ALBANIAN JOURNAL
OF MATHEMATICS
Volume 2, Number 4, Pages 265–275
ISSN 1930-1235: (2008)

SOME ITERATIVE ALGORITHMS FOR EXTENDED GENERAL VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and analyze a new class of three-step projection iterative methods for solving the extended general variational inequalities, which are obtained using the updating technique of the solution in conjunction with projection technique. We also consider the convergence criteria of these new iterative methods under some mild conditions. Since the extended general variational inequalities include the general variational inequalities and other related optimization problems as special cases, results obtained in this paper continue to hold for these problems. Results obtained in this paper may be viewed as a refinement and improvement of the known results.

1. Introduction

Extended general variational inequality, which was introduced and studied by Noor[20-23,26], is an important and useful generalization of variational inequalities. It has been shown that the extended general variational inequalities provide us with a unified, simple and natural framework to study a wide class of problems including unilateral, moving, obstacle, free, equilibrium and economics arising in various areas of pure and applied sciences. Noor [20,21,26] has shown that the minimum of a differentiable nonconvex functions on the nonconvex sets can be characterized by the extended general variational inequalities. It has been shown in [21,22, 26] that the extended general variational inequalities are equivalent to the fixed point problems. This equivalent alternative equivalent has been used to discuss the uniqueness of the solution as well as to suggest some iterative methods for solving the extended general variational inequalities, see Noor [20-23,26] and the references therein.

Noor[13,15] has suggested and analyzed some three steps forward-backward splitting algorithms for solving variational inequalities by using the updating techniques of the solution and auxiliary principle. These forward-backward splitting algorithms

Received by the editors October 20, 2008 and, in revised form, October 25, 2008.
2000 *Mathematics Subject Classification.* Primary 49J40; Secondary 90C33.
Key words and phrases. Variational inequalities; nonconvex functions; fixed-point problem, convergence.

are similar to that of the θ-scheme of Glowinski and Le Tellec[6], which they suggested by using the Lagrangian technique. It is known that three step schemes are versatile and efficient, see [3, 6]. These three-step schemes are natural generalization of the splitting methods for solving partial differential equations. We would like to point out that the iterative methods serve to solve a variety of problems which are either of the feasibility or the optimization type. This class of algorithms has witnessed great progress in recent years. Apart from theoretical interest, the main advantage of these iterative methods, which make them use of in real world problems, is computational. These methods have the ability to handle large-size problems of dimensions beyond which other methods cease to be efficient. In short, the field if the iterative type methods is vast, see [1-35] and the references therein.

Inspired and motivated by the usefulness and applications of the splitting type methods, we suggest and analyze a new class of three step approximation schemes for solving the extended general variational inequalities and related problems. These new methods include the Mann and Ishikawa iterative schemes and modified forward-backward splitting methods of Noor[13,15] as special cases. We also study the convergence criteria of these new methods under some mild conditions. Our results represent an improvement and refinement of the previously known results in these fields. We hope that the interested reader may be able to explore the novel and innovative applications of these extended general variational inequalities in other branches of pure and applied sciences. This is may open other window of future research in this growing and dynamic field.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $< \cdot, \cdot >$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T, g, h : H \to H$, consider the problem of finding $u \in H, h(u) \in K$ such that

$$\langle Tu, g(v) - h(u)\rangle \geq 0, \qquad \forall v \in H : g(v) \in K. \tag{1}$$

Inequality of type (1) is called the *extended general variational inequality involving three opertors.* Noor [20-23,26] has shown that the minimum of a class of differentiable nonconvex functions on hg-convex set K in H can be characterized by extended general variational inequality (1).

For this purpose, we recall the following well known concepts, see [3].

Definition 2.1[3,21]. Let K be any set in H. The set K is said to be hg-convex, if there exist a function $g, h : H \longrightarrow H$ such that

$$h(u) + t(g(v) - h(u)) \in K, \quad \forall u, v \in H : h(u), g(v) \in K, \quad t \in [0, 1].$$

Note that every convex set is hg-convex, but the converse is not true, see[22]. If $g = h$, then the hg-convex set K is called the g-convex set, which was introduced by Youness [34].

From now onward, we assume that K is a hg-convex set, unless otherwise specified.

Definition 2.2[22,26]. The function $F : K \longrightarrow H$ is said to be hg-convex, iff, there exists two functions h, g such that

$$F(h(u) + t(g(v) - h(u))) \leq (1 - t)F(h(u)) + tF(g(v))$$

for all $u, v \in H : h(u), g(v) \in K, \quad t \in [0,1]$. Clearly every convex function is hg-convex, but the converse is not true. For $g = h$, definition 2.2 is due to Youness [34].

We now show that the minimum of a differentiable hg-convex function on the hg-convex set K in H can be characterized by the extended general variational inequality (1). This result is due to Noor [21,22,26]. We include all the details for the sake of completeness and to convey the main idea.

Lemma 2.1[22,26]. Let $F : K \longrightarrow H$ be a differentiable hg-convex function. Then $u \in H : h(u) \in K$ is the minimum of hg-convex function F on K if and only if $u \in H : h(u) \in K$ satisfies the inequality

$$\langle F'(h(u)), g(v) - h(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2}$$

where $F'(u)$ is the differential of F at $h(u) \in K$.

Proof. Let $u \in H : h(u) \in K$ be a minimum of hg-convex function F on K. Then

$$F(h(u)) \leq F(g(v)), \quad \forall v \in H : g(v) \in K. \tag{3}$$

Since K is a hg-convex set, so, for all $u, v \in H : h(u), g(v) \in K, t \in [0,1], g(v_t) = h(u) + t(g(v) - h(u)) \in K$. Setting $g(v) = g(v_t)$ in (3), we have

$$F(h(u)) \leq F(h(u) + t(g(v) - h(u)).$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'(h(u)), g(v) - h(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is the required result(2).

Conversely, let $u \in H : h(u) \in K$ satisfy the inequality (2). Since F is a hg-convex function, $\quad \forall u, v \in H : h(u), g(v) \in K, \quad t \in [0,1], \quad h(u) + t(g(v) - h(u)) \in K$ and

$$F(h(u) + t(g(v) - h(u))) \leq (1-t)F(h(u)) + tF(g(v)),$$

which implies that

$$F(g(v)) - F(h(u)) \geq \frac{F(h(u) + t(g(v) - g(u))) - F(h(u))}{t}.$$

Letting $t \longrightarrow 0$, we have

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle \geq 0, \quad \text{using (2)},$$

which implies that

$$F(h(u)) \leq F(g(v)), \quad \forall v \in H : g(v) \in K$$

showing that $u \in K$ is the minimum of F on K in H. □

Lemma 2.1 implies that hg-convex programming problem can be studied via the extended general variational inequality (1) with $Tu = F'(h(u))$. In a similar way, one can show that the extended general variational inequality is the Fritz-John condition of the inequality constrained optimization problem.

We would like to emphasize that problem (1) is equivalent to finding $u \in H : h(u) \in K$ such that

$$\langle \rho Tu + h(u) - g(u), g(v) - h(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{4}$$

This equivalent formulation is also useful from the applications point of view.

We now list some special cases of the extended general variational inequalities.

I. I f $g = h$, then Problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \qquad \forall v \in H : g(v) \in K, \tag{5}$$

which is known as general variational inequality, introduced and studied by Noor [7] in 1988. It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via general variational inequalities, see [8-10,13-15, 18-23] and the references therein.

II. For $h = I$, the identity operator, then problem (1) ie equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{6}$$

which is also called the general variational inequalities, introduced and studied by Noor [24].

III. For $g \equiv I$, the identity operator, the extended general variational inequality (1) collapses to: find $u \in H : h(u) \in K$ such that

$$\langle Tu, v - h(u)\rangle \geq 0, \quad \forall v \in K, \tag{7}$$

which is also called the general variational inequality, see Noor [8].

IV. For $g = h = I$, the identity operator, the extended general variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \forall v \in K, \tag{8}$$

which is known as the classical variational inequality and was introduced in 1964 by Stampacchia [33]. For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulations of variational inequalities, see [1-35] and the references therein.

V. If $K^* = \{u \in H; \langle u, v\rangle \geq 0, \quad \forall v \in K\}$ is a polar(dual) convex cone of a closed convex cone K in H, then problem (1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^*, \quad \langle g(u), Tu\rangle = 0, \tag{9}$$

which is known as the general complementarity problem, see[15]. If $g = I$, the identity operator, then problem (9) is called the generalized complementarity problem. For $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem (9) is called the quasi(implicit) complementarity problem, see [15,30] and the references therein.

From the above discussion, it is clear that the extended general variational inequalities (1) is most general and includes several previously known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences, see the references.

We also need the following concepts and results.

Lemma 2.2. Let K be a closed convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the projection of H onto the closed convex set K in H.

It is well known that the projection operator P_K is a nonexpansive operator, that is,

$$\|P_K u - P_K v\| \leq \|u - v\|, \quad \forall u, v \in H.$$

Definition 2.3. An operator $T : H \rightarrow H$ is said to be:

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha ||u - v||^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta ||u - v||, \quad \forall u, v \in H..$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

Definition 2.4 A mapping $T : H \rightarrow H$ is called relaxed cocoercive, if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma ||Tx - Ty||^2, \quad \forall x, y \in H.$$

Definition 2.5. A mapping $T : H \rightarrow H$ is called relaxed co-coercive strongly monotone, if there exist constants $\gamma > 0, \alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma ||Tx - Ty||^2 + \alpha ||x - y||^2 \quad \forall x, y \in H.$$

It is clear that, if T is Lipschitz continuous, then the relaxed co-coercive strongly monotone operator is strongly monotone with a constant $(\alpha - \gamma\beta^2)$. However, the converse is not true. Thus it is obvious that class of relaxed cocoercive strongly monotone operator is more general than the class of strongly monotone operators.

3. Main Results

In this section, we suggest and analyze some new approximation schemes for solving the extended general variational inequality (4). One can prove that the extended general variational inequality (4) is equivalent to the fixed point problem by invoking Lemma 2.2.

Lemma 3.1[22]. The function $u \in H : h(u) \in K$ is a solution of the extended general variational inequality (4) if and only if $u \in H : h(u) \in K$ satisfies the relation

$$h(u) = P_K[g(u) - \rho Tu], \tag{10}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the extended general variational inequality (4) is equivalent to the fixed point problem (10). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. Zhao and Sun [35] used the concept of the exceptional family to study the existence of a solution of the nonlinear projection equations (10).

We rewrite the the relation (10) in the following form

$$F(u) = u - h(u) + P_K[g(u) - \rho Tu], \tag{11}$$

which is used to study the existence of a solution of the extended general variational inequalities (4).

We now study those conditions under which the extended general variational inequality (4) has a unique solution and this is the main motivation of our next result.

Theorem 3.1. Let the operators $T, g, h : H \longrightarrow H$ be relaxed co-coercive strongly monotone with constants $(\gamma > 0, \alpha > 0)$, $(\gamma_1 > 0, \sigma > 0)$, $(\gamma_2 > 0, \mu > 0)$ and Lipschitz continuous with constants with $\beta > 0$, $\delta > 0$, $\eta > 0$ respectively. If

$$|\rho - \frac{(\alpha - \gamma\beta^2)}{\beta^2}| < \frac{\sqrt{(\alpha - \gamma\beta^2)^2 - \beta^2 k(2-k)}}{\beta^2}, \qquad \alpha > \gamma\beta^2 + \beta\sqrt{k(2-k)}, \quad k < 1, \tag{12}$$

where

$$k = \sqrt{1 - 2(\sigma - \gamma_1\delta^2) + \delta^2} + \sqrt{1 - 2(\mu - \gamma_2\eta^2) + \eta^2}, \tag{13}$$

then, there exists a unique solution $u \in H : h(u) \in K$ of the extended general variational inequality (4).

Proof. From Lemma 3.1, it follows that problems (10) and (4) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (11), has a fixed point. For all $u \neq v \in H$,

$$\begin{aligned} ||F(u) - F(v)|| &= ||u - v - (h(u) - h(v)) + P_K[g(u) - \rho Tu] - P_K[g(v) - \rho Tv]|| \\ &\leq ||u - v - (h(u) - h(v))|| + ||P_K[g(u) - \rho Tu] - P_K[g(v) - \rho Tv]|| \\ &\leq ||u - v - (g(u) - g(v))|| + ||u - v - (h(u) - h(v))|| \\ &\quad + ||u - v - \rho(Tu - Tv)||, \end{aligned} \tag{14}$$

where we have used the fact that the operator P_K is nonexpansive.

Since the operator T is relaxed co-coercive strongly monotone with constants $\gamma > 0, \alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} ||u - v - \rho(Tu - Tv)||^2 &\leq ||u - v||^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2||Tu - Tv||^2 \\ &\leq (1 - 2\rho(\alpha - \gamma\beta^2) + \rho^2\beta^2)||u - v||^2. \end{aligned} \tag{15}$$

In a similar way, we have

$$||u - v - (g(u) - g(v))||^2 \leq (1 - 2(\sigma - \gamma_1\delta^2) + \delta^2)||u - v||^2, \tag{16}$$

$$||u - v - (h(u) - h(v))||^2 \leq (1 - 2(\mu - \gamma_2\eta^2) + \eta^2)||u - v||^2, \tag{17}$$

where $\gamma_1 > 0, \sigma > 0$, $\gamma_2 > 0, \mu > 0$ and $\delta > 0$, $\eta > 0$ are the relaxed co-coercive strongly monotonicity and Lipschitz continuity constants of the operator g and h respectively.

From (13), (14), (15),(16) and (17), we have

$$\begin{aligned} ||F(u) - F(v)|| &\leq \Big(\sqrt{1 - 2(\sigma - \gamma_1\delta^2) + \delta^2} + \sqrt{1 - 2(\mu - \gamma_2\eta^2) + \eta^2} \\ &\quad + \sqrt{1 - 2\rho(\alpha - \gamma\beta^2) + \beta^2\rho^2)}||u - v|| \\ &= (k + t(\rho))||u - v||, \\ &= \theta||u - v||, \end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\rho(\alpha - \gamma\beta^2) + \rho^2\beta^2}. \tag{18}$$

and

$$\theta = k + t(\rho). \tag{19}$$

From (12), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by has a fixed point, which is a unique solution of (4). □

Using the fixed point formulation (10), Noor [22] has suggested and analyzed the following iterative method for solving the extended general variational inequalities (4).

Algorithm 3.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - h(u_n) + P_K[g(u_n) - \rho T u_n]\}, \quad n = 0, 1, \ldots$$

which is known as the Mann iteration process for solving the extended general variational inequalities (4).

Note that if $h = g$, then Algorithm 3.1 reduces to the following iterative method for solving the general variational inequalities (5).

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_K[g(u_n) - \rho T u_n]\}, \quad n = 0, 1, \ldots$$

which is due to Noor [21]. For the convergence analysis of Algorithm 3.2 and Algorithm 3.2, see Noor [13,15].

Using the technique of updating the solution, we now suggest and analyze some iterative three-step iterative schemes for solving the extended general variational inequalities (2.4) and this is the main motivation of this paper.

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solutions $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} h(y_n) &= P_K[g(u_n) - \rho T u_n] \\ h(w_n) &= P_K[g(y_n) - \rho T y_n] \\ h(u_{n+1}) &= P_K[g(w_n) - \rho T w_n], \quad n = 0, 1, 2, \ldots.. \end{aligned}$$

Using Lemma 2.2, Algorithm 3.3 can be written as

Algorithm 3.4. For a given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$\begin{aligned} &\langle \rho T u_n + h(y_n) - g(u_n), g(v) - h(y_n)\rangle \geq 0, \quad \forall g(v) \in K \\ &\langle \rho T y_n + h(w_n) - g(y_n), g(v) - h(w_n)\rangle \geq 0, \quad \forall g(v) \in K \\ &\langle \rho T w_n + h(u_{n+1}) - g(w_n), g(v) - h(u_{n+1})\rangle \geq 0, \quad \forall g(v) \in K \end{aligned}$$

Invoking Algorithm 3.3, we now suggested another three step scheme for solving the extended general variational inequality (4).

Algorithm 3.5. For a given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$y_n = (1-\gamma_n)u_n + \gamma_n\{u_n - h(u_n) + P_K[g(u_n) - \rho T u_n]\} \tag{20}$$

$$w_n = (1-\beta_n)u_n + \beta_n\{y_n - h(y_n) + P_K[g(y_n) - \rho T y_n]\} \tag{21}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{w_n - h(w_n) + P_K[g(w_n) - \rho T w_n]\}. \tag{22}$$

For $\gamma_n = 0$, Algorithm 3.5 reduces to:

Algorithm 3.6. For a given $u_0 \in H$, compute $\{u_n\}$ by the iterative schemes

$$\begin{aligned} w_n &= (1-\beta_n)u_n + \beta_n\{u_n - h(u_n) + P_K[g(u_n) - \rho T u_n]\} \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{w_n - h(w_n) + P_K[g(w_n) - \rho T w_n]\}, \quad n = 0, 1, 2, \ldots.. \end{aligned}$$

which is known as the Ishikawa iterative scheme for the extended general variational inequality (4). Note that for $\gamma_n = 0$ and $\beta_n = 0$, Algorithm 3.5 is called the Mann iterative method.

For $g = h = I$, the identity operator, Algorithm 3.5 collapses to the following algorithm for variational inequality (8), which are mainly due to Noor [13,15].

Algorithm 3.7. For a given $u_0 \in K$, compute $\{u_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n P_K[u_n - \rho T u_n] \\ w_n &= (1-\beta_n)u_n + \beta_n P_K[y_n - \rho T y_n] \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n P_K[w_n - \rho T w_n], \quad n = 0,1,2,\ldots\ldots \end{aligned}$$

Now we suggest a perturbed iterative scheme for solving the extended general variational inequality (4).

Algorithm 3.8. For a given $u_o \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n\{u_n - h(u_n) + P_{K_n}[g(u_n) - \rho T u_n]\} + \gamma_n h_n \\ w_n &= (1-\beta_n)u_n + \beta_n\{y_n - h(y_n) + P_{K_n}[g(y_n) - \rho T y_n]\} + \beta_n f_n \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{w_n - h(w_n) + P_{K_n}[g(w_n) - \rho T w_n]\} + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of H introduced to take into account possible inexact computations and P_{K_n} is the corresponding perturbed projection operator; and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy $0 \le \alpha_n, \beta_n, \gamma_n \le 1$; for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving the extended general variational inequality (4). If $g = h$, , we obtain the perturbed iterative method for solving the general variational inequalities (5), which is mainly due to Noor [13,15].

If $g = h = I$, and $K = H$, then Algorithm 3.8 is equivalent to the following three-step scheme for the nonlinear equations $Tu = 0$, which is known as Noor three-step iterative method, see [13,15] and the references therein.

Algorithm 3.9. For a given $u_0 \in H$, find the approximate solution $\{u_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)u_n + \gamma_n T u_n + \gamma_n h_n \\ w_n &= (1-\beta_n)u_n + \beta_n T y_n + \beta_n f_n \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T w_n + \alpha_n e_n, \quad n = 0,1,2,\ldots \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$ and $\{h_n\}$ are sequences of the elements of H introduced to take into account possible inexact computations and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy $0 \le \alpha_n, \beta_n, \gamma_n \le 1$; for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

In brief, for suitable and appropriate choice of the operators T, g and the space H, one can obtain a number of new and previously known iterative schemes for solving variational inequalities and related problems. This clearly shows that Algorithm 3.5 and Algorithm 3.9 are quite general and unifying ones.

We now study the convergence criteria of Algorithms 3.5. In a similar way, one can analyze the convergence criteria of other algorithms.

Theorem 3.2. Let the operators T, g, h satisfy all the assumptions of Theorem 3.1. If the condition (12) holds, then the approximate solution $\{u_n\}$ obtained from

Algorithm 3.5 converges to an exact solution u of the extended general variational inequality (4) strongly in H.

Proof. From Theorem 3.1, we see that there exists a unique solution $u \in H$ of the extended general variational inequality (4). Let $u \in H$ be a unique solution of (4). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{u - h(u) + P_K[g(u) - \rho Tu]\} \tag{23}$$
$$= (1-\beta_n)u + \beta_n\{u - h(u) + P_K[g(u) - \rho Tu]\} \tag{24}$$
$$= (1-\gamma_n)u + \gamma_n\{u - h(u) + P_K[g(u) - \rho Tu]\}. \tag{25}$$

From (20),(23),(15), (16) and (17), we have

$$\begin{aligned} ||u_{n+1}-u|| &= ||(1-\alpha_n)(u_n-u) + \alpha_n(w_n - u - (h(w_n)-h(u))) \\ &+ \alpha_n\{P_K[g(w_n)-\rho Tw_n] - P_K[g(u)-\rho Tu]\}|| \\ &\leq (1-\alpha_n)||u_n-u|| + \alpha_n||w_n-u-(g(w_n)-g(u))|| \\ &\quad + + \alpha_n||w_n-u-(h(w_n)-h(u))|| \\ &+ \alpha_n||w_n-u-\rho(Tw_n-Tu)|| \\ &\leq (1-\alpha_n)||u_n-u|| + \alpha_n(k+t(\rho))||w_n-u||, \\ &= (1-\alpha_n)||u_n-u|| + \alpha_n\theta||w_n-u||, \end{aligned} \tag{26}$$

In a similar way, from (21),(24) and using (13), (15), and (16), we have

$$\begin{aligned} ||w_n-u|| &\leq (1-\beta_n)||u_n-u|| + 2\beta_n\theta||y_n-u-(g(y_n)-g(u))|| \\ &+ \beta_n||y_n-u-\rho(Ty_n-Tu)|| \\ &\leq (1-\beta_n)||u_n-u|| + \beta_n(k+t(\rho))||y_n-u||, \\ &\leq (1-\beta_n)||u_n-u|| + \beta_n\theta||y_n-u||, \end{aligned} \tag{27}$$

and from (16), (25) and (17), we obtain

$$\begin{aligned} ||y_n-u|| &\leq (1-\gamma_n)||u_n-u|| + \gamma_n\theta||u_n-u||, \\ &\leq (1-(1-\theta)\gamma_n)||u_n-u|| \\ &\leq ||u_n-u||. \end{aligned} \tag{28}$$

From (27) and (28), we obtain

$$\begin{aligned} ||w_n-u|| &\leq (1-\beta_n)||u_n-u|| + \beta_n\theta||u_n-u|| \\ &= (1-(1-\theta)\beta_n)||u_n-u|| \\ &\leq ||u_n-u||. \end{aligned} \tag{29}$$

Combining (26) and (29), we have

$$\begin{aligned} ||u_{n+1}-u|| &\leq (1-\alpha_n)||u_n-u|| + \alpha_n\theta||u_n-u|| \\ &= [1-(1-\theta)\alpha_n]||u_n-u|| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]||u_0-u||. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\longrightarrow\infty}\prod_{i=0}^{n}[1-(1-\theta)\alpha_i] = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. From (28), and (29), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to u strongly in H. This completes the proof. □

Acknowledgement. The author would like to express his deepest gratitude to Dr. S. M. Junaid Zaidi, Rector, CIIT, Islamabad, for the excellent research facilities and support in his research endeavors.

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