

## IMPLICIT WIENER-HOPF EQUATIONS AND QUASI VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce and consider a new class of Wiener-Hopf equations involving the nonlinear operator and nonexpansive operators, which is called the implicit Wiener-Hopf equations. Essentially using the projection technique, we establish the equivalence between the implicit Wiener-Hopf equations and quasi variational inequalities. Using this alternative equivalent formulation, we suggest and analyze an iterative method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of the quasi variational inequalities. We also study the convergence criteria of iterative methods under some mild conditions. Our results include the previous results as special cases and may be considered as an improvement and refinement of the previously known results.

### 1. INTRODUCTION

Quasi variational inequalities are being used as a mathematical programming tool in modelling various equilibrium problems in economics, operations research, optimization, regional, ecology and network analysis, see [1-33]. It is well known that the quasi variational inequalities include variational inequalities, implicit complementarity problems and optimization problems as special cases. It combines novel theoretical and algorithmic advances with new domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. As a result of such interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various numerical methods including projection technique and its variant forms, auxiliary principle and Wiener-Hopf equations for solving variational inequalities and related optimization problems. Essentially using the projection technique, one can establish the equivalence between the variational inequalities and the Wiener-Hopf equations. This equivalence has played an important and significant role in studying various problems associated with variational inequalities. Related to the quasi variational inequalities and the implicit Wiener-Hopf equations, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these different problems.

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Motivated and inspired by the research going on in this direction, we first introduce a new class of the Wiener-Hopf equations involving a nonexpansive operator  $S$ , which is called the implicit Wiener-Hopf equations. Using the projection technique, we show that the implicit Wiener-Hopf equations are equivalent to the quasi variational inequalities. We use this alternative equivalence to suggest and analyze an iterative scheme for finding the common solutions of the variational inequalities and nonexpansive mappings using the Wiener-Hopf equation technique. We also prove the convergence criteria of these new iterative schemes under some mild conditions. Since the quasi variational inequalities include variational inequalities and the implicit(quasi) complementarity problems as special cases, results proved in this paper continue to hold for these problems. In this respect, results proved in this paper may be viewed as significant and improvement of the previously known results.

## 2. FORMULATIONS AND BASIC FACTS

Let  $H$  be a real Hilbert space, whose inner product and norm are denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ , respectively. Let  $K(u)$  be a closed and convex-valued set in  $H$  and  $T : H \rightarrow H$  be a nonlinear operator.

A *quasi variational inequality* consists in finding  $u \in K(u)$ , such that

$$(1) \quad \langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u).$$

It is well known [1- 28] that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in economics, finance, physics, mathematics, engineering and applied sciences can be studied in the unifying and general framework of (1).

To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding  $u$  such that

$$(2) \quad \left. \begin{aligned} -u'' &\geq f(x) && \text{on } \Omega = [a, b] \\ u &\geq M(u) && \text{on } \Omega = [a, b] \\ [-u'' - f(x)][u - M(u)] &= 0 && \text{on } \Omega = [a, b] \\ u(a) &= 0, \quad u(b) = 0. \end{aligned} \right\}$$

where  $f(x)$  is a continuous function and  $M(u)$  is the cost (obstacle) function. The prototype encountered [2] is

$$(3) \quad M(u) = k + \inf_i \{u^i\}.$$

In (3),  $k$  represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator  $M$  provides the coupling between the unknowns  $u = (u^1, u^2, \dots, u^i)$ . We study the problem (2) in the framework of the quasi variational inequality approach. To do so, we first define the set  $K(u)$  as

$$(4) \quad K(u) = \{v : v \in H_0^1(\Omega) : v \geq M(u), \quad \text{on } \Omega\},$$

which is a closed convex-valued set in  $H_0^1(\Omega)$ , where  $H_0^1(\Omega)$  is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem

(2) is

$$\begin{aligned}
 I[v] &= - \int_a^b \left( \frac{d^2 v}{dx^2} \right) v dx - 2 \int_a^b f(x) (v) dx, \quad \forall v \in K(u) \\
 &= \int_a^b \left( \frac{dv}{dx} \right)^2 dx - 2 \int_a^b f(x) (v) dx \\
 (5) \quad &= \langle Tv, v \rangle - 2\langle f, v \rangle
 \end{aligned}$$

where

$$\begin{aligned}
 (6) \quad \langle Tu, v \rangle &= \int_a^b \left( \frac{d^2 u}{dx^2} \right) (v) dx = \int_a^b \frac{du}{dx} \frac{dv}{dx} dx \\
 \langle f, v \rangle &= \int_a^b f(x)(v) dx.
 \end{aligned}$$

It is clear that the operator  $T$  defined by (6) is linear, symmetric and positive. Using the technique of Noor [13,14,20], one can show that the minimum of the functional  $I[v]$  defined by (5) associated with the problem (2) on the closed convex-valued set  $K(u)$  can be characterized by the inequality of type (1). See also [1-29] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

If  $K^*(u)$  is the dual (polar) cone of the convex-valued cone  $K(u)$ , then the quasi variational inequalities (2.1) are equivalent to finding  $u$  such that

$$(7) \quad u \in K(u), \quad Tu \in K^*(u), \quad \text{and} \quad \langle u, Tu \rangle = 0,$$

which are called the quasi (implicit) complementarity problems. It is well known that a wide class of problems arising in various branches of pure and applied sciences can be studied via the implicit complementarity problems (7). For the applications, numerical methods and physical formulation, see the references.

If the convex-valued set  $K(u)$  is independent of the solution  $u$ , that is,  $K(u) = K$ , a closed convex set, then problem (1) is equivalent to finding  $u \in K$ , such that

$$(8) \quad \langle Tu, v - u \rangle \geq 0, \quad \forall v \in K,$$

which is known as the classic variational inequality introduced and studied by Stampacchia [32] in 1964. For the state of the art in this theory; see [1- 33].

We also need the following well-known concepts and results.

**Lemma 2.1.** Let  $K(u)$  be a closed convex-valued set in  $H$ . Then, for a given  $z \in H$ ,  $u \in K(u)$  satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K(u),$$

if and only if

$$u = P_{K(u)} z,$$

where  $P_{K(u)}$  is the projection of  $H$  onto the closed convex-valued set  $K(u)$ .

It is worth mentioning that the implicit projection operator  $P_{K(u)}$  is not an nonexpansive operator. This fact motivates us to consider the following assumption on the projection operator  $P_{K(u)}$  as:

**Assumption 2.1.** The projection operator  $P_{K(u)}$  satisfies the following relation.

$$\|P_{K(u)} w - P_{K(v)} w\| \leq \nu \|u - v\|, \quad \forall v, u, w \in H,$$

where  $\nu > 0$  is a constant.

We remark that Assumption 2.1 is true for the special case,

$$(9) \quad K(u) = m(u) + K,$$

which appears in many important applications [2], where  $m$  is a point-to-point mapping and  $K$  is a closed convex set in  $H$ . It is well known that

$$(10) \quad P_{K(u)}w = P_{m(u)+K}w = m(u) + P_K[w - m(u)], \quad \forall w, u \in H.$$

We remark that if the mapping  $m(u)$  is a Lipschitz continuous with constant  $\nu_1 > 0$ , then, from (9) and (10), we have

$$\begin{aligned} \|P_{m(u)+K}w - P_{m(v)+K}w\| &= \|m(u) - m(v) + P_K[w - m(u)] - P_K[w - m(v)]\| \\ &\leq 2\|m(u) - m(v)\| \leq 2\nu_1\|u - v\|. \end{aligned}$$

This shows that the projection operator  $P_{m(u)+K}$  is Lipschitz continuous with constant  $2\nu_1 > 0$ . and satisfies the Assumption 2.1 with  $\nu = 2\nu_1$ .

We now show that the quasi variational inequalities (1) are equivalent to the implicit fixed point problem. This result can be proved by using Lemma 2.1. See also Noor [9].

**Lemma 2.2.** The function  $u \in K(u)$  is a solution of the quasi variational inequality (1) if and only if  $u \in K(u)$  satisfies the relation

$$u = P_{K(u)}[u - \rho T u],$$

where  $\rho > 0$  is a constant.

Lemma 2.2 implies that quasi variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the quasi variational inequalities and related optimization problems.

We now state the problem.

**Remark 2.3.** Let  $S$  be a nonexpansive mapping. We denote the set of the fixed points of  $S$  by  $F(S)$  and the set of the solutions of the quasi variational inequalities (2.1) by  $QVI(K, T)$ . If  $x^* \in F(S) \cap VI(K, T)$ , then  $x^* \in F(S)$  and  $x^* \in VI(K, T)$ . Thus from Lemma 2.2, it follows that

$$(11) \quad x^* = Sx^* = P_{K(u)}[x^* - \rho T x^*] = SP_{K(u)}[x^* - \rho T x^*],$$

where  $\rho > 0$  is a constant.

This fixed point formulation is used to suggest the following iterative method for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings and the variational inequalities.

**Algorithm 2.1.** For a given  $u_0 \in K(u)$ , compute the approximate solution  $x_n$  by the iterative schemes

$$u_{n+1} = (1 - a_n)u_n + a_n SP_{K(u_n)}[u_n - \rho T u_n],$$

where  $a_n \in [0, 1]$  for all  $n \geq 0$  and  $S$  is the nonexpansive operator. Algorithm 2.1 is also known as a Mann iteration. For the convergence analysis of Algorithm 2.1, see Huang and Noor [24] and Noor [16,17].

Related to the variational inequalities, we have the problem of solving the Wiener-Hopf equations. To be more precise, let  $Q_{K(u)} = I - SP_{K(u)}$ , where  $P_{K(u)}$  is the

projection of  $H$  onto the closed convex set  $K(u)$ ,  $I$  is the identity operator and  $S$  is the nonexpansive operator. We consider the problem of finding  $z \in H$  such that

$$(12) \quad TSP_{K(u)}z + \rho^{-1}Q_{K(u)}z = 0,$$

which is called the implicit Wiener-Hopf equation involving the nonexpansive operator  $S$ . For  $S = I$ , the identity operator, we obtain the implicit Wiener-Hopf equation, introduced by Noor [14]. If  $S = I$ , and  $K(u) = K$ , then the implicit Wiener-Hopf equations (12) reduce to the original Wiener-Hopf equations considered and studied by Shi [31] in relation with the classical variational inequalities. Using essentially the technique of the projection operator, one can establish the equivalence between the Wiener-Hopf equations and variational inequalities. This alternative equivalence has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities and related optimization problems. The Wiener-Hopf equation technique has been used to study the sensitivity analysis and asymptotical stability of the variational inequalities, see [11-27,30,31]. It has been shown that the Wiener-Hopf equation technique is more flexible and general than the projection method and its variant form.

**Definition 2.1.** An operator  $T : H \rightarrow H$  is called  $\mu$ -Lipschitzian if, there exists a constant  $\mu > 0$ , such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \quad \forall x, y \in H.$$

**Definition 2.2.** An operator  $T : H \rightarrow H$  is called  $\alpha$ -inverse strongly monotone (or co-coercive) if, there exists a constant  $\alpha > 0$ , such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

**Definition 2.3.** An operator  $T : H \rightarrow H$  is called  $r$ -strongly monotone if, there exists a constant  $r > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

**Definition 2.4.** An operator  $T : H \rightarrow H$  is called relaxed  $(\gamma, r)$ -cocoercive if, there exists constants  $\gamma > 0, r > 0$ , such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma\|Tx - Ty\|^2 + r\|x - y\|^2, \quad \forall x, y \in H.$$

**Remark 2.1.** Clearly a  $r$ -strongly monotone operator or a  $\gamma$ -inverse strongly monotone operator must be a relaxed  $(\gamma, r)$ -cocoercive operator, but the converse is not true. Therefore the class of the relaxed  $(\gamma, r)$ -cocoercive operators is the most general class, and hence definition 2.4 includes both the definition 2.2 and the definition 2.3 as special cases.

**Remark 2.2.** From definition 2.2, it follows that if  $T$  is  $\alpha$ -inverse strongly monotone (or co-coercive), then  $T$  is also Lipschitz continuous with constant  $\frac{1}{\alpha}$ .

**Lemma 2.3 [34].** Suppose  $\{\delta_k\}_{k=0}^{\infty}$  is a nonnegative sequence satisfying the following inequality:

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \quad k \geq 0,$$

with  $\lambda_k \in [0, 1]$ ,  $\sum_{k=0}^{\infty} \lambda_k = \infty$ , and  $\sigma_k = o(\lambda_k)$ . Then  $\lim_{k \rightarrow \infty} \delta_k = 0$ .

## 3. MAIN RESULTS

In this section, we use the Wiener-Hopf equations to suggest and analyze an iterative method for finding the common element of the nonexpansive mappings and the variational inequalities QVI(T,K). For this purpose, we need the following result, which can be proved by using Lemma 2.2. However, for the sake of completeness, we include its proof.

**Lemma 3.1.** The element  $u \in K(u)$  is a solution of quasi variational inequality (1) if and only if  $z \in H$  satisfies the implicit Wiener-Hopf equation (12), where

$$(13) \quad u = P_{K(u)}z,$$

$$(14) \quad z = u - \rho Tu,$$

where  $\rho > 0$  is a constant.

**Proof.** Let  $u \in K(u)$  be a solution of VI(K,T). Then, from Lemma 2.3 and Remark 2.3, we have

$$(15) \quad u = SP_{K(u)}[u - \rho Tu].$$

Let

$$(16) \quad z = u - \rho Tu.$$

Form (15) and (16), we have

$$u = SP_{K(u)}z, \quad z = u - \rho Tu,$$

from which, we have

$$z = SP_{K(u)}z - \rho TSP_{K(u)}z,$$

which is exactly the implicit Wiener-Hopf equation (12), the required result.  $\square$

From Lemma 3.1, it follows that the quasi variational inequality (1) and the implicit Wiener-Hopf equation (12) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving variational inequalities and related optimization problems, see [3-16] and the references therein. We denote the set of the solutions of the Wiener-Hopf equations by IWHE(H,T,S).

Using Lemma 3.1 and Remark 2.3, we now suggest and analyze a new iterative algorithm for finding the common element of the solution sets of the quasi variational inequalities and nonexpansive mappings  $S$  and this is the main motivation of this paper.

**Algorithm 3.1.** For a given  $z_0 \in H$ , compute the approximate solution  $z_{n+1}$  by the iterative schemes

$$(17) \quad u_n = SP_{K(u_n)}z_n$$

$$(18) \quad z_{n+1} = (1 - a_n)z_n + a_n\{u_n - \rho Tu_n\}$$

where  $a_n \in [0, 1]$  for all  $n \geq 0$  and  $S$  is a nonexpansive operator. For  $S = I$ , the identity operator, Algorithm 3.1 reduces to the following iterative method for solving quasi variational inequalities (1) and appears to be a new one.

**Algorithm 3.2.** For a given  $z_0 \in H$ , compute the approximate solution  $z_{n+1}$  by the iterative schemes

$$\begin{aligned} u_n &= P_{K(u_n)}z_n \\ z_{n+1} &= (1 - a_n)z_n + a_n\{u_n - \rho Tu_n\}. \end{aligned}$$

For  $a_n = 1$  and  $S = I$ , the identity operator, Algorithm 3.1 collapses to the following iterative method for solving quasi variational inequalities (1).

**Algorithm 3.3.** For a given  $z_0 \in H$ , compute the approximate solution  $z_{n+1}$  by the iterative schemes

$$\begin{aligned} u_n &= P_{K(u_n)} z_n \\ z_{n+1} &= u_n - \rho T u_n. \end{aligned}$$

If  $K(u) = K$ , the convex set in  $H$ , then Algorithms 3.1-3.3 reduce to the following algorithms for solving variational inequalities (8) and nonexpansive mapping, which are due to Noor and Huang [25].

**Algorithm 3.4.** For a given  $z_0 \in H$ , compute the approximate solution  $z_{n+1}$  by the iterative schemes

$$\begin{aligned} u_n &= S P_K z_n \\ z_{n+1} &= (1 - a_n) z_n + a_n \{u_n - \rho T u_n\} \end{aligned}$$

where  $a_n \in [0, 1]$  for all  $n \geq 0$  and  $S$  is a nonexpansive operator. For  $S = I$ , the identity operator, Algorithm 3.4 reduces to the following iterative method for solving variational inequalities (8) and appears to be a new one.

**Algorithm 3.5.** For a given  $z_0 \in H$ , compute the approximate solution  $z_{n+1}$  by the iterative schemes

$$\begin{aligned} u_n &= P_K z_n \\ z_{n+1} &= (1 - a_n) z_n + a_n \{u_n - \rho T u_n\}. \end{aligned}$$

For  $a_n = 1$  and  $S = I$ , the identity operator, Algorithm 3.4 collapses to the following iterative method for solving variational inequalities (2.8).

**Algorithm 3.6.** For a given  $z_0 \in H$ , compute the approximate solution  $z_{n+1}$  by the iterative schemes

$$\begin{aligned} u_n &= P_K z_n \\ z_{n+1} &= u_n - \rho T u_n. \end{aligned}$$

We now study the conditions under the approximate solution obtained from Algorithm 3.1

**Theorem 3.1.** Let  $T$  be a relaxed  $(\gamma, r)$ -cocoercive and  $\mu$ -Lipschitzian mapping and  $S$  be a nonexpansive mapping such that  $F(S) \cap IWHE(H, T, S) \neq \emptyset$ . Let  $\{z_n\}$  be a sequence defined by Algorithm 2.1, for any initial point  $z_0 \in H$ . If Assumption 2.1 holds and

$$(19) \quad \left| \rho - \frac{r - \gamma \mu^2}{\mu^2} \right| \leq \frac{\sqrt{(r - \gamma \mu^2)^2 - \mu^2 \nu (2 - \nu)}}{\mu^2},$$

$$r > \gamma \mu^2 + \mu \sqrt{\nu (2 - \nu)}, \quad \nu \in (0, 1),$$

$a_n \in [0, 1]$  and  $\sum_{n=0}^{\infty} a_n = \infty$ , then  $z_n$  converges strongly to  $z^* \in F(S) \cap IWHE(H, T, S)$ .

**Proof.** Let  $z^* \in H$  be a solution of  $F(S) \cap IWHE(H, T, S)$ . Then, from Lemma 3.1, we have

$$(20) \quad u^* = a_n SPK_{(u^*)} z^*$$

$$(21) \quad z^* = (1 - a_n) z^* + a_n \{u^* - \rho T u^*\}$$

where  $a_n \in [0, 1]$  and  $u^* \in K$  is a solution of QVI(K, I). To prove the result, we need first to evaluate  $\|z_{n+1} - z^*\|$  for all  $n \geq 0$ . From (18) and (21), we have

$$(22) \quad \begin{aligned} \|z_{n+1} - z^*\| &= \|(1 - a_n)z_n + a_n\{u_n - \rho T u_n\} \\ &\quad - (1 - a_n)z^* - a_n\{u^* - \rho T u^*\}\| \\ &\leq (1 - a_n)\|z_n - z^*\| + a_n\|u_n - u^* - \rho(Tu_n - Tu^*)\|. \end{aligned}$$

From the relaxed  $(\gamma, r)$ -cocoercive and  $\mu$ -Lipschitzian definition on  $T$ , we have

$$(23) \quad \begin{aligned} &\|u_n - u^* - \rho(Tu_n - Tu^*)\|^2 \\ &= \|u_n - u^*\|^2 - 2\rho\langle Tu_n - Tu^*, u_n - u^* \rangle + \rho^2\|Tu_n - Tu^*\|^2 \\ &\leq \|u_n - u^*\|^2 - 2\rho[-\gamma\|Tu_n - Tu^*\|^2 + r\|u_n - u^*\|^2] \\ &\quad + \rho^2\|Tu_n - Tu^*\|^2 \\ &\leq \|u_n - u^*\|^2 + 2\rho\gamma\mu^2\|u_n - u^*\|^2 - 2\rho r\|u_n - u^*\|^2 + \rho^2\mu^2\|u_n - u^*\|^2 \\ &= [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|u_n - u^*\|^2 \\ &= \theta_1^2\|u_n - u^*\|^2, \end{aligned}$$

where

$$(24) \quad \theta_1 = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}.$$

Combining (22) and (23), we have

$$(25) \quad \|z_{n+1} - z^*\| \leq (1 - a_n)\|z_n - z^*\| + a_n\theta_1\|u_n - u^*\|.$$

From (17), (20) and the Assumption 2.1., we have

$$\begin{aligned} \|u_n - u^*\| &\leq a_n\|SPK_{(u_n)}z_n - SPK_{(u^*)}z^*\| \\ &\leq \|P_{K(u_n)}z_n - P_{K(u_n)}z^*\| + \|P_{K(u_n)}z^* - P_{K(u^*)}z^*\| \\ &\leq \nu\|u_n - u^*\| + \|z_n - z^*\|, \end{aligned}$$

which implies that

$$(26) \quad \|u_n - u^*\| \leq \frac{1}{1 - \nu}\|z_n - z^*\|.$$

From (25) and (26), we obtain that

$$\begin{aligned} \|z_{n+1} - z^*\| &\leq (1 - a_n)\|z_n - z^*\| + a_n\theta\|z_n - z^*\| \\ &= [1 - a_n(1 - \theta)]\|z_n - z^*\|, \end{aligned}$$

where

$$\theta = \frac{\sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}}{1 - \nu} < 1, \quad \text{using (19),}$$

and hence by Lemma 2.3,  $\lim_{n \rightarrow \infty} \|z_n - z^*\| = 0$ , completing the proof.  $\square$

We now prove the strong convergence of Algorithm 3.1 under the  $\alpha$ -inverse strongly monotonicity.



**Theorem 3.2.** Let  $K(u)$  be a closed convex subset of a real Hilbert space  $H$ . Let  $T$  be an  $\alpha$ -inverse strongly monotonic mapping with constant  $\alpha > 0$  and  $S$  be a nonexpansive mapping such that  $F(S) \cap IWHE(H, T) \neq \emptyset$ . If

$$(27) \quad |\rho - \alpha| \leq \alpha(1 - \nu), \quad \nu \in (0, 1),$$

then the approximate solution obtained from Algorithm 3.1 converges strongly to  $z^* \in F(S) \cap IWHE(H, T)$ .

**Proof.** Let  $T$  be  $\alpha$ -inverse strongly monotone with the constant  $\alpha > 0$ , then  $T$  is  $\frac{1}{\alpha}$ -Lipschitzian continuous. Consider

$$(28) \quad \begin{aligned} & \|u_n - u^* - \rho[Tu_n - Tu^*]\|^2 \\ &= \|u_n - u^*\|^2 + \rho^2\|Tu_n - Tu^*\|^2 - 2\rho\langle Tu_n - Tu^*, u_n - u^* \rangle \\ &\leq \|u_n - u^*\|^2 + \rho^2\|Tu_n - Tu^*\|^2 - 2\rho\alpha\|Tu_n - Tu^*\|^2 \\ &= \|u_n - u^*\|^2 + (\rho^2 - 2\rho\alpha)\|Tu_n - Tu^*\|^2 \\ &\leq \|u_n - u^*\|^2 + (\rho^2 - 2\rho\alpha) \cdot \frac{1}{\alpha^2}\|u_n - u^*\|^2 \\ &= \left(1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2}\right) \|u_n - u^*\|^2 = \theta_2 \|u_n - u^*\|^2, \end{aligned}$$

where

$$(29) \quad \theta_2 = \left(1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2}\right)^{1/2}.$$

From (27), (28) and (29), we have

$$\begin{aligned} \|z_{n+1} - z^*\| &\leq (1 - a_n)\|z_n - z^*\| + a_n\|u_n - u^* - \rho(Tu_n - Tu^*)\| \\ &\leq (1 - a_n)\|z_n - z^*\| + a_n\theta_2\|u_n - u^*\| \\ &= [1 - a_n(1 - \theta_3)]\|z_n - z^*\|, \end{aligned}$$

where

$$\theta_3 = \frac{\sqrt{1 + \frac{\rho^2 - 2\rho\alpha}{\alpha^2}}}{1 - \nu} < 1, \quad \text{using (26).}$$

Therefore, it follows  $\lim_{n \rightarrow \infty} \|z_n - z^*\| = 0$  from Lemma 2.3, completing the proof.  $\square$

#### 4. COMPUTATIONAL ASPECTS

In this paper, we have shown that the variational inequalities are equivalent to a new class of Wiener-Hopf equations involving the nonexpansive operator. This equivalence is used to suggest and analyze an iterative method for finding the common element of set of the solutions of the variational inequalities and the set of the fixed-points of the nonexpansive operator. It is worth mentioning that Pitonyak, Shi and Schiller [30] and Noor, Wang and Xiu [28] used the Wiener-Hopf equations technique to develop some very efficient and numerically implementable iterative methods for solving variational inequalities and related optimization problems. The results are encouraging and perform better than other methods. It is interesting to use the techniques and ideas of this paper to develop other new iterative methods for solving the quasi variational inequalities involving the nonexpansive operators. This is another direction for future work.

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