ALBANIAN JOURNAL
OF MATHEMATICS
Volume 3, Number 3, Pages 107–116
ISSN 1930-1235: (2009)

RESOLVENT EQUATIONS METHOD FOR GENERAL VARIATIONAL INCLUSIONS

EMAN AL-SHEMAS *Mathematics Department,*
College of Basic Education
Main Campus, Shamiya, Kuwait

ABSTRACT. In this paper, we introduce a new class of variational inclusions involving three operator. Using the resolvent operator technique, we establish the equivalence between the general variational inclusions and the resolvent equations. We use this alternative equivalent formulation to suggest and analyze some iterative methods for solving the general variational inclusions. We also consider the criteria of these iterative methods under suitable conditions. Since the general variational inclusions include the variational inequalities and the related optimization problems as special cases, our results continue to hold for these problems.

1. INTRODUCTION

Variational inclusions involving three operators are useful and important extensions and generalizations of the the general variational inequalities with a wide range of applications in industry, mathematical finance, economics, decision sciences, ecology, mathematical and engineering sciences, see [1-45] and the references therein. It is well known that the projection method and its variant forms including the Wiener-Hopf equations can not be extended and modified for solving the variational inclusions. These facts and comments have motivated to use the technique of the resolvent operators. This technique can lead to the development of very efficient and robust methods since one can treat each part of the original operator independently. A useful feature of these iterative methods for solving the general variational inclusion is that the resolvent step involves the the maximal monotone operator only, while other parts facilitates the problem decomposition. Essentially using the resolvent technique, one can show that the variational inclusions are equivalent to the fixed point problems. This alternative equivalent formulation has played very crucial role in developing some very efficient methods for solving the variational inclusions and related optimization problems, see [15-38] and the references therein. Related to the variational inclusions, we have the problem of solving the resolvent equations, which are mainly due to Noor [20,21,23]. Essentially using the resolvent operator technique, we can establish the equivalence between the resolvent equations and the variational inclusions. This equivalence formulations is more general and flexible than the resolvent operator method. Resolvent equations

Received by the editors Feb 15, 2008 and, in revised form, June 15, 2008.
2000 *Mathematics Subject Classification.* Primary 49J40; Secondary 90C33.
Key words and phrases. Variational inclusion; approximation methods; resolvent equations; variational inequalities.

technique has been used to suggest and analyze several iterative methods for solving variational inclusions and related problems, see [24-27,32,34-38] and the references therein.

Motivated and inspired by the recent research activities in these areas, we introduce some new classes of variational inclusions and resolvent equations. Essentially using the resolvent operator methods, we establish the equivalence between the resolvent equations and the general variational inclusions. This alternative equivalent formulation is used to suggest some iterative methods for solving the general variational inclusions. We study the convergence criteria of the new iterative method under some mild conditions. Since the variational inclusions include the mixed variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems.

2. Basic Results

Let K be a nonempty closed and convex set in a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let $T, A, g : H \longrightarrow H$ be three nonlinear operators.

We consider the problem of finding $u \in H$ such that

$$0 \in \rho Tu + u - g + \rho A(u), \quad \rho > 0, \quad \text{a constant}, \tag{1}$$

which is known as the general variational inclusion $GVI(T, A, g)$. Problem (1) is also known as finding the zero of the sum of two (or more) monotone operators. Variational inclusions and related problems are being studied extensively by many authors and have important applications in operations research, optimization, mathematical finance, decision sciences and other several branches of pure and applied sciences, see [2-45] and the references therein.

If $A(.) \equiv \partial\varphi(.)$, where $\partial\varphi(.)$ is the subdifferential of a proper, convex and lower-semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$, then the problem (1) reduces to finding $u \in H$ such that

$$0 \in \rho Tu + u - g(u) + \rho\partial\varphi(u),$$

or equivalently, finding $u \in H$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle + \rho\varphi(g(v)) - \rho\varphi(u) \geq 0, \quad \forall v \in H. \tag{2}$$

The inequality (2) is called the general mixed variational inequality or the general variational inequality of the second kind. It has been shown that a wide class of linear and nonlinear problems arising in various branches of pure and applied sciences can be studied in the unified framework of mixed variational inequalities, see [2-38].

We note that if φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if} \quad u \in K \\ +\infty, & \text{otherwise}, \end{cases}$$

then the general mixed variational inequality (2) is equivalent to finding $u \in K$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{3}$$

which is called the general variational inequality introduced and studied by Noor [29] in connection with nonconvex functions. See also Noor and Noor [31,32] for more details.

If $g \equiv I$, the identity operator, then problem (3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{4}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [44] in 1964. For the recent trends and developments in variational inclusions and inequalities, see [2-45] and the references therein.

We also need the following well known concepts and results.

Definition 2.1 [5]. If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|.$$

Remark 2.1. It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to R\cup\{+\infty\}$ is a maximal monotone operator, we denote by

$$J_\varphi(u) = (I + \rho\partial\varphi)^{-1}(u), \quad \text{for all } u \in H,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H. In particular, the resolvent operator J_φ has the following interesting characterization.

Lemma 2.1 [5]. For a given $z \in H$, $u \in H$ satisfies the inequality

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \text{for all } v \in H,$$

if and only if

$$u = J_\varphi z,$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

This property of the resolvent operator J_φ plays an important part in developing the numerical methods for solving the mixed variational inequalities.

If the function $\varphi(.)$ is the indicator function of a closed convex set K in H, then it is well known that $J_\varphi = P_K$, the projection operator of H onto the closed convex set K.

Related to the variational inclusions, we consider the problem of solving the resolvent equations. To be more precise, let $R_A = I - gJ_A$, where J_A is the resolvent operator associated with the maximal monotone operator A, and I is the identity operator. For a given operator T, we consider the problem of finding $z \in H$ such that

$$TJ_Az + \rho^{-1}R_Az = 0, \tag{5}$$

which is called the general resolvent equation. If φ is the indicator function of a closed convex set K, then $J_\varphi = P_K$, the projection of H onto the closed convex set K. and $Q_K = I - gP_K$. In this case, resolvent equations (5) are equivalent to find $z \in H$ such that

$$TP_Kz + \rho^{-1}Q_Kz = 0,$$

which are called the Wiener-Hopf equations which were introduced and considered by Noor [29]. If $g \equiv I$, the identity operator, then obtain the original Wiener-Hopf equations introduced and studied by Shi [43]. It is well known that the Wiener-Hopf equations are equivalent to the variational inequalities. This equivalence alternative formulation is more general and flexible that the projection fixed point problem. For the formulation, numerical methods and applications of the Wiener-Hopf equations, see [14,25,28,33,37,41,43] and the references therein.

Using the definition of the resolvent operator J_A, one can easily prove the following well known result. For the sake of completeness and to convey an idea, we include its proof.

Lemma 2.2. The function $u \in H$ is a solution of the variational inclusion (1) if and only if $u \in H$ satisfies the relation

$$u = J_A[g(u) - \rho Tu],$$

where $\rho > 0$ is a constant and $J_A = (I + \rho A)^{-1}$ is the resolvent operator associated with the maximal monotone operator.

Proof. Let $u \in H$ be a solution of (1). Then

$$\begin{aligned} 0 &\in \rho Tu + u - g(u) + \rho A(u) \\ &\Longleftrightarrow -(g(u) - \rho Tu) + (I + \rho A)(u) \\ &\Longleftrightarrow u = (I + \rho A)^{-1}[g(u) - \rho Tu] = J_A[g(u) - \rho Tu], \end{aligned}$$

the required result.

□

It is clear from Lemma 2.2 that variational inclusion (1) and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Algorithm 2.1. For a given $x_0 \in H$, compute the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = (1 - a_n)x_n + a_n J_A[g(x_n) - \rho Tx_n].$$

where $a_n \in [0, 1]$ for all $n \geq 0$. Algorithm 2.1 is also known as Mann iteration.

We now discuss some special cases of Algorithm 2.1 for solving the mixed variational inequalities (2).

I. If $A(.) \equiv \varphi(.)$, the subdifferential of a proper lower-semicontinuous and convex function φ, then $J_A = J_\varphi = (I + \rho\partial\varphi)^{-1}$ and consequently Algorithm 2.1 collapses to:

Algorithm 2.2. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes

$$x_{n+1} = (1 - a_n)x_n + a_n J_\varphi[g(x_n) - \rho Tx_n],$$

where $a_n \in [0, 1]$ for all $n \geq 0$. Algorithm 2.2 is called one-step method for solving the general mixed variational inequalities (2) and appears to be a new one.

II. If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto the closed convex set K. In this case Algorithm 2.1 reduces to the following method.

Algorithm 2.3. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes

$$x_{n+1} = (1 - a_n)x_n + a_n P_K[g(x_n) - \rho T x_n],$$

where $a_n \in [0, 1]$ for all $n \geq 0$. Algorithm 2.3 is a one-step method for solving the general variational inequalities (3). Noor [29] has studied the convergence analysis of Algorithm 2.3 and its various special cases.

From the above discussion, it is clear that Algorithm 2.1 is quite general and it includes several new and previously known algorithms for solving variational inequalities and related optimization problems.

We now recall some well known concepts and notions.

Definition 2.2. A mapping $T : H \to H$ is called μ-Lipschitz if for all $x, y \in H$, there exists a constant $\beta > 0$, such that

$$||Tx - Ty|| \leq \beta ||x - y||.$$

Definition 2.4. A mapping $T : H \to H$ is called $\in K$, there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha ||x - y||^2.$$

Lemma 2.3 [46]. Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \; k \geq 0$$

with $\lambda_k \in [0, 1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k\to\infty} \delta_k = 0$.

3. Main Results

In this section, we use the general resolvent equation technique to suggest and analyze some iterative methods for solving the general variational inclusion (1). For this purpose, we need the following result, which can be proved by using Lemma 2.2. However, for the sake of completeness and to convey an idea, we include its proof.

Lemma 3.1. The element $u \in H$ is a solution of (1), if and only if, $z \in H$ satisfies the resolvent equations (5), where

$$\begin{aligned} u &= J_A z, \\ z &= g(u) - \rho T u. \end{aligned}$$

Proof. Let $u \in H$ be a solution of (1). Then, from Lemma 2.3, we have

$$u = J_A[g(u) - \rho T u]. \tag{6}$$

$$z = g(u) - \rho T u. \tag{7}$$

Form (6) and (7), we have

$$\begin{aligned} u &= J_A z, \\ z &= g(u) - \rho T u, \end{aligned}$$

from which, we have

$$z = gJ_A z - \rho T J_A z,$$

which is exactly the resolvent equation (5), the required result.

□

From Lemma 3.1, it follows that the variational inclusion (1) and the resolvent equation (5) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving variational inclusions and related optimization problems, see [3-16] and the references therein.

Using Lemma 3.1, we now suggest and analyze a new iterative algorithm for solving the general variational inclusion (1) and this is the main motivation of this paper.

Algorithm 3.1. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$u_n = (1 - a_n)z_n + a_n J_A z_n \tag{8}$$

$$z_{n+1} = (1 - a_n)z_n + a_n\{g(u_n) - \rho T u_n\} \tag{9}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

If $g \equiv I$, the identity operator, then Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= (1 - a_n)z_n + a_n J_A z_n \\ z_{n+1} &= (1 - a_n)z_n + a_n\{u_n - \rho T u_n\}. \end{aligned}$$

For $a_n = 1$, Algorithm 3.1 collapses to the following iterative method for solving variational inclusions (1).

Algorithm 3.3. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= J_A z_n \\ z_{n+1} &= g(u_n) - \rho T u_n. \end{aligned}$$

If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto the closed convex set K. In this case Algorithm 2.1 reduces to the following method for solving general variational inequalities (3). These iterative methods are mainly due to Noor [29].

Algorithm 3.4. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= (1 - a_n)z_n + a_n P_K z_n \\ z_{n+1} &= (1 - a_n)z_n + a_n\{g(u_n) - \rho T u_n\} \end{aligned}$$

where $a_n \in [0, 1]$ for all $n \geq 0$.

In brief, Algorithm 3.1 is quite general and includes several iterative methods for solving mixed variational inequalities and related optimization problems as special cases.

We now study those conditions under which the approximate solution obtained from Algorithm 3.1 to a solution of the variational inclusion (1).

Theorem 3.1. Let T be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$ and let g be a strongly monotone with constant

$\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. If

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \mu^2(2k - k^2)}}{\beta^2}, \tag{10}$$

$$\alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{11}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2},$$

and $a_n \in [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, then the approximate solution z_{n+1} obtained from Algorithm 3.1 converges to a solution z of the general resolvent equation (5).

Proof. Let $z^* \in H$ be a solution of (5). Then, from Lemma 3.1, we have

$$u^* = (1 - a_n)z^* + a_n J_A z^* \tag{12}$$

$$z^* = (1 - a_n)z^* + a_n\{g(u^*) - \rho T u^*\} \tag{13}$$

where $a_n \in [0,1]$ and $u^* \in H$ is a solution of (1). To prove the result, we need first to evaluate $||z_{n+1} - z^*||$ for all $n \geq 0$. From (9) and (14), we have

$$\begin{aligned} ||z_{n+1} - z^*|| &= ||(1-a_n)z_n + a_n\{g(u_n) - \rho T u_n\} - (1-a_n)z^* - a_n\{g(u^*) - \rho T u^*\}|| \\ &\leq (1-a_n)||z_n - z^*|| + a_n||g(u_n) - g(u^*) - \rho(Tu_n - Tu^*)|| \\ &\leq (1-a_n)\|z_n - z^*\| + a_n\|u_n - u^* - (g(u_n) - g(u^*))\| \\ (14) \qquad & a_n\|u_n - u^* - \rho(Tu_n - Tu^*)\|. \end{aligned}$$

From the strongly monotonicity and Lipschitz continuity of the operator T, we have

$$\begin{aligned} & ||u_n - u^* - \rho(Tu_n - Tu^*)||^2 \\ = \; & ||u_n - u^*||^2 - 2\rho\langle Tu_n - Tu^*, u_n - u^*\rangle + \rho^2||Tu_n - Tu^*||^2 \\ \leq \; & ||u_n - u^*||^2 - 2\rho\|u_n - u^*\|^2 + \beta^2\rho^2||u_n - u^*||^2] \\ = \; & \theta_1^2\|u_n - u^*\|^2, \end{aligned} \tag{15}$$

where

$$\theta_1 = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}. \tag{16}$$

In a similar way, we have

$$\begin{aligned} \|u_n - u^* - (g(u_n) - g(u^*))\| &\leq [1 - 2\sigma + \delta^2]\|u_n - u^*\|^2 \\ &= k^2\|u_n - u^*\|^2, \end{aligned} \tag{17}$$

where k is defined by (12).

Combining (15), (16) and (18), we have

$$\|z_{n+1} - z^*\| \leq (1 - a_n)\|z_n - z^*\| + a_n\theta\|u_n - u^*\|, \tag{18}$$

where $\theta = \theta_1 + k$.

From (8), (13) and the nonexpansivity of the operators J_A, we have

$$\begin{aligned} \|u_n - u^*\| &\leq (1 - a_n)\|z_n - z^*\| + a_n\|J_A z_n - J_A z^*\| \\ &\leq (1 - a_n)\|z_n - z^*\| + a_n\|z_n - z^*\| \\ &= \|z_n - z^*\|. \end{aligned} \tag{19}$$

From (19) and (20), we obtain that

$$\begin{aligned} ||z_{n+1} - z^*|| &\leq (1 - a_n)||z_n - z^*|| + a_n\theta\|z_n - z^*\| \\ &= [1 - a_n(1 - \theta)]||z_n - z^*||, \end{aligned}$$

and hence by Lemma 2.3, $\lim_{n\to\infty} ||z_n - z^*|| = 0$, completing the proof.

□

4. Computational Aspects

In this paper, we have shown that the general variational inclusions are equivalent to a new class of resolvent equations . This equivalence is used to suggest and analyze an iterative method for solving the general variational inclusions. It is worth mentioning that a special case of Algorithm 3.1 has been used by Pitonyak, Shi and Schiller [41] to find the numerical solutions of the obstacle problems. The results are encouraging and perform better than other methods. Noor, Wang and Xiu [39] has developed a very efficient and robust method using the technique of the Wiener-Hopf equations for solving the variational inequalities. It is interesting to use the technique and idea of this paper to develop other new iterative methods for solving the variational inequalities involving the nonexpansive operators. This is another direction for future work.

References

(1) W. F. Ames, Numerical Methods for Partial Differential Equations, Third Edition, Academic Press, New York, 1992.

(2) C. Baiocchi and A. Capelo, Variational and Quasi-Variational Inequalities, J. Wiley and Sons, New York, London, 1984.

(3) A. Bnouhachem, M. Aslam Noor and T. M. Rassias, Three-step iterative algorithms for mixed variational inequalities. Appl. Math. Comput. **183**(2006), 436-446.

(4) A. Bnouhachem and M. Aslam Noor, Numerical comparison between prediction-correction methods for general variational inequalities, Appl. Math. Comput. **186**(2007), 496-505.

(5) H. Brezis, Operateurs Maximaux Monotone et Semigroups de Contractions dan Espaces de Hilbert, North-Holland, Amsterdam, 1973.

(6) R.W. Cottle, F. Giannessi and J.L. Lions, Variational Inequalities and Complementarity Problems: Theory and Applications, J. Wiley and Sons, New York, 1980.

(7) J. Eckstein and B. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, Math. Prog. 55 (1992) 293-318.

(8) M. Fukushima, The primal Douglas-Rachford splitting algorithm for a class of monotone operators with applications to the traffic equilibrium problem, Math. Prog. 72 (1996) 1-15.

(9) F. Giannessi and A. Maugeri, Variational Inequalities and Network Equilibrium Problems, Plenum Press, New York, 1995.

(10) R. Glowinski, J.L. Lions and R. Trémolières, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

(11) S. Haubruge, V. H. Nguyen and J. J. Strodiot, Convergence analysis and applications of the Glowinski-Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators, J. Optim. Theory Appl. 97 (1998) 645-673.

(12) P. L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, SIAM J. Numer. Anal. 16 (1979) 69-76.

(13) A. Moudafi and M. Thera, Finding a zero of the sum of two maximal monotone operators, J. Optim. Theory Appl. 97 (1997) 425-448.
(14) A. Moudafi and M. Aslam Noor, Sensitivity analysis of variational inclusions by the Wiener-Hopf equations technique, J. Appl. Math. Stochastic Anal. 12 (1999) 223-232.
(15) M. Aslam Noor, Some algorithms for general monotone mixed variational inequalities, Math. Computer Modelling 29(7)(1999) 1-9.
(16) M. Aslam Noor, Algorithms for general monotone mixed variational inequalities, J. Math. Anal. Appl. 229(1999) 330-343.
(17) M. Aslam Noor, An extraresolvent method for monotone mixed variational inequalities, Math. Computer Modelling 29(1999) 95-100.
(18) M. Aslam Noor, Some recent advances in variational inequalities, Part I, basic concepts, New Zealand J. Math. 26 (1997) 53-80.
(19) M. Aslam Noor, Some recent advances in variational inequalities, Part II, other concepts, New Zealand J. Math. 26 (1997) 229-255.
(20) M. Aslam Noor, Generalized set-valued variational inclusions and resolvent equations, J. Math. Anal. Appl. 228 (1998) 206-220.
(21) M. Aslam Noor, Set-valued mixed quasi variational inequalities and implicit resolvent equations, Math. Computer Modelling, 29(1999), 1-11.
(22) M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217-229.
(23) M. Aslam Noor, Equivalence of variational inclusions with resolvent equations, Nonl. Anal., **42**(2000), 963-970.
(24) M. Aslam Noor, Three-step iterative algorithms for multivalued quasi variational inclusions, J. Math. Anal. Appl. 255(2001), 589-604.
(25) M. Aslam Noor, A Wiener-Hopf dynamical system for variational inequalities, New Zealand J. Math. **31**(2002), 173-182.
(26) M. Aslam Noor, Resolvent dynamical systems for mixed variational inequalities, Korean J. Comput. Appl. Math., **9**(2002), 15-26.
(27) M. Aslam Noor, Fundamentals of mixed quasi variational inequalities, Inter. J. Pure Appl. Math. 15(2004), 137-258.
(28) M.Aslam Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199-277.
(29) M. Aslam Noor, Differentiable nonconvex functions and general variational inequalities, Appl. Math. Comput., **199**(2008), 623-630.
(30) M. Aslam Noor and A. Bnouhachem, On an iterative algorithm for general variational inequalities, Appl. Math. Comput., **185**(2007), 155-168.
(31) M. Aslam Noor and K. Inayat Noor, On sensitivity analysis of general variational inequalities, Math. Comm. **13**(2008), 75-83.
(32) M. Aslam Noor and K. Inayat Noor, Projection algorithms for solving a system of general variational inequalities, Nonl. Anal. (2008).
(33) M. Aslam Noor and K. Inayat Noor, Three-step iterative methods for general variational inclusions in L^p-spaces, J. Appl. Math. Computing, **27**(2008), 281-291.
(34) M. Aslam Noor and K. Inayat Noor, Multivalued variational inequalities and resolvent equations, Math. Computer Modelling, **26**
(35) M. Aslam Noor and K. Inayat Noor, Sensitivity analysis for quasi variational inclusions, J. Math. Anal. Appl. 236(1999) 290-299.

(36) M. Aslam Noor and Th. M. Rassias, Resolvent equations for set-valued variational inequalities, Nonl. Anal., **42**(2000), 71-83.
(37) M. Aslam Noor, K. Inayat Noor and Th. M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math. 47 (1993) 285-312.
(38) M. Aslam Noor, K. Inayat Noor and Th. M. Rassias, Set-valued resolvent equations and mixed variational inequalities, J. Math. Anal. Appl., 220 (1998) 741- 759.
(39) M. Aslam Noor, Y. J. Wang and N. Xiu, Some new projection methods for variational inequalities, Appl. Math. Computation, **137** (2003), 423-435.
(40) M. Patriksson, Nonlinear Programming and Variational Inequalities: A Unified Approach, Kluwer Academic Publishers, Dordrecht, 1998.
(41) A. Pitonyok, P. Shi and M. Shillor, On an iterative method for variational inequalities, Numer. Math. , **58**(1990), 231-242.
(42) R. T. Rockafellar, Monotone operators and the proximal point algorithms, SIAM J. Control Optim. 14 (1976) 877-898.
(43) P. Shi, Equivalence of Wiener-Hopf equations with variational inequalities, Proc. Amer. Math. Soc. , **111**(1991), 339- 346.
(44) G. Stampacchia, Formes bilineaires coercivities sur les ensembles convexes, C.R. Acad. Sci. Paris, 258 (1964) 4413-4416.
(45) W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2) (2003) 417-428.
(46) X.L. Weng, Fixed point iteration for local strictly pseudocontractive mappings, Proc. Amer. Math. Soc. 113 (1991) 727-731.