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ON THE INSTABILITY OF SOLUTIONS TO A CERTAIN CLASS OF NON-AUTONOMOUS AND NON-LINEAR ORDINARY VECTOR DIFFERENTIAL EQUATIONS OF SIXTH ORDER

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Abstract. The aim of the present paper is to establish a new result, which guarantees the instability of zero solution to a certain class of non-autonomous ordinary differential equations of sixth order. Our result improves some known results in the literature for non- autonomous case (see, [20,Theorem 4], [31, Theorem 1]).

1. Introduction

Consider the non-autonomous and non-linear vector differential equation of sixth order:

$$
\begin{aligned}
(1)\qquad & X^{(6)}+AX^{(5)}+BX^{(4)}+C\dddot{X}+\Phi(t,X,\dot{X},\ddot{X},\dddot{X},X^{(4)},X^{(5)})\ddot{X}\\
& +\Psi(X)\dot{X}+H(t,X,\dot{X},\ddot{X},\dddot{X},X^{(4)},X^{(5)})X=0,
\end{aligned}
$$

in which $t\in\Re_+$, $\Re_+=[0,\infty)$ and $X\in\Re^n$; A , B and C are constant $n\times n$ -real symmetric matrices; Φ , Ψ and H are continuous $n\times n$ -symmetric real matrix functions depending, in each case, on the arguments shown in (1). Let $J\left(\Psi(X)X\left|X\right.\right)$ denote the linear operator from the matrix function $\Psi(X)$ to the matrix

$$
J\left(\Psi(X)X\left|X\right.\right)=\left(\frac{\partial}{\partial x_j}\sum_{k=1}^{n}\psi_{ik}x_k\right)=\Psi(X)+\left(\sum_{k=1}^{n}\frac{\partial\psi_{ik}}{\partial x_j}x_k\right),
$$

where $(x_1,x_2,...,x_n)$ and (ψ_{ik}) are components of X and Ψ , respectively. It is assumed that the matrix $J\left(\Psi(X)X\left|X\right.\right)$ exists and is symmetric and continuous. From the relevant literature, it can be followed that, so far, many problems about the instability of solutions of various scalar and vector linear or nonlinear differential equations of third-, fourth-, fifth-, sixth-, seventh and eighth order have been investigated by researchers. For some papers carried out on the topic, one can refer to the book of Reissig et al [15] and the papers of Bereketoğlu [2], Ezeilo ([3], [4], [5], [6], [7]), Liao and Lu [9], Li and Yu [10], Li and Duan [11], Lu and Liao [12], Lu [13], Sadek ([16], [17]), Skrapek ([18], [19]), Tejumola [20], Tiryaki ([21], [22], [23]), Tunç ([24], [25], [26], [27], [28], [29], [30], [31], [32]), C.Tunç and E. Tunç ([33], [34], [35], [36]), C. Tunç and H. Sevli [37], E. Tunç [38] and the references listed therein. Throughout all the papers mentioned above the Lyapunov's second

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(or direct) method [14] is used as a basic tool to prove the result established there, and it will be also used to verify our result, which will be given hereafter. The motivation for the present work has been inspired especially by the papers in [20], [31] and the papers mentioned above. It should be noted that in [20], Tejumola investigated the instability of the trivial solution of the following sixth order scalar nonlinear differential equation of the type

$$x^{(6)} + a_1x^{(5)} + a_2x^{(4)} + a_3\dddot{x} + \varphi_4(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)})\ddot{x}$$

$$+\varphi_5(x)\dot{x} + \varphi_6(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) = 0.$$

He proved a result on the subject. Recently, in [31], Tunç investigated the instability of the trivial solution of sixth order nonlinear vector differential equation of the form

$$X^{(6)}+AX^{(5)} + BX^{(4)} + C\dddot{X} + \Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})\ddot{X}$$

$$+\Psi(X)\dot{X} + H(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})X = 0.$$

Clearly, (1) is a non-autonomous differential equation, that is, (1) is different from the above equations and ones considered in the literature, see also the papers mentioned above.

Throughout this paper, the symbol $\langle X, Y\rangle$ is used to denote the usual scalar product in $\Re^n$, that is, $\langle X, Y\rangle = \sum\limits_{i=1}^{n} x_iy_i$, thus $\langle X, X\rangle = \|X\|^2$, and $\lambda_i(A)$, $(i = 1, 2, ..., n)$, are the eigenvalues of the $n \times n$ - matrix A .

We take into consideration, in place of (1), the equivalent differential system

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = S, \dot{S} = T, \dot{T} = U,$$

$$\dot{U} = -AU - BT - CS - \Phi(t, X, Y, Z, S, T, U)Z \tag{2}$$

$$-\Psi(X)Y - H(t, X, Y, Z, S, T, U)X,$$

which was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = S$, $X^{(4)} = T$, $X^{(5)} = U$ by (1).

2. Preliminaries

In order to reach our main result, we will give a basic theorem for the general non-autonomous differential system and two well-known lemmas which play an essential role in the proof of our main result. Consider the differential system

$$\dot{x} = f(t, x), x(t_0) = x_0, t \geq 0, \tag{3}$$

where $f \in C[R_+ \times S_{(\rho)},\ \Re^n]$ and $S_{(\rho)} = [x \in \Re^n : |x| < \rho]$. Assume, for convenience that a solution $x(t) = x(t, t_0, x_0)$ of (3) exists and is unique for $t \geq t_0$ and $f(t, 0) = 0$ so that we have trivial solution $x = 0$. Let K denote a class of the functions as $K = [\sigma \in C[[t_0, \rho), \Re_+]]$ such that $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$.

Now, we state the following fundamental instability theorem.

Theorem1. Assume that there exists a $t_0 \in \Re_+$ and an open set $U \subset S_{(\rho)}$ such that

$V \in C^1[[t_0, \infty) \times S_{(\rho)},\ \Re_+]$ for (t, x) from $[t_0, \infty) \times U$,

(i) $0 < V(t,x) \leq a(|x|)$, $a \in K$;

(ii) either $V'(t,x) \geq b(|x|)$, $b \in K$, $K = [\sigma \in C[[t_0, \rho), \Re_+]]$ such that $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$ or $V'(t,x) = CV(t,x) + \omega(t,x)$, where $C > 0$ and $\omega \in C[[t_0, \infty) \times U, \Re_+]$;

(iii) $V(t,x) = 0$ on $[t_0, \infty) \times (\partial U \bigcap S_{(\rho)})$, ∂U denotes boundary of U and $0 \in \partial U$.

Then the trivial solution $x = 0$ of system (3) is unstable.

Proof. See Lakshmikantham et al. [Theorem 1.1.9, 8].

Lemma 1. Let A be a real symmetric $n \times n$ -matrix and
$a' \geq \lambda_i(A) \geq a > 0$ $(i = 1, 2, ..., n)$, where a' , a are constants.
Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle$$

Proof. See Bellman[1].

Lemma 2. Let Q , D be any two real $n \times n$ commuting symmetric matrices. Then

(i) the eigenvalues $\lambda_i(QD)$, $(i = 1, 2, ..., n)$, of the product matrix QD are real and satisfy
$\max_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D)$;

(ii) the eigenvalues $\lambda_i(Q + D)$, $(i = 1, 2, ..., n)$, of the sum of matrices Q and D are real and satisfy
$\left\{ \max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D) \right\} \geq \lambda_i(Q + D) \geq \left\{ \min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D) \right\}$,
where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of matrices Q and D .

Proof. See Bellman[1].

3. Main result

We establish the following theorem:

Theorem 2. In addition to the basic assumptions imposed on A , B , C , Φ , Ψ and H that appeared in (2), we assume that the following conditions hold:There are constants a_1 , a_2 and a_5 such that
$\lambda_i(A) \geq a_1 > 0$, $\lambda_i(B) \leq a_2 < 0$, $|\lambda_i(\Psi(X)| \leq a_5$, ($a_5 > 0$),
and
$\lambda_i(H(t, X, Y, Z, S, T, U)) < \frac{1}{4a_2} [\lambda_i(\Phi(t, X, Y, Z, S, T, U))]^2$, $(i = 1, 2, ..., n)$,
for all $t \in \Re_+$ and X , Y , Z , S , T , $U \in \Re^n$.

Then the zero solution of the system (2) is unstable.

Remark. It should be noted that there is no restriction on the eigenvalues of the matrix C in the system (2).

Proof. To prove Theorem 2, we construct a scalar differentiable Lyapunov function $V_0 = V_0(t, X, Y, Z, S, T, U)$. This function, V_0 , is defined as follows:

$$\begin{aligned} V_0 = \; & \langle X, U\rangle + \langle X, AT\rangle + \langle X, BS\rangle + \langle X, CZ\rangle \\ & - \langle Y, T\rangle - \langle Y, AS\rangle - \langle Y, BZ\rangle + \langle Z, S\rangle - \tfrac{1}{2}\langle Y, CY\rangle \\ & + \tfrac{1}{2}\langle Z, AZ\rangle + \int_0^1 \langle \Psi(\sigma X)X, X\rangle \, d\sigma. \end{aligned}$$

Clearly, $V_0(t, 0, 0, 0, 0, 0, 0) = 0$ on $[t_0, \infty)$. Now, subject to the assumptions of Theorem 2, it is a straightforward calculation to see that

$$\begin{aligned} V_0(t, 0, 0, \varepsilon, \varepsilon, 0, 0) & = \tfrac{1}{2}\langle \varepsilon, A\varepsilon\rangle + \langle \varepsilon, \varepsilon\rangle \\ & \geq \tfrac{1}{2}\langle \varepsilon, a_1\varepsilon\rangle + \langle \varepsilon, \varepsilon\rangle \\ & = (\tfrac{1}{2}a_1 + 1)\left\|\varepsilon\right\|^2 > 0 \end{aligned}$$

for all arbitrary, $\varepsilon \neq 0$, $\varepsilon \in \Re^n$. In view of the function $V_0 = V_0(t, X, Y, Z, S, T, U)$, the assumptions of Theorem 2, the properties of symmetric matrices, Lemma 1, Lemma 2 and Cauchy-Schwarz inequality $|\langle X, Y\rangle| \leq \|X\| \; \|Y\|$, one can easily obtain that there is a positive constant K_1 such that

$$V_0(t, X, Y, Z, S, T, U) \leq K_1 \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|S\|^2 + \|T\|^2 + \|U\|^2\right).$$

These show that assumption (i) of Theorem 1 holds.

Now, let $(X, Y, Z, S, T, U) = (X(t), Y(t), Z(t), S(t), T(t), U(t))$ be an arbitrary solution of system (2). By an elementary differentiation along the solution paths of the system (2), it can be verified that

$$\begin{aligned} \dot{V}_0 = \tfrac{d}{dt}V_0(t, X, Y, Z, S, T, U) = \; & - \langle \Phi(t, X, Y, Z, S, T, U)Z, X\rangle \\ & - \langle H(t, X, Y, Z, S, T, U)X, X\rangle \\ & - \langle BZ, Z\rangle + \langle S, S\rangle - \langle \Psi(X)Y, X\rangle \\ & + \tfrac{d}{dt}\int_0^1 \langle \Psi(\sigma X)X, X\rangle \, d\sigma. \end{aligned} \tag{4}$$

Check that

$$\begin{aligned} \tfrac{d}{dt}\int_0^1 \langle \Psi(\sigma X)X, X\rangle \, d\sigma & = \int_0^1 \langle \Psi(\sigma X)X, Y\rangle \, d\sigma + \int_0^1 \langle \sigma J\left(\Psi(\sigma X)X \,|\sigma X\,\right)Y, X\rangle \, d\sigma \\ & = \int_0^1 \langle \Psi(\sigma X)X, Y\rangle \, d\sigma + \int_0^1 \sigma \langle J\left(\Psi(\sigma X)X \,|\sigma X\,\right)X, Y\rangle \, d\sigma \\ & = \int_0^1 \langle \Psi(\sigma X)X, Y\rangle \, d\sigma + \int_0^1 \sigma \tfrac{\partial}{\partial\sigma}\langle \Psi(\sigma X)X, Y\rangle \, d\sigma \\ & = \sigma \langle \Psi(\sigma X)X, Y\rangle \,\Big|_0^1 = \langle \Psi(X)X, Y\rangle \,. \end{aligned} \tag{5}$$

Combining the estimate (5) with (4), we deduce that

$$\dot{V}_0 = \begin{array}{l} -\langle \Phi(t,X,Y,Z,S,T,U)Z,X\rangle - \langle H(t,X,Y,Z,S,T,U)X,X\rangle \\ -\langle BZ,Z\rangle + \langle S,S\rangle \end{array}.$$

Hence, the assumptions of Theorem 2 and the fact $\langle S,S\rangle = \|S\|^2$ imply that

$$\begin{aligned} \dot{V}_0 \quad & \geq -\langle \Phi(t,X,Y,Z,S,T,U)Z,X\rangle - \langle H(t,X,Y,Z,S,T,U)X,X\rangle - a_2\langle Z,Z\rangle \\ & = -a_2\left\|Z + \tfrac{1}{2a_2}\Phi(t,X,Y,Z,S,T,U)X\right\|^2 - \langle H(t,X,Y,Z,S,T,U)X,X\rangle \\ & + \tfrac{1}{4a_2}\langle \Phi(t,X,Y,Z,S,T,U)X,\ \Phi(t,X,Y,Z,S,T,U)X\rangle \\ & \geq -\langle H(t,X,Y,Z,S,T,U)X,X\rangle \\ & + \tfrac{1}{4a_2}\langle \Phi(t,X,Y,Z,S,T,U)X,\Phi(t,X,Y,Z,S,T,U)X\rangle > 0. \end{aligned}$$

Thus, the assumptions of the theorem imply that $\dot{V}_0(t) \geq K_2\|X\|^2$ for all $t \geq 0$, where K_2 is a positive constant, say infinite inferior limit of the function $\dot{V}_0$. Besides, $\dot{V}_0 = 0$ $(t \geq 0)$ necessarily implies that $X = 0$ for all $t \geq 0$, and therefore also that $Z = \dot{Y} = 0$, $S = \ddot{Y} = 0$, $T = \dddot{Y} = 0$, $U = Y^{(4)} = 0$ for all $t \geq 0$. Hence

$$X = Y = Z = S = T = U = 0 \text{ for all} \geq 0.$$

Therefore, subject to the assumptions of the theorem the function V_0 has the entire the criteria of Theorem 1, Lakshmikantham et al. [Theorem 1.1.9, 8]. Thus, the basic properties of the function $V_0(t,X,Y,Z,S,T,U)$, which are proved just above verify that the zero solution of the system (2) is unstable, see also Lakshmikantham et al. [Theorem 1.1.9, 8]. The system of equations (2) is equivalent to differential equation (1) and the proof of Theorem 2 is now complete.

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