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A NOTE ON LIGHT INDUCED MAPPINGS

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Abstract. Let a mapping $f : X \to Y$ between continua X and Y be given. We shall prove: a) if the induced mapping $2^f : 2^X \to 2^Y$ is light, then $w(X) = w(Y)$. In particular, if Y is metrizable, then X is metrizable, b) if the induced mapping $C(f) : C(X) \to C(Y)$ is light and X is a D-continuum, then $w(X) = w(Y)$.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by $card(A)$.

Let X be a space. We define its hyperspaces as the following sets:

$$2^X = \{F \subseteq X : F \text{ is closed and nonempty}\},$$
$$C(X) = \{F \in 2^X : F \text{ is connected}\},$$
$$X(n) = \{F \in 2^X : F \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}.$$

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n\rangle = \left\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, ..., U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ $\}$, and $C(X)$ is a subspace of 2^X.

Given a mapping $f : X \to Y$ between continua X and Y, we let $2^f : 2^X \to 2^Y$ to denote the corresponding *induced* mapping defined by $2^f(F) = f(F)$ for $F \in 2^X$. By [8, 5.10] 2^f is continuous and $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

A continuous mapping $f : X \to Y$ is *light (zero-dimensional)* if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [3, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one

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($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

In this paper we shall prove that the lightness of $C(f)$ or 2^f implies the equality of the weights of continua.

It is clear that the lightness of $2^f : 2^X \to 2^Y$ implies the lightness of $C(f) : C(X) \to C(Y)$, but not conversely. The following result is known.

THEOREM 1.1. [1, Theorem 5.4]. *Let continua X and Y and a mapping $f : X \to Y$ be given. Consider the following conditions:*

- **(a):** *$C(f) : C(X) \to C(Y)$ is light*;
- **(b):** *for every two continua $P, Q \in C(X) \setminus X(1)$ with $P \cap Q = \emptyset$ the inequality $f(P) \setminus f(Q) \neq \emptyset$ holds*;
- **(c):** *$2^f : 2^X \to 2^Y$ is light.*

Then (c) *implies* (b)*, and* (b) *implies* (a)*. Consequently,* (c) *implies* (a)*. The other implications do not hold.*

A family $\mathcal{N} = \{M_s : s \in S\}$ of a subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [3, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form $card(\mathcal{N})$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $nw(X)$.

Remark. It is known that for every compact space X we have $nw(X) = w(X)$ [3, p. 171, Theorem 3.1.19].

2. Lightness of $2^f : 2^X \to 2^Y$ implies $w(X) = w(Y)$

In this section we shall prove the following result.

THEOREM 2.1. *Let a mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $C(f) : C(X) \to C(Y)$ satisfies the condition that for every two continua $C, D \in C(X) \setminus X(1)$ with $C \cap D = \emptyset$ the inequality $f(C) \setminus f(D) \neq \emptyset$ holds, then $w(X) = w(Y)$.*

Proof. It is obvious that $w(Y) \leq w(X)$ [3, p. 171, Theorem 3.1.22]. Let us prove that $w(Y) \geq w(X)$. The proof is broken into several steps.

Step 1. *$C(f) : C(X) \to C(Y)$ is one-to-one on $C(X) \setminus X(1)$. Moreover, $C(f)$ is a homeomorphism of $C(X) \setminus X(1)$ onto $C(f)(C(X) \setminus X(1))$.* Suppose that $C(f)$ is not one-to-one. Then there exists a continuum F in Y and two continua C, D in X such that $f(C) = f(D) = F$. We have to consider the following cases.

a) $C \cap D = \emptyset$. Now $f(C) \setminus f(D) = \emptyset$. This is impossible because of the condition (b) of Theorem 1.1.

b) *$C \subset D$ or $D \subset C$*. Suppose that $C \subset D$. The proof is similar if $D \subset C$. By [7, p. 1209, Theorem] we infer that there exists an order arc $L \subset C(X)$ from C to D. If a subcontinuum E of X is in L then $f(E) = F$ since $f(C) = f(D) = F$. This means that $C(f)(L) = F$, i.e., $(C(f))^{-1}(F)$ contains a non-degenerate continuum L. This is impossible since $C(f)$ is light (see Theorem 1.1).

c) $C \cap D \neq \emptyset$ and $C \setminus D \neq \emptyset, C \setminus D \neq \emptyset$. Let $C \cup D = K$. It is clear $f(K) = f(C) = f(D) = F$.

Moreover $C \subset K$. By b) this is impossible.

Hence, the proof of Step 1 is completed.

We infer that $C(f)^{-1}[Y\setminus Y(1)] = C(X)\setminus X(1)$. It follows that the restriction $P = C(f)|(C(X)\setminus X(1))$ is one-to-one and closed [3, p. 95, Proposition 2.1.4]. From $C(f)^{-1}[Y\setminus Y(1)] = C(X)\setminus X(1)$ it follows that P is surjective. Hence, P is a homeomorphism.

Step 2. $w(C(X)\setminus X(1)) \leq w(Y)$. Now we have

$$w(C(X)\setminus X(1)) = w(C(f)|(C(X)\setminus X(1))) \leq w(C(Y)\setminus Y(1)) \leq w(2^X) = w(Y)$$

since $w(2^X) = w(Y)$ [3, p. 306, Problem 3.12.26 (a)].

Step 3. $w(X) \leq w(Y)$. Let $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ be a base of $C(X)\setminus X(1)$. For each B_α let $C_\alpha = \{x \in X : x \in B,\ B \in B_\alpha\}$, i.e., the union of all continua B contained in B_α.

Claim 1. *The family $\{C_\alpha : \alpha \in A\}$ is a network of X.* Let X be a point of X and let U be an open subsets of X such that $x \in U$. There exists and open set V such that $x \in V \subset \mathrm{Cl}V \subset U$. Let K be a component of $\mathrm{Cl}V$ containing x. By Boundary Bumping Theorem [10, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X)\setminus X(1)$. Now, $\langle U\rangle \cap (C(X)\setminus X(1))$ is a neighbourhood of K in $C(X)\setminus X(1)$. It follows that there exists a $B_\alpha \in \mathcal{B}$ such that $K \in B_\alpha \subset \langle U\rangle \cap (C(X)\setminus X(1))$. It is clear that $C_\alpha \subset U$ and $x \in C_\alpha$ since $x \in K$. Hence, the family $\{C_\alpha : \alpha \in A\}$ is a network of X.

Claim 2. $nw(X) = w(C(X)\setminus X(1))$. Apply Claim 1. Moreover, by Remark at the end of Introduction, it follows that $w(X) = w(C(X)\setminus X(1))$. Finally, from Step 2 we obtain $w(X) \leq w(Y)$. □

The condition that for every two continua $C, D \in C(X)\setminus X(1)$ with $C \cap D = \emptyset$ the inequality $f(C)\setminus f(D) \neq \emptyset$ holds, used in the proof of Theorem 2.1, is actually condition (b) of Theorem 1.1. Hence, Theorems 2.1 and 1.1 imply the following result.

COROLLARY 2.2. *Let a mapping $f : X \to Y$ between continua X and Y be given. If the induced mapping $2^f : 2^X \to 2^Y$ is light, then $w(X) = w(Y)$.*

3. The lightness of $C(f) : C(X) \to C(Y)$ implies $w(X) = w(Y)$ for D-continua

A continuum X is called a *D-continuum* if for every pair C, D of its disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D)\setminus E \neq \emptyset$.

The class of D-continua is very large. Each arcwise connected continuum and each locally connected continuum is a D-continuum. Moreover, each aposyndetic continuum is a D-continuum.

In the proof of Theorem 2.1 only the subspace $C(X)$ of 2^X and the lightness of the mapping $C(f) : C(X) \to C(Y)$ is used. If X is a D-continuum then the lightness of the mapping $2^f : 2^X \to 2^Y$ can be omitted. In this case the condition (b) of Theorem 1.1 is replaced by the assumption that X is a D-continuum. This assumption enables to prove the step of the new proof similar to proof of Step1 of the proof of Theorem 2.1. The remaining part of the proof of Theorem 2.1 works in a new situation. Hence we have the following result.

THEOREM 3.1. *Let X be a D-continuum and let $f : X \to Y$ be a mapping such that $C(f) : C(X) \to C(Y)$ is light. Then $w(X) = w(Y)$.*

4. The lightness of $C(f) : C(X) \to C(Y)$ and Whitney map for $C(X)$

The lightness of the mapping $C(f) : C(X) \to C(Y)$ play important role in theory of continua, in particular, in the study of Whitney maps.

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [9, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([9, pp. 24-26], [4, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [2]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [2].

The following theorem explains the role of light mappings in the study of Whitney maps for continua.

THEOREM 4.1. *A continuum X admits a Whitney map for $C(X)$ if and only if there exists a light mapping $f : C(X) \to Y$ onto a metric continuum Y.*

Proof. **a)** Suppose that X admits a Whitney map for $C(X)$. By [5, Theorem 1.8] there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim \mathbf{X}$. Now we have a σ-directed inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ of metric continua such that $C(X)$ is homeomorphic to $\lim C(\mathbf{X})$. From [6, Corollary 3.2.] it follows that the projections $C(p_b) : C(\lim \mathbf{X}) \to C(X_b)$ are light for every $b \in B$, where B is cofinal subset of A. Hence, each $C(p_b)$ is a required light mapping onto a metric continuum $Y = C(p_b)(\lim C(\mathbf{X}))$.

b) Suppose now that there exist a light mapping $f : C(X) \to Y$ onto a metric continuum Y. Consider, as in a), a σ-directed inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ of metric continua such that $C(X)$ is homeomorphic to $\lim C(\mathbf{X})$. There is a subset B cofinal in A such that there exists a mapping $f_b : C(p_b)(\lim C(X)) \to Y$ such that $f = f_b C(p_b)$ since $C(\mathbf{X})$ is σ-directed and Y is metric. Let us prove that $C(p_b)$ is light. Suppose that it is not light. Then there exist a point $z \in C(p_b)(\lim C(X))$ such that $C(p_b)^{-1}(z)$ contains a continuum Z. It follows that $f^{-1}(f_b(z))$ contains Z since $C(p_b)^{-1}(z) \subset f^{-1}(f_b(z))$. This is impossible since f is light. Hence, $C(p_b)$ is light. By [6, Corollary 3.2.] we infer that X admits a Whitney map for $C(X)$. The proof of Theorem is completed. □

The notion of an irreducible mapping was introduced by Whyburn [11, p. 162]. If X is a continuum, a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f. Some theorems for the case when X is semi-locally-connected are given in [11, p. 163].

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [9, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

Proposition 1. [9, p. 204, (1.212.3)]. *If $f : X \to Y$ is a mapping between continua, then $C(f) : C(X) \to C(Y)$ is light if and only if f is hereditarily irreducible.*

Proposition 1 and Theorem 4.1 imply the following result.

COROLLARY 4.2. *A continuum X admits a Whitney map for $C(X)$ if and only if there exists a hereditarily irreducible mapping $f : X \to Y$ onto a metric continuum Y.*

COROLLARY 4.3. *Let $f : C(Y) \to C(Y)$ be a light mapping. If Y admits a Whitney map for $C(Y)$, then X admits a Whitney map for $C(X)$.*

Proof. Consider a σ-directed inverse system $C(\mathbf{Y}) = \{C(Y_a), C(q_{ab}), A\}$ of metric continua such that $C(Y)$ is homeomorphic to $\lim C(\mathbf{Y})$. There is a subset B cofinal in A such that the projections $C(q_b)$ are light. Now, the composition $C(q_b)f : C(X) \to Y_b$ is light since the composition of light mappings is light. By Theorem 4.1 X admits a Whitney map for $C(X)$. □

We close this Section with theorem which shows that the existence of Whitney map for $C(X)$ is equivalent to the metrizability of X.

THEOREM 4.4. *A D-continuum X admits a Whitney map for $C(X)$ if and only if X is metrizable. In particular, if X is either an arcwise connected, or locally connected or aposyndetic continuum, then X admits a Whitney map for $C(X)$ if and only if X is metrizable.*

Proof. By Theorem 4.1 a continuum X admits a Whitney map for $C(X)$ if and only if there exists a light mapping $f : C(X) \to Y$ onto a metric continuum Y. Moreover, from Theorem 3.1 it follows that $w(X) = w(Y)$. Hence, X is metrizable since $w(Y) = \aleph_0$. IF X is either an arcwise connected continuum or a locally connected continuum or aposyndetic continuum, then X is a D-continuum and, consequently, metrizable. □

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