ALBANIAN JOURNAL
OF MATHEMATICS
Volume 2, Number 2, Pages 111–124
ISSN 1930-1235: (electronic version)

NUMERICAL BLOW-UP AND ASYMPTOTIC BEHAVIOR FOR A SEMILINEAR PARABOLIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. This paper concerns the study of the numerical approximation for the following initial-boundary value problem:

$$(P)\begin{cases} u_t(x,t) = u_{xx}(x,t) + au^p(x,t), & 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) + bu^q(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x) > 0, & 0 \le x \le 1, \end{cases}$$

where $a > 0$, $b > 0$ and $p > q > 1$. We show that under some conditions, the solution of a semidiscrete form of (P) either decays uniformly to zero or blows up in a finite time. When the blow-up occurs, we estimate the semidiscrete blow-up time and prove that under some assumptions, the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. When the semidiscrete solution goes to zero as t goes to infinity, we give its asymptotic behavior. Finally, we give some numerical experiments to illustrate our analysis.

1. Introduction

Consider the following initial-boundary value problem:

$$u_t(x,t) = u_{xx}(x,t) + au^p(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) + bu^q(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{3}$$

where $a > 0$, $b > 0$, $p > q > 1$, $u_0 \in C^2([0,1])$,

$$u_0''(x) + au_0^p(x) > 0 \quad \text{in} \quad [0,1], \tag{4}$$

$$u_0'(0) = 0, \quad u_0'(1) + bu_0^q(1) = 0. \tag{5}$$

The particularity of this kind of problem is that the solution u of (1)–(3) may develop singularities in a finite time. In other words, under some assumptions, there exists a finite time T such that $\|u(\cdot,t)\|_\infty < +\infty$ for $t \in (0,T)$ but $\lim_{t\to T}\|u(\cdot,t)\|_\infty = +\infty$ where $\|u(\cdot,t)\|_\infty = \sup_{x\in[0,1]}|u(x,t)|$. In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u. When T is infinite, we say that the solution u exists globally. The theoretical study of blow-up and asymptotic behavior of solutions for semilinear parabolic equations

1991 *Mathematics Subject Classification.* 35B40, 35B50, 35K60, 65M06.
Key words and phrases. semidiscretizations, semilinear parabolic equation, asymptotic behavior, convergence.

with nonlinear boundary conditions has been the subject of investigation of many authors (see [2]–[5], [7], [13], [14] and the references cited therein).
The fact that $p > 1$, $q > 1$ and the condition (5) ensure the local in time existence and the uniqueness of the solution of (1)–(3) which is regular (see for instance [2], [3], [7], [9], [13]).
Since $a > 0$, $b > 0$, $p > q > 1$, under the condition given in (4), it is also proved that the solution u of (1)–(3) blows up in a finite time and we have an upper bound of the blow-up time (see [2], [3], [7]).
Finally, it is shown that the solution u of (1)–(3) exists globally and decays uniformly to zero for small initial data (see [2], [4], [7]).

In this paper, we are interesting in the numerical study of (1)–(3). Let I be a positive integer and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. We approximate the solution u of the problem (1)–(3) by the solution $U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{d}{dt}U_i(t) = \delta^2 U_i(t) + a(U_i(t))^p, \quad 0 \le i \le I-1, \quad t > 0, \tag{6}$$

$$\frac{d}{dt}U_I(t) = \delta^2 U_I(t) + a(U_I(t))^p - \frac{2b}{h}(U_I(t))^q, \quad t > 0, \tag{7}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{8}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

For the initial data $\varphi_h = (\varphi_0, ..., \varphi_I)^T$, one may take $\varphi_i = u_0(x_i)$, $0 \le i \le I$ but this is not necessary. In fact, we shall see later that if φ_h is close to $u_0(x)$, then the semidiscrete solution $U_h(t)$ approaches the continuous one (see Theorem 3.2 below).

We need the following definition.

Definition 1.1. *We say that the solution U_h of (6)–(8) blows up in a finite time if there exists a finite time T_h such that*

$$\|U_h(t)\|_\infty < +\infty \text{ for } t \in [0, T_h) \text{ but } \lim_{t \to T_h} \|U_h(t)\|_\infty = +\infty,$$

where $\|U_h(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|$. The time T_h is called the semidiscrete blow-up time of the solution $U_h(t)$.

In this paper, under some assumptions on the initial data, we show that the solution $U_h(t)$ of (6)–(8) either blows up in a finite time or exists globally and decays uniformly to zero. In the case where the blow-up occurs, we show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. When the solution decays uniformly to zero, we give its asymptotic behavior.

Our work was motived by the papers in [1], [6] and [11]. In [1] and [11], the authors have studied numerical blow-up for semilinear parabolic equations with Dirichlet boundary conditions. In this paper, the results obtained in the case of blow-up solutions generalize those found in [1] and [11] but this is not a simple generalization because of the nonlinearity of boundary conditions. Let us illustrate this fact. In the case where the semidiscrete solution blows up in a finite time, for the convergence of the semidiscrete blow-up time, our proof is based on an idea of Friedman and McLeod in [8] and on the construction of an upper solution. In [1], an upper solution has been also used to prove the convergence of the semidiscrete blow-up time but in the present paper, because of the nonlinearity of boundary conditions, the upper solution constructed is not usual. Indeed, we construct a continuous upper solution and show after a semidiscretization that the discrete version of the above solution is a good candidate as an upper solution for the semidiscrete problem. Let us also notice that in [11], the author has proved the convergence of the discrete blow-up time for a solution which blows up in L^p norm with $1 \leq p < +\infty$. This condition is restrictive because in general, one deals with solutions which blow up in L^∞ norm. In [6], the phenomenon of extinction is investigated using some semidiscrete and discrete schemes (we say that a solution extincts in a finite time if it reaches the value zero in a finite time).

The rest of the paper is written in the following manner. In the next section, we prove some lemmas about the discrete maximum principle. In the third section, we show that under some assumptions, the solution $U_h(t)$ of (6)–(8) blows up in a finite time and estimate its semidiscrete blow-up time. We also prove that the blow-up time of the semidiscrete problem converges to the one of the continuous problem when the mesh size goes to zero. In the fourth section, we show that the solution of the semidiscrete problem goes to zero for small initial data and determine its asymptotic behavior. Finally in the last section, we construct two schemes and give some numerical results.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later.
The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let $a_h(t) \in C^0([0,T),\mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T),\mathbb{R}^{I+1})$ such that*

$$\frac{d}{dt}V_i(t) - \delta^2 V_i(t) + a_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0,T), \tag{9}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{10}$$

Then we have $V_i(t) \geq 0$ for $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and introduce the vector $Z_h(t) = e^{\lambda t}V_h(t)$ where λ is such that $a_i(t) - \lambda > 0$, $0 \leq i \leq I$, $t \in [0,T_0]$. Let $m = \min_{0\leq i\leq I, 0\leq t\leq T_0} Z_i(t)$. Since for $i \in \{0,...,I\}$, $Z_i(t)$ is a continuous function, there exists $t_0 \in [0,T_0]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in \{0,...,I\}$. It is not hard to see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \leq 0, \tag{11}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0, \tag{12}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1, \tag{13}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = I. \tag{14}$$

Using (9), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{15}$$

According to (11)–(15), we arrive at $(a_{i_0}(t) - \lambda)Z_{i_0}(t_0) \geq 0$, which implies that $m = Z_{i_0}(t_0) \geq 0$. Therefore, $V_h(t) \geq 0$ for $t \in [0, T_0]$ and we have the desired result. □

Another version of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.2. *Let $V_h(t)$, $U_h(t) \in C^1([0,\infty), \mathbb{R}^{I+1})$ and $f \in C^0(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ such that for $t \in (0,\infty)$*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t), t) < \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_i(t), t), \quad 0 \leq i \leq I, \tag{16}$$

$$V_i(0) < U_i(0), \quad 0 \leq i \leq I. \tag{17}$$

Then we have $V_i(t) < U_i(t)$, $0 \leq i \leq I$, $t \in (0,\infty)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, ..., I\}$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = I,$$

which implies that $\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \leq 0$. But this inequality contradicts (16) and the proof is complete.

□

3. Blow-up solutions

In this section, under some assumptions, we show that the solution U_h of (6)–(8) blows up in a finite time and estimate its semidiscrete blow-up time. In addition, we prove that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero.

We need the following result.

Lemma 3.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $U_h \geq 0$. Then we have*

$$\delta^2 U_i^q \geq q U_i^{q-1} \delta^2 U_i, \quad 0 \leq i \leq I.$$

Proof. Apply Taylor's expansion to obtain

$$\delta^2 U_0^q = q U_0^{q-1} \delta^2 U_0 + (U_1 - U_0)^2 \frac{q(q+1)}{h^2} \theta_0^{q-2},$$

$$\delta^2 U_i^q = q U_i^{q-1} \delta^2 U_i + (U_{i+1} - U_i)^2 \frac{q(q+1)}{2h^2} \theta_i^{q-2} + (U_{i-1} - U_i)^2 \frac{q(q+1)}{2h^2} \eta_i^{q-2}$$
$$if \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_I^q = qU_I^{q-1}\delta^2 U_I + (U_{I-1} - U_I)^2 \frac{q(q+1)}{h^2}\eta_I^{q-2},$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. Use the fact that $U_h \geq 0$ to complete the proof.

□

Now let us state a result on blow-up.

Theorem 3.1. *Let U_h be the solution of (6)–(8). Suppose that there exists a positive constant A such that*

(18) $$\delta^2\varphi_i + a\varphi_i^p \geq A\varphi_i^q, \quad 0 \leq i \leq I.$$

Then the solution U_h of (6)–(8) blows up in a finite time T_b^h with the following estimation

(19) $$T_b^h \leq \frac{1}{A}\frac{\|\varphi_h\|_\infty^{1-q}}{(q-1)}.$$

Proof. Let $(0, T_b^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty < +\infty$. Our aim is to show that T_b^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

(20) $$J_i = \frac{d}{dt}U_i - AU_i^q, \quad 0 \leq i \leq I.$$

A direct calculation yields

$$\frac{d}{dt}J_i - \delta^2 J_i = \frac{d}{dt}(\frac{d}{dt}U_i - \delta^2 U_i) - AqU_i^{q-1}\frac{d}{dt}U_i + A\delta^2 U_i^q.$$

From Lemma 3.1 $\delta^2 U_i^q \geq qU_i^{q-1}\delta^2 U_i$ which implies that

$$\frac{d}{dt}J_i - \delta^2 J_i \geq \frac{d}{dt}(\frac{d}{dt}U_i - \delta^2 U_i) - AqU_i^{q-1}(\frac{d}{dt}U_i - \delta^2 U_i),\ 0 \leq i \leq I.$$

It follows from (6)–(7) that

$$\frac{d}{dt}J_i - \delta^2 J_i \geq apU_i^{p-1}J_i, \quad 0 \leq i \leq I-1,$$

$$\frac{d}{dt}J_I - \delta^2 J_I \geq (-2qb\frac{U_I^{q-1}}{h} + apU_I^{p-1})J_I.$$

The relation (18), implies that $J_h(0) \geq 0$. It follows from Lemma 2.1 that $J_h(t)$ is nonnegative, which implies $\frac{d}{dt}U_i \geq AU_i^q$, $0 \leq i \leq I$. We observe that

(21) $$\frac{dU_i}{U_i^q} \geq Adt, \quad 0 \leq i \leq I.$$

Integrating these inequalities over (t, T_b^h), we arrive at

(22) $$T_b^h - t \leq \frac{1}{A}\frac{(U_i(t))^{1-q}}{(q-1)}, \quad 0 \leq i \leq I.$$

Let i_0 such that $\|U_h(t)\|_\infty = U_{i_0}(t)$. If we replace i by i_0 and the time t by 0 in the above inequalities, we get the following estimation $T_b^h \leq \frac{1}{A}\frac{\|U_h(0)\|_\infty^{1-q}}{(q-1)}$. This implies that the solution $U_h(t)$ blows up in a finite time because the quantity on the right hand side of the above inequality is finite. Use the fact that $\|U_h(0)\|_\infty = \|\varphi_h\|_\infty$ to complete the rest of the proof. □

Remark 3.1. *The inequalities (22) imply that*

$$T_b^h - t_0 \le \frac{1}{A}\frac{\|U_h(t_0)\|_\infty^{1-q}}{(q-1)} \quad \text{if} \quad 0 < t_0 < T_b^h.$$

Remark 3.2. *Let us notice that the condition (18) is the discrete version of the one given in (4) for the continuous solution.*

In the following theorem, we show that for each fixed time interval $[0,T]$ where u is defined, the solution $U_h(t)$ of (6)–(8) approximates u when the mesh parameter h goes to zero.

Theorem 3.2. *Assume that (1)-(3) has a solution $u \in C^{4,1}([0,1]\times[0,T])$ and the initial condition at (8) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as} \quad h \to 0, \tag{23}$$

where $u_h(t) = (u(x_0,t),...,u(x_I,t))^T$. Then, for h sufficiently small, the problem (6)–(8) has a unique solution $U_h \in C^1([0,T],\mathbb{R}^{I+1})$ such that

$$\max_{0\le t\le T}\|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad \text{as} \quad h \to 0. \tag{24}$$

Proof. The problem (6)–(8) has for each h, a unique solution $U_h \in C^1([0,T_q^h),\mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t>0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 \quad \text{for} \quad t \in (0,t(h)). \tag{25}$$

The relation (23) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h),T\}$. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \le \|u(\cdot,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0,t^*(h)),$$

which implies that $U_h(t)$ is bounded on the interval $(0,t^*(h))$. Let $e_h(t) = U_h(t) - u_h(x,t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0,t^*(h))$,

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) = \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t) + ap\xi_i^{p-1}e_i(t),$$

$$\frac{d}{dt}e_I(t) - \delta^2 e_I(t) = \frac{2}{h}q\theta_I^{q-1}e_I + \frac{2h^2}{3}u_{xxx}(\widetilde{x}_I,t) + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_I,t) - ap\xi_I^{p-1}e_I(t),$$

where θ_I is an intermediate value between $U_I(t)$ and $u(x_I,t)$ and ξ_i the one between $U_i(t)$ and $u(x_i,t)$. Since $U_i(t)$ is bounded and $u \in C^{4,1}$, there exist two positive constants K and L such that

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) \le L|e_i(t)| + Kh^2, \quad 0 \le i \le I-1, \tag{26}$$

$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) \le \frac{L|e_I(t)|}{h} + L|e_I(t)| + Kh^2. \tag{27}$$

Consider the function $z(x,t) = e^{((M+1)t+Cx^2)}(\|\varphi_h - u_h(0)\|_\infty + Qh^2)$ where M, C, Q are constants which will be determined later. A direct calculation yields

$$z_t(x,t) - z_{xx}(x,t) = (M+1-2C-4C^2x^2)z(x,t),$$

$$z_x(0,t) = 0, \quad z_x(1,t) = 2Cz(1,t),$$

$$z(x,0) = e^{Cx^2}(\|\varphi_h - u_h(0)\|_\infty + Qh^2).$$

By a semidiscretization of the above problem, we may choose M, C, Q large enough that

$$\frac{d}{dt}z(x_i,t)-\delta^2 z(x_i,t)>L|z(x_i,t)|+Kh^2,\quad 0\le i\le I-1, \tag{28}$$

$$\frac{d}{dt}z(x_I,t)-\delta^2 z(x_I,t)>\frac{L}{h}|z(x_I,t)|+L|z(x_I,t)|+Kh^2, \tag{29}$$

$$z(x_i,0)>e_i(0),\quad 0\le i\le I. \tag{30}$$

It follows from Lemma 2.2 that $z(x_i,t)>e_i(t)$ for $t\in(0,t^*(h))$, $0\le i\le I$. By the same way, we also prove that $z(x_i,t)>-e_i(t)$ for $t\in(0,t^*(h))$, $0\le i\le I$, which implies that

$$\|U_h(t)-u_h(t)\|_\infty\le e^{(Mt+C)}(\|\varphi_h-u_h(0)\|_\infty+Qh^2),\quad t\in(0,t^*(h)).$$

Let us show that $t^*(h)=T$. Suppose that $T>t(h)$. From (25), we obtain

$$1=\|U_h(t(h))-u_h(t(h))\|_\infty\le e^{(MT+C)}(\|\varphi_h-u_h(0)\|_\infty+Qh^2). \tag{31}$$

Since the term in the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $1\le 0$, which is impossible. Consequently $t^*(h)=T$, and we obtain the desired result. ☐

Remark 3.3. *Let us notice that if for the semidiscrete scheme in (6)–(8) we take as initial data $\varphi_i=u_0(x_i)$, $0\le i\le I$, then we easily see that*

$$u_h(0)=(u(x_0,0),...,u(x_I,0))^T=(u_0(x_0),...,u_0(x_I))^T=\varphi_h.$$

In this case $\|\varphi_h-u_h(0)\|_\infty=0$ and the condition (23) is valid. We also observe that if we take $\varphi_i=u_0(x_i)+ih^2$, $0\le i\le I$ then the condition (23) remains valid. The advantage to choose this kind of initial data is that if for instance the initial data u_0 of the continuous problem is nondecreasing, taking $\varphi_i=u_0(x_i)+ih^2$, $0\le i\le I$, we remark that $\varphi_{i+1}>\varphi_i$, $0\le i\le I-1$. This is sometimes very important when we want to treat certain problems.

Now, we are in a position to prove the main result of this section

Theorem 3.3. *Suppose that the problem (1)–(3) has a solution u which blows up in a finite time T_b such that $u\in C^{4,1}([0,1]\times[0,T_b))$ and the initial condition at (8) satisfies*

$$\|\varphi_h-u_h(0)\|_\infty=o(1)\quad\text{as}\quad h\to 0.$$

Assume that there exists a constant $A>0$ such that

$$\delta^2\varphi_i+a\varphi_i^p\ge A\varphi_i^q,\quad 0\le i\le I.$$

Then the problem (6)–(8) has a solution U_h which blows up in a finite time T_b^h and

$$\lim_{h\to 0}T_b^h=T_b.$$

Proof. Letting $\varepsilon>0$, there exists a positive constant N such that

$$\frac{1}{A}\frac{x^{1-q}}{(q-1)}\le\frac{\varepsilon}{2}<\infty\quad\text{for}\quad x\in(N,+\infty). \tag{32}$$

Since u blows up at the time T_b, there exists T_1 such that $|T_1-T_b|\le\frac{\varepsilon}{2}$ and $\|u(\cdot,t)\|_\infty\ge 2N$ for $t\in[T_1,T_b]$. Let $T_2=\frac{T_1+T_b}{2}$, then $\sup_{t\in[0,T_2]}|u(\cdot,t)|<\infty$. It follows from Theorem 3.2 that the problem (6)–(8) has a solution $U_h(t)$

and $\sup_{t\in[0,T_2]}|U_h(t)-u_h(t)|_\infty \leq N$. Applying the triangle inequality, we get $\|U_h(t)\|_\infty \geq \|u_h(t)\|_\infty - \|U_h(t)-u_h(t)\|_\infty$, which leads to $\|U_h(t)\|_\infty \geq N$ for $t\in[0,T_2]$. From Theorem 3.1, $U_h(t)$ blows up at the time T_b^h. We deduce from Remark 3.1 and (32) that

$$|T_b^h - T_b| \leq |T_b^h - T_2| + |T_2 - T_b| \leq \frac{\varepsilon}{2} + \frac{1}{A}\frac{\|U_h(T_2)\|_\infty^{1-q}}{(q-1)} \leq \varepsilon,$$

and the proof is complete. □

4. Asymptotic behavior

In this section, we show that for small initial data, the solution U_h of (6)–(8) goes to zero as $t\to+\infty$ and give its asymptotic behavior.

Theorem 4.1. *Let $U_h(t)$ be the solution of (6)–(8). There exists a constant $C>0$ such that if the initial condition defined in (8) satisfies $\|\varphi_h\|_\infty \leq C$ then $U_h(t)$ goes to zero as $t\to+\infty$. Moreover, the following relation holds*

$$\lim_{t\to\infty} t^{\frac{1}{q-1}}\|U_h(t)\|_\infty = C_0,$$

where $C_0 = (\frac{1}{b(q-1)})^{\frac{1}{q-1}}$.

The proof of the above theorem is based on the lemmas below. Introduce the function

$$\mu(x) = -\lambda(C_0+\varepsilon) + b(C_0+\varepsilon)^q,$$

where $\lambda = \frac{1}{q-1}$. This function is crucial for the proof of the above theorem.
Let us state our first lemma which gives us an upper bound of the semidiscrete solution.

Lemma 4.1. *Let U_h be the solution of (6)-(8). There exists a positive constant C such that if the initial condition defined in (8) satisfies $\|\varphi_h\|_\infty \leq C$, then U_h goes to zero when t tends to infinity. In addition for any $\varepsilon>0$, there exist two positive times T and τ such that*

$$U_i(t+\tau) \leq (C_0+\varepsilon)(t+T)^{-\lambda} + \varphi_i(t+T)^{-\lambda-1}, \quad 0\leq i\leq I,$$

where $\varphi_i = -\frac{b}{2}(C_0+\varepsilon)^q i^2h^2$.

Proof. Since $\mu(0)=0$ and $\mu'(0)=1$, let $\eta>0$ such that $\mu(\eta)>0$. Define the vector W_h such that

$$W_i(t) = (C_0+\eta)t^{-\lambda} + \varphi_i t^{-\lambda-1}, \quad 0\leq i\leq I.$$

Our idea is to show that the vector W_h is an upper solution of (6)-(8). A direct calculation reveals that

$$\begin{aligned}\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p &= -\lambda(C_0+\eta)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2}\varphi_i \\ &+ at^{-\lambda p}((C_0+\eta)+t^{-\lambda-1})^p - t^{-\lambda-1}\delta^2\varphi_i,\ 0\leq i\leq I-1,\end{aligned}$$

$$\begin{aligned}\frac{dW_I}{dt} - \delta^2 W_I + aW_I^P - \frac{2b}{h}W_I^q &= -\lambda(C_0+\eta)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2}\varphi_I \\ &+ at^{-\lambda p}((C_0+\eta)+\varphi_I t^{-1})^p + \\ &+ t^{-\lambda-1}\delta^2\varphi_I \\ &- \frac{2b}{h}t^{-\lambda-1}(C_0+\eta+\varphi_I t^{-1})^q,\end{aligned}$$

because $\lambda q = \lambda + 1$. By the mean value theorem, we get $(C_0 + \eta + \varphi_I t^{-1})^q = (C_0 + \eta)^q + \chi_I t^{-1}$ where $\chi_I(t)$ is a bounded function. We deduce that

$$\begin{aligned}\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p &= t^{-\lambda-1}(\mu(\eta) - (\lambda+1)t^{-1}\varphi_i \\ &+ at^{-\lambda p+\lambda+1}(C_0+\eta+t^{-1}\varphi_i)^p),\end{aligned}$$

$$\begin{aligned}\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p - \frac{2b}{h}W_I^q &= t^{-\lambda-1}(-\lambda\mu(\eta) - (\lambda+1)t^{-1}\varphi_I \\ &+ at^{-\lambda p+\lambda+1}(C_0+\eta+t^{-1}\varphi_I)^p + \frac{2b}{h}\chi_I t^{-1}),\end{aligned}$$

we observe that $-\lambda p + \lambda + 1 = \frac{q-p}{q-1} < 0$. Since $\mu(\eta) > 0$, there exists a time $T > 0$ such that

$$\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p > 0, \quad 0 \le i \le I-1, \quad t \ge T,$$

$$\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p - \frac{2b}{h}W_I^q > 0, \quad t \ge T,$$

$$W_i(T) > \frac{T^{-\lambda}C_0}{2}.$$

Suppose that $U_i(0) < \frac{T^{-1}C_0}{2} < W_i(T)$. Let us introduce the vector $Z_h(t)$ such that $Z_h(t) = U_h(t-T)$. It is not hard to see that

$$\frac{dZ_i}{dt} - \delta^2 Z_i + aZ_i^p = 0, \quad 0 \le i \le I-1, \quad t \ge T,$$

$$\frac{dZ_I}{dt} - \delta^2 Z_I + aZ_I^p - \frac{2b}{h}Z_I^q = 0, \quad t \ge T,$$

$$Z_i(T) = U_i(0) < W_i(T), \quad 0 \le i \le I.$$

We deduce from Comparison Lemma 2.2 that $U_h(t-T)) \le W_h(t)$ for $t \ge T$. Since $W_h(t)$ decays to zero when t tends to infinity, we deduce that $U_h(t)$ goes to zero when t approaches infinity. Now introduce the vector $V_h(t)$ defined as follows

$$V_i(t) = (C_0+\varepsilon)t^{-\lambda} + \varphi_i t^{-\lambda-1}, \quad 0 \le i \le I.$$

By an analogous argument as in the proof of the first part of the lemma, we obtain

$$\begin{aligned}\frac{dV_i}{dt} - \delta^2 V_i + aV_i^p &= t^{-\lambda-1}(\mu(\varepsilon) - (\lambda+1)t^{-1}\varphi_i \\ &+ at^{-\lambda p+\lambda+1}(C_0+\varepsilon+t^{-1}\varphi_i)^p),\ 0 \le i \le I-1,\end{aligned}$$

$$\frac{dV_I}{dt} - \delta^2 V_I + aV_I^p - \frac{2b}{h}V_I^q = t^{-\lambda-1}(-\lambda\mu(\varepsilon) - (\lambda+1)\varphi_I + at^{-\lambda p+\lambda+1}(C_0 + \varepsilon + t^{-1}\varphi_i)^p + \frac{2b}{h}\chi_i t^{-1}).$$

Since $\mu(\varepsilon) > 0$ and $-p\lambda + \lambda + 1 < 0$, there exists a positive time T such that

$$\frac{dV_i}{dt} - \delta^2 V_i + aV_i^p > 0, \quad 0 \le i \le I-1, \quad t \ge T,$$

$$\frac{dV_I}{dt} - \delta^2 V_I + aV_I^p - \frac{2b}{h}V_I^q > 0, \quad t \ge T,$$

$$V_i(T) > \frac{T^{-\lambda}C_0}{2}.$$

Since $U_h(t)$ goes to zero as t approaches infinity, there exists a time $\tau > T$ such that $U_i(\tau) < \frac{T^{-1}C_0}{2} < V_i(T)$. Let the vector $Z_h(t)$ such that $Z_h(t) = U_h(t+\tau-T)$. A routine computation reveals that

$$\frac{dZ_i}{dt} - \delta^2 Z_i + aZ_i^p = 0, \quad 0 \le i \le I-1, \quad t \ge T,$$

$$\frac{dZ_I}{dt} - \delta^2 Z_I + aZ_I^p - \frac{2b}{h}Z_I^q = 0, \quad t \ge T,$$

$$Z_i(T) = V_i(\tau) < U_i(T).$$

It follows from Comparison Lemma 2.2 that $U_h(t-T) \ge V_h(t)$, $t \ge T$, which leads us to the result. □

The following lemma establishes a lower bound of the solution $U_h(t)$ of (6)–(8)

Lemma 4.2. *For any $\varepsilon > 0$ there exists a positive time τ such that*

$$U_i(t+1) \ge (C_0 - \varepsilon)(t+\tau)^{-\lambda} + \psi_i(t+\tau)^{-\lambda-1}, \quad 0 \le i \le I,$$

where $\psi_i = \frac{-b(C_0-\varepsilon)^q}{2} i^2 h^2$.

Proof. Define the vector W_h such that

$$W_i(t) = (C_0 - \varepsilon)t^{-\lambda} + \psi_i t^{-\lambda-1}, \quad 0 \le i \le I.$$

As in the proof of Lemma 4.1, we find that

$$\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p = t^{-\lambda-1}(\mu(-\varepsilon) - (\lambda+1)t^{-1}\psi_i + at^{-\lambda p+\lambda+1}(C_0 - \varepsilon + t^{-1}\psi_i)^p), \ 0 \le i \le I-1,$$

$$\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p - \frac{2b}{h}W_I^q = t^{-\lambda-1}(\mu(-\varepsilon) - (\lambda+1)\varphi_I t^{-1} + at^{-\lambda p+\lambda+1}(C_0 - \varepsilon + t^{-1}\psi_I)^p + \frac{2b}{h}\chi_I t^{-1}),$$

where $\chi_I(t)$ is a bounded function. Since $\mu(0) = 0$ and $\mu'(0) = 1$, we observe that $\mu(-\varepsilon) < 0$. Using the fact that $-p\lambda + \lambda + 1 < 0$, we deduce that there exists a positive time τ such that

$$\frac{dW_i}{dt} - \delta^2 W_i + aW_i^p < 0, \quad 0 \le i \le I-1, \quad t \ge \tau,$$

$$\frac{dW_I}{dt} - \delta^2 W_I + aW_I^p - \frac{2b}{h} W_I^q < 0, \quad t \geq \tau,$$

Since $W_h(t)$ goes to zero when t approaches infinity, there exists a time $T \geq \tau$ such that $W_h(T) < U_h(1)$. Introduce the vector $Z_h(t)$ such that $Z_h(t) = U_h(t - \tau + 1)$. A straightforward computation gives

$$\frac{dZ_i}{dt} - \delta^2 Z_i + aZ_i^p = 0, \quad 0 \leq i \leq I - 1, \quad t \geq \tau,$$

$$\frac{dZ_I}{dt} - \delta^2 Z_I + aZ_I^p - \frac{2b}{h} Z_I^q = 0, \quad t \geq \tau,$$

$$Z_i(\tau) = U_i(1) > W_h(T).$$

It follows from Comparison Lemma 2.2 that $U_h(t - \tau + 1)) \geq W_h(t)$, $t \geq \tau$, which leads us to the result. □

With the above lemmas, we are ready to prove the main result of this section.

Proof of Theorem 4.1. From Lemmas 4.1 and 4.2, we deduce that

$$(C_0 - \varepsilon) \leq \lim_{t\to\infty} \inf(\frac{U_i(t)}{t^\lambda}) \leq \lim_{t\to\infty} \sup(\frac{U_i(t)}{t^\lambda}) \leq (C_0 + \varepsilon),$$

for any $\varepsilon > 0$ and we have the desired result. □

5. Numerical results

In this section, we give some numerical results. Firstly, we approximate the solution $u(x,t)$ of (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} + a(U_i^{(n)})^p, \quad 0 \leq i \leq I - 1, \tag{33}$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} - \frac{2b}{h}(U_I^{(n)})^{q-1} U_I^{(n+1)} + a(U_I^{(n)})^p, \tag{34}$$

$$U_i^{(0)} = \phi_i > 0, \quad 0 \leq i \leq I, \tag{35}$$

where $n \geq 0$, $\Delta t_n = \min\{\frac{h^2}{2}, \frac{h^2}{\|U_h^{(n)}\|_\infty^{p-1}}\}$. Let us notice that the restriction on the time step $\Delta t_n \leq \frac{h^2}{2}$ guarantees the positivity of the discrete solution.

Secondly, we approximate the solution $u(x,t)$ of (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following implicit scheme

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n+1)} + a(U_i^{(n)})^p, \quad 0 \leq i \leq I - 1, \tag{36}$$

$$\delta_t U_I^{(n)} = \delta^2 U_I^{(n+1)} - \frac{2b}{h}(U_I^{(n)})^{q-1} U_I^{(n+1)} + a(U_I^{(n)})^p, \tag{37}$$

$$U_i^{(0)} = \phi_i > 0, \quad 0 \leq i \leq I, \tag{38}$$

where $n \geq 0$, $\quad \Delta t_n^i = \frac{h^2}{\|U_h^{(n)}\|_\infty^{p-1}}$.

The above equations may be rewritten in the following form

$$A^{(n)} U_h^{(n+1)} = a(U_h^{(n)})^p$$

where $A^{(n)}$ is a tridiagonal matrix defined as follows

$$A^{(n)} = \begin{pmatrix} d_0 & \frac{-2\Delta t_n}{h^2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-\Delta t_n}{h^2} & d_1 & \frac{-\Delta t_n}{h^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-\Delta t_n}{h^2} & d_2 & \frac{-\Delta t_n}{h^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-\Delta t}{h^2} & d_{I-2} & \frac{-\Delta t_n}{h^2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{-\Delta t_n}{h^2} & d_{I-1} & \frac{-\Delta t_n}{h^2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-2\Delta t_n}{h^2} & d_I \end{pmatrix},$$

with

$$d_i = 1 + 2\frac{\Delta t_n}{h^2}, \quad 0 \leq i \leq I-1,$$

$$d_I = 1 + 2\frac{\Delta t_n}{h^2} + \frac{2b}{h}|U_I^{(n)}|^{q-1}\Delta t_n.$$

We remark that the tridiagonal matrix $A^{(n)}$ satisfies the following properties

$$A_{ii}^{(n)} > 0, \quad A_{ij}^{(n)} < 0, \quad i \neq j,$$

$$|A_{ii}^{(n)}| > \sum_{i \neq j} |A_{ij}^{(n)}|.$$

These properties imply that $U_h^{(n)}$ exists for all n and $U_h^{(n)} \geq 0$ (See for instance [6]).
We suppose that $p = 3$, $q = 2$, $a = 1$, $b = 1$. In the following tables, in rows, we present the numerical blow-up times or numerical times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

5.1. **Blow-up solutions.** Here we take $U_i^{(0)} = 2 * (hi)^4$. The numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = |t_n - t_{n-1}| \leq 10^{-16}$.

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

I	T^n	n	CPU_t	s
16	0.0012719	3150	0.6	-
32	0.0012669	11879	3	-
64	0.0012657	44690	31.6	2.06
128	0.0012654	167504	839.7	2.01

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.00126726	3138	1.40	-
32	0.00126571	11868	6.9	-
64	0.00126545	44680	11.4	2.58
128	0.00126539	167490	256.2	2.12

5.2. **Solutions which go to zero.** Here we take $U_i^{(0)} = \frac{1}{2} * (hi)^{\frac{1}{4}}$. The numerical time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\|t_n^{\frac{1}{q-1}} U_h^{(n)} - 1\|_\infty < 10^{-2}$.

Table 3: Numerical times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method.

I	T^n	n	CPU_t	s
16	0.655822	335	-	-
32	0.654190	1339	0.5	-
64	0.654048	5358	3	3.53
128	0.653946	21431	55	0.47

Table 4: Numerical times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	T^n	n	CPU_t	s
16	0.654296	334	0.12	-
32	0.653808	1338	1	-
64	0.653730	5356	11.4	2.65
128	0.653661	21429	179	0.17

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