general variational inequality (2.1). Let $w_1 \neq w_2$ be two solutions of (2.13) related to $u_1, u_2 \in H$ respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$(respectively w_1) in (3.4) related to u_1 (respectively u_2), adding the resultant, we have

$$\langle h(w_1) - h(w_2), h(w_1) - h(w_2)\rangle \leq \langle g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2), h(w_1) - h(w_2)\rangle,$$

from which we have

$$\begin{aligned} \|h(w_1) - h(w_2)\| &\leq \|g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2)\| \\ &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))| + \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|. \end{aligned} \tag{3.5}$$

Since T is both strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively, it follows that

$$\begin{aligned} \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 &\leq \|u_2 - u_2\|^2 - 2\rho\langle u_1 - u_2, Tu_1 - Tu_2\rangle + \rho^2\|Tu_1 - Tu_2\|^2 \\ &\leq \left(1 - 2\rho\alpha + \rho^2\beta^2\right)\|u_1 - u_2\|^2. \end{aligned} \tag{3.6}$$

In a similar way, using the strongly monotonicity with constant $\sigma > 0$ and Lipschitz continuity with constant $\delta > 0$, we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\| \leq \sqrt{1 - 2\sigma + \delta^2}\|u_1 - u_2\|. \tag{3.7}$$

From (3.5), (5.6), (3.7) and using the fact that the operator h is firmly expanding, we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \left\{k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right\}\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned}$$

From (3.1) and (3.2), it follows that $\theta < 1$ showing that the mapping defined by (3.4) has a fixed point belonging to K, which is the solution of (2.1), the required result. □ □

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