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TORIC FIBRATIONS AND MIRROR SYMMETRY

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Abstract. The relation between the quantum $\mathcal{D}$-modules of a smooth variety X and a toric bundle is studied here. We describe the relation completely when X is a semi-ample complete intersection in a toric variety. In this case, we obtain all the relations in the small quantum cohomology ring of the bundle.

1. Introduction and Goals

For a smooth, projective variety Y we denote by $Y_{k,\beta}$ the moduli stack of rational stable maps of class $\beta \in H_2(Y, \mathbb{Z})$ with k-markings (Fulton et al [8]) and $[Y_{k,\beta}]$ its virtual fundamental class (Behrend et al [3], Li et al [13]). Genus zero Gromov-Witten invariants are defined as appropriate integrals over $[Y_{k,\beta}]$. We let $e : Y_{1,\beta} \to Y$ be the evaluation map, ψ - the first chern class of the cotangent line bundle on $Y_{1,\beta}$ and $\mathrm{ft} : Y_{1,\beta} \to Y_{0,\beta}$ - the forgetful morphism.
The formal completion of an arbitrary ring $\mathcal{R}$ along the semigroup MY of the rational curves of Y is defined to be

$$\mathcal{R}[[q^\beta]] := \{ \sum_{\beta \in \mathrm{MY}} a_\beta q^\beta, \quad a_\beta \in \mathcal{A}, \quad \beta - \text{effective}\}. \tag{1}$$

where $\beta \in H_2(Y, \mathbb{Z})$ is *effective* if it is a positive linear combination of rational curves. For each β, the set of α such that α and $\beta - \alpha$ are both effective is finite, hence $\mathcal{R}[[q^\beta]]$ behaves like a power series. Alternatively, we may define

$$q^\beta := q_1^{d_1} \cdot ... \cdot q_k^{d_k} = \exp(t_1 d_1 + ... + t_k d_k)$$

where $\{d_1, d_2, ..., d_k\}$ are the coordinates of β relative to the dual of a nef basis $\{p_1, ..., p_k\}$ of $H^2(Y, \mathbb{Q})$..
Let $*$ denote the small quantum product of Y. The small quantum cohomology ring

$$(QH_s^* Y, *)$$

is a deformation of the cohomology ring $(H^*(Y, \mathbb{Q}[q^\beta]), \cup)$. Its structural constants are three point Gromov-Witten invariants of genus zero. Let $\hbar$ be a formal variable and

$$J_\beta(Y) := e_*\left(\frac{[Y_{1,\beta}]}{\hbar(\hbar - \psi)}\right) = \sum_{k=0}^{\infty} \frac{1}{\hbar^{2+k}} e_*(\psi^k \cap [Y_{1,\beta}]).$$

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The sum is finite for dimension reasons. For $t = (t_0, t_1, ..., t_k)$, let

$$tp := t_0 + \sum_{i=1}^{k} t_i p_i.$$

The $\mathcal{D}$-module for the quantum differential equation of Y

$$1 \leq i \leq k, \ \hbar\partial/\partial t_i = p_i *,$$

is generated by (Givental [10])

$$J(Y) = \exp\left(\frac{tp}{\hbar}\right) \sum_{\beta \in H_2(Y,\mathbb{Z})} q^\beta J_\beta(Y)$$

where we use the convention $J_0 = 1$. The generator $J(Y)$ encodes *all* of the genus zero, one marking Gromov-Witten invariants and gravitational descendants of Y. The generator $J(Y)$ is an element of the completion $H^*(Y, \mathbb{Q})[t][[q^\beta]]$ that may be used to produce relations in QH_s^*Y in the following way: let

$$\mathcal{P}(\hbar, \hbar\partial/\partial t_i, q_i)$$

be a polynomial differential operator where q_i and $\hbar$ act via multiplication and $q_i = e^{t_i}$ are on the left of derivatives. If

$$\mathcal{P}(\hbar, \hbar\partial/\partial t_i, q_i)J(Y) = 0$$

then

$$\mathcal{P}(0, p_i, q_i) = 0$$

is a relation in the small quantum cohomology ring QH_s^*Y.
If Y is a complete intersection in a toric variety, $J(Y)$ is related to an explicit hypergeometric series $I(Y)$ via a change of variables (Givental [8], Lian et al [12],[13]). Furthermore, if Y is Fano then the change of variables is trivial, i.e.

$$J(Y) = I(Y).$$

Since $I(Y)$ is known explicitly, this yields two immediate benefits.

(1) The one point Gromov-Witten invariants and gravitational descendants of Y are determined completely.
(2) Differential operators that annihilate $I(Y)$ are easy to find, hence producing relations in the small quantum cohomology ring of Y.

In this paper we seek to relativize these results for Fano toric bundles, hence extending the results of the papers Elezi [6],[7]

2. Toric Bundles and Mirror Theorems

Toric varieties and bundles. We follow the approach and the terminology of Oda [15]. Let $\mathbb{M} \simeq \mathbb{Z}^m$ be a free abelian group of rank m, $\mathbb{N} = \mathrm{Hom}(\mathbb{M}, \mathbb{Z})$ its dual, and $<,>: \mathbb{M} \times \mathbb{N} \mapsto \mathbb{Z}$ the pairing between them. Let Y be an m-dimensional smooth, toric variety determined by a fan $\Sigma \subset \mathbb{N} \otimes \mathbb{R}$. Denote by

$$\Sigma(1) = \{\rho_1, ..., \rho_m, \rho_{m+1}, ..., \rho_{r=m+k}\}$$

the one dimensional cones of Σ and $D_1, ..., D_r$ the corresponding toric divisors. Let $v_i = (v_{i1}, ..., v_{im})$ be the first lattice point along the ray ρ_i. Let

$$\{a_1, a_2, ..., a_k\}$$

with $a_j := (a_{1j}, a_{2j}, ..., a_{mj}, a_{m+1j}, ..., a_{rj})$ be a basis of the lattice of relations Λ between $v_1, ..., v_r$. There is a short exact sequence

$$0 \to \Lambda \to \mathbb{Z}^{\Sigma(1)} \xrightarrow{h} \mathbb{N} \to 0, \tag{2}$$

where $h(c_1, c_2, ...c_r) = c_1 v_1 + ... + c_r v_r$. The lattice Λ may be identified with $\mathrm{Hom}(A_{m-1}(Y), \mathbb{Z}) \simeq H_2(Y, \mathbb{Z})$. Under this isomorphism, a_{ij} is the intersection of a_j, when interpreted as a two dimensional cycle, with the toric divisor D_i. We choose a_j so that $\{a_1, ..., a_k\}$ is a generating set for the Mori cone of classes of effective curves. Then $a_{i1}, ..., a_{ik}$ are the coordinates of D_i with respect to the nef basis $\{p_1, ..., p_k\}$ dual to $\{a_1, ..., a_k\}$.

Assume that $\rho_1, ..., \rho_m$ generate a maximal dimensional cone in Σ. Since Y is smooth, $\{v_1, v_2, ..., v_m\}$ forms a $\mathbb{Z}$-basis of $\mathbb{N}$ and the absolute value of the matrix

$$(a_{ij});\ i = m+1, ..., r;\ j = 1, 2, ..., k$$

is 1.

The cohomology ring $H^*(Y, \mathbb{Z})$ is generated by the divisors $D_1, ..., D_r$ subject to the following two types of relations:

Type One: Whenever $\{\rho_{j_1}, ..., \rho_{j_s}\}$ do not generate a cone in Σ, the intersection

$$D_{j_1} \cdot ... \cdot D_{j_s} = 0. \tag{3}$$

Type Two: For each $1 \leq i \leq m$,

$$D_i = \sum_{j=1}^{k} a_{ij} p_j \tag{4}$$

From the short exact sequence (2) we obtain

$$0 \to \mathbb{T}^k \xrightarrow{\alpha} \mathbb{T}^r \xrightarrow{\beta} \mathbb{T}^m \to 0, \tag{5}$$

where the maps are defined as follows:

$$\alpha(t_1, t_2, ...t_k) = (\prod_{i=1}^{k} t_i^{a_{1i}}, ..., \prod_{i=1}^{k} t_i^{a_{ri}}),\ \beta(t_1, ...t_r) = (\prod_{i=1}^{r} t_i^{v_{i1}}, ..., \prod_{i=1}^{r} t_i^{v_{im}}).$$

Let $Z(\Sigma) \subset \mathbb{C}^r$ be the variety whose ideal is generated by the products of those variables which do *not* generate a cone in Σ. The toric variety Y is the geometric quotient (Cox [5])

$$\mathbb{C}^r - Z(\Sigma)//\mathbb{T}^k$$

where the torus acts as follows

$$t \cdot x = \left(\prod_{i=1}^{k} t_i^{a_{1i}} x_1, ..., \prod_{i=1}^{k} t_i^{a_{ri}} x_r \right). \tag{6}$$

The short exact sequence (5) yields an action of the quotient $\mathbb{T} := \mathbb{T}^m$ on Y.

The first chern class of the tangent bundle to Y is equal to

$$\sum_{i=1}^{r} D_i = \sum_{i=1}^{k} n_i p_i.$$

The toric variety Y is Fano iff $n_i > 0$ for all i.

We relativize the previous construction as follows. Consider the principal $\mathbb{T}$-bundle

$$\mathbb{E} := \oplus_{i=1}^{m}(L_i - \{0\}) \to X,$$

where L_i are line bundles over a smooth, projective variety X. Let $\mathbb{T}$ act fibrewisely on $\mathbb{E}$ and the diagonally on the first m-homogeneous coordinates of Y. The quotient space

$$Y(\mathbb{E}) := \mathbb{E} \times_{\mathbb{T}} Y$$

is a toric bundles over X with fiber isomorphic to Y. The bundle $Y(\mathbb{E})$ inherits a $\mathbb{T}$-action.
There is a projection map $\pi : Y(\mathbb{E}) \to Y$. The maximal cone generated by $\{\rho_1, \rho_2, ..., \rho_m\}$ determines a $\mathbb{T}$ fixed point q in Y whose homogeneous coordinates are $(0, 0, ..., 0, 1, 1, ...1)$. In the relativized setting, the $\mathbb{T}$-equivariant inclusion

$$q \hookrightarrow Y$$

yields a map

$$q(\mathbb{E}) \simeq X \overset{s}{\hookrightarrow} Y(\mathbb{E})$$

which is a section of π. This is also a fixed point component for the action of $\mathbb{T}$ on $Y(\mathbb{E})$. The other $\mathbb{T}$-fixed points of Y yield sections of π and these are all the fixed point components.
Toric divisors lift to divisors in $Y(\mathbb{E})$; these liftings will be denoted by the same letter in this paper. It was shown in Sankaran and Uma [17] that the two types of relations (3) and (4) lift in a natural way in $H^*(Y(\mathbb{E}), \mathbb{Z})$; namely

$$D_{j_1} \cdot ... \cdot D_{j_s} = 0$$

whenever $\{\rho_{j_1}, ..., \rho_{j_s}\}$ do not generate a cone in Σ, and

$$D_i = \sum_{j=1}^{k} a_{ij} p_j + c_1(L_i)$$

for each $1 \leq i \leq m$, where as in the case of $H^*(Y, \mathbb{Z})$ the divisors

$$p_1, ..., p_k$$

generate freely $H^*(Y(\mathbb{E}), \mathbb{Z})$. In fact, there is a simple relation between the $\mathbb{T}$-equivariant cohomology of Y and the cohomology of $Y(\mathbb{E})$ which will be used throughout this paper. Recall, that the rational cohomology of the classifying space $B\mathbb{T}$ is $\mathbb{Q}[\lambda_1, ...\lambda_m]$ where λ_i is the first chern class of the equivariant line bundle corresponding to the character

$$\nu_i : \mathbb{T} \to \mathbb{C}^* \ \nu_i(t_1, ..., t_m) = t_i.$$

A relation in the equivariant cohomology ring of Y becomes a relation in $H^*(Y(\mathbb{E}))$ after substituting $c_1(L_i)$ for λ_i.

We may assume that $L_i = \mathcal{O}_X, i > m$ without loss of generality. This is due to the fact that $\rho_1, ..., \rho_m$ generate a maximal cone in Σ.

The quantum $\mathcal{D}$-module structure of a toric bundle. The generator J of a quantum $\mathcal{D}$-structure is weighted by the lattice points of the Mori cone. Hence we first study the relation between the Mori cones of Y and $Y(\mathbb{E})$.

Lemma 1. *If L_i^* are generated by global sections, then the liftings of the nef divisors $p_1, ..., p_k$ in $Y(\mathbb{E})$ are also nef. Furthermore, the Mori cone of $Y(\mathbb{E})$ is a direct sum of the Mori cone of X, embedded via the section s, and the Mori cone of the fiber Y.*

Proof. In toric varieties, every nef divisor p is generated by global sections (Oda [14]). Let $x_1, x_2, ..., x_r$ be homogeneous coordinates in Y. The vector space of global sections $H^0(\mathcal{O}(p))$ has a monomial basis

$$\prod_{i=1}^{r} x_i^{m_i}.$$

Let $\{\phi_{ij}\}$ be a collection of generating sections for the line bundles L_i^*. The "monomials"

$$\prod_{i=1}^{r} (x_i \phi_{ij})^{m_i}$$

are generating sections the line bundle

$$\prod_{i=1}^{r} (\mathcal{O}(D_i) \otimes (L_i^*))^{m_i}$$

which is isomorphic to $\mathcal{O}(p)$ in $Y(\mathbb{E})$. Thus p lifts to a nef divisor in $Y(\mathbb{E})$.
This shows that the addition of $p_1, ..., p_k$ to a nef basis $\{p_{k+1}, ..., p_l\}$ of X yields a nef basis

$$\{p_1, ..., p_l\}$$

of $Y(\mathbb{E})$. Now for a curve $C \subset Y(\mathbb{E})$ we have

$$\pi_* \left([C] - s_*(\pi_*([C]))\right) = 0.$$

Notice that the restrictions of the divisors $p_1, p_2, ..., p_k$ in the section $q(\mathbb{E})$ are all zero since they may be written as $\mathbb{Z}$-linear combinations of $D_{m+1}, ..., D_{m+k}$. Hence $\forall i = 1, 2, ..., k,\ p_i \cdot ([C] - s_*(\pi_*([C]))) \geq 0$ and we have a unique decomposition

$$[C] = s_*(\pi_*([C])) + [C'],$$

where $[C']$ and $\pi_*([C])$ are curve classes respectively in the fiber of π and X. □

We introduce a "mixed" $I(Y(\mathbb{E}))$ that admits contributions from both $J(X)$ and an $\mathbb{E}$-twisted $J(Y)$. Let (ν, d) denote a curve class in the Mori cone of $Y(\mathbb{E})$, with ν a curve class in the fiber of π and d a curve class in X.
Define

$$I(Y(\mathbb{E})) := \exp\left(\frac{tp}{\hbar}\right) \sum_{(d,\nu)} {q_1}^{\nu} {q_2}^{d} \prod_{i=1}^{m} \frac{\prod_{m=0}^{\infty}(D_i + m\hbar)}{\prod_{m=0}^{D_i(\nu,d)}(D_i + m\hbar)} \pi^*(J_d(X)).$$

If X is a point then $Y(\mathbb{E}) = Y$. Furthermore, as mentioned in the introduction $J(Y) = I(Y)$ if Y is a Fano toric variety. In this paper we show that the same holds for the relativized $Y(\mathbb{E})$.

Proposition 1. *If X is a semi-ample complete intersection in a toric variety, and both Y and $Y(\mathbb{E})$ are Fano, then $J(Y(\mathbb{E})) = I(Y(\mathbb{E}))$.*

Proposition 1 will follow as a corollary of another statement which we now formulate and prove.
Let Z be a toric variety, $\tilde{L}_i,\ i=0,1,...,n$ toric line bundles over Z and $\tilde{\mathbb{E}}=\oplus_{i=0}^n \tilde{L}_i$. The bundle

$$\pi : Y(\tilde{\mathbb{E}}) \to Z$$

is also a toric variety (Oda [15]). The edges of the fan for $Y(\tilde{\mathbb{E}})$ corresponds to the liftings $B_1,...,B_r$ to $Y(\mathbb{E})$ of the toric base divisors $b_1,...,b_r$ and the divisors D_i from Y.
Let $\mathcal{L}_a : a = 1,2,...,l$ be globally generated line bundles over Z and X the zero locus of a generic section s of

$$V = \oplus_{a=1}^l \mathcal{L}_a.$$

Such an X will be called *a semi-ample complete intersection.* Denote by L_i and $\mathbb{E}$ the restrictions of $\tilde{L}_i$ and $\tilde{\mathbb{E}}$ to X. The total space of $Y(\mathbb{E})$ is easily seen to be the zero locus of the section $\pi^*(s)$ of the pull back bundle $\pi^*(V)$.

Assume that the line bundles $\tilde{L}_i^*$ are globally generated and $-K_Z - \sum_{a=1}^l c_1(\mathcal{L}_a) + \sum_{i=0}^n c_1(\tilde{L}_i)$ is ample. (This will ensure that the conditions of Proposition 1 for the bundle $Y(\mathbb{E})$ over X are satisfied.)

Let V_d be the bundle on $Z_{1,d}$ whose fiber over the moduli point (C, x_1, f) is $\oplus_a H^0(f^*(\mathcal{L}_a))$. Denote by s_V its canonical section induced by s, i.e.

$$s_V((C, x_1, f)) = f^*(s).$$

The stack theoretic zero section of s_V is the disjoint union

$$Z(s_V) = \coprod_{i_*(\beta)=d} X_{1,\beta}. \tag{7}$$

The map $i_* : H_2X \to H_2Z$ is not injective in general, hence the zero locus $Z(s_V)$ may have more then one connected component. An example is the quadric surface in $\mathbb{P}^3$. The sum of the virtual fundamental classes $[X_{1,\beta}]$ is the refined top Chern class of V_d with respect to s_V.
Let $\tilde{V}_{\nu,d}$ and $\tilde{s}_V$ be the pull backs of V_d and s_V via the stack morphism

$$Y(\tilde{\mathbb{E}})_{1,(\nu,d)} \to Z_{1,d}.$$

The zero section of $\tilde{s}_V$ is the disjoint union

$$z(\tilde{s}_V) = \coprod_{i_*(\beta)=d} Y(\mathbb{E})_{1,(\nu,\beta)}.$$

It follows that

$$\sum_{i_*(\beta)=d} [Y(\mathbb{E})_{1,(\nu,\beta)}] = c_{\text{top}}(\tilde{V}_{\nu,d}) \cap [Y(\tilde{\mathbb{E}})_{1,(\nu,d)}].$$

Recall that the nef basis $\{p_1, p_2, ..., p_k, p_{k+1}, ...p_l\}$ of $Y(\mathbb{E})$ is obtained by completing a nef basis $\{p_{k+1}, ..., p_l\}$ of X. We will use tp to denote both $\sum_{i=1}^l t_i p_i$ and $\sum_{i=k+1}^l t_i p_i$. The difference will be clear from the context.
Consider the following generating functions

$$J^V(Y(\tilde{\mathbb{E}})) = \exp\left(\frac{tp}{\hbar}\right) \sum_{(\nu,d)} {q_1}^{\nu} {q_2}^{d} e_* \left(\frac{c_{\text{top}}(\tilde{V}_{\nu,d}) \cap [Y(\tilde{\mathbb{E}})_{1,(\nu,d)}]}{\hbar(\hbar - c)}\right)$$

and

$$\tilde{I}^V(Y(\tilde{\mathbb{E}})) = \exp\left(\frac{tp}{\hbar}\right) \sum_{(\nu,d)} q_1^{\nu} {q_2}^d \; \Omega_{\nu,d} \; \pi^* e_* \left(\frac{c_{\rm top}(V_d) \cap [Z_{1,d}]}{\hbar(\hbar - c)}\right),$$

where

$$\Omega_{\nu,d} = \prod_{i=1}^{m} \frac{\prod_{m=0}^{\infty}(D_i + m\hbar)}{\prod_{m=0}^{D_i(\nu,d)}(D_i + m\hbar)}.$$

Proposition 2. *If $-K_Y - \sum_{a=1}^{l} c_1(\mathcal{L}_a) - \sum_{i=0}^{n} c_1(\tilde{L}_i)$ is ample then*

$$J^V((\tilde{\mathbb{E}})) = \tilde{I}^V(Y(\tilde{\mathbb{E}}))$$

Proof. Let

$$I_d^V(Z) = \prod_a \frac{\prod_{m=-\infty}^{\mathcal{L}_a(d)}(\mathcal{L}_a + m\hbar)}{\prod_{m=-\infty}^{0}(\mathcal{L}_a + m\hbar)} \prod_i \frac{\prod_{m=-\infty}^{0}(B_i + m\hbar)}{\prod_{m=-\infty}^{B_i(d)}(B_i + m\hbar)}.$$

From Givental [9], Lian et al [12], Lian et al [13] we know that $J^V(Y(\tilde{E}))$ is related via a mirror transformation to

$$I^V(Y(\tilde{\mathbb{E}})) = \exp\left(\frac{tp}{\hbar}\right) \cdot \sum {q_1}^{\nu} {q_2}^d \Omega_{\nu,d} I_d^V(Z).$$

Likewise

$$J^V(Z) = \exp\left(\frac{tp}{\hbar}\right) \sum {q_2}^d e_* \left(\frac{c_{\rm top}(V_d) \cap [Z_{1,d}]}{\hbar(\hbar - c)}\right)$$

is related to

$$I^V(Z) = \exp\left(\frac{tp}{\hbar}\right) \sum {q_2}^d I_d^V(Z).$$

Since $-K_{Y(\tilde{E})} - \sum_a c_1(\mathcal{L}_a)$ and $-K_Z - \sum_a c_1(\mathcal{L}_a)$ are ample, the mirror transformations are particularly simple. Indeed, both series can be written as power series of $\hbar^{-1}$ as follows:

$$I^V(Y(\tilde{E})) = 1 + \frac{P_1(q_1, q_2)}{\hbar} + o(\hbar^{-1}), \; I^V(Z) = 1 + \frac{P_2(q_2)}{\hbar} + o(\hbar^{-1}),$$

where $P_1(q_1, q_2), P_2(q_2)$ are both polynomials supported respectively in

$$\Lambda_1 := \{(\nu, d) \mid (-K_{Y(\tilde{E})} - \sum c_1(\mathcal{L}_a)) = 1; \; D_j \geq 0, \; \forall j; \; B_i \geq 0, \; \forall i\},$$

and

$$\Lambda_2 := \{d \mid (-K_Z - \sum c_1(\mathcal{L}_a)) = 1; B_i \geq 0 \; \forall i\}.$$

Then

$$J^V(Y(\tilde{E})) = \exp\left(\frac{-P_1(q_1, q_2)}{\hbar}\right) I^V(Y(\tilde{E}))$$

and

$$J^V(Z) = \exp\left(\frac{-P_2(q_2)}{\hbar}\right) I^V(Z).$$

Simple algebraic manipulations show that

- $c_1(\tilde{L}_j) \cdot d = 0, \forall d \in \Lambda_2, \forall j = 1, 2, ..., n$
- $\Lambda_1 = \{(0, d) \mid d \in \Lambda_2\}$.

It follows that $\Omega_{0,d} = 1, \forall d \in \Lambda_2$ hence $P_1(q_1, q_2) = P_2(q_2)$. Notice also that if we expand

$$\exp\left(\frac{-P_2(q_2)}{\hbar}\right) = \sum_{\alpha} c_\alpha q_2^\alpha$$

then

$$c_1(\tilde{L}_j) \cdot \alpha = 0, \forall j = 1, 2, ..., n.$$

Hence for each $(\nu, d) \in M\mathbb{P}(\tilde{V})$ we have $\Omega_{\nu,d} = \Omega_{\nu,d+\alpha}$. Now the proposition follows easily. □

Proof. of Proposition 1. We know return to the proof of Proposition 1. Recall that the map

$$i_* : H_2(X) \to H_2(Z) \tag{8}$$

is not necessarily injective in general. If it is, then

$$[X_{1,\beta}] = c_{\text{top}}(V_{i_*(\beta)}) \cap [Y_{1,i_*(\beta)}]$$

and

$$[Y(\mathbb{E})_{1,(\nu,\beta)}] = c_{\text{top}}(\tilde{V}_{\nu,i_*(\beta)}) \cap [Y(\tilde{\mathbb{E}})_{1,(\nu,i_*(\beta))}].$$

In this case one can easily show that

$$i_*(J_{\nu,\beta}(Y(\mathbb{E}))) = J^V_{\nu,i_*(\beta)}(Y(\tilde{\mathbb{E}}))$$

and

$$i_*(I_{\nu,\beta}(Y(\mathbb{E}))) = \tilde{I}^V_{\nu,i_*(\beta)}(Y(\tilde{\mathbb{E}})).$$

Proposition 2 shows that Proposition 1 holds for complete intersection in toric varieties for which the map (8) is injective.

□

3. Lifting the Quantum Cohomology Structure

In this section we use Proposition 1 to study small quantum cohomology ring of $Y(\mathbb{E})$. As explained in the introduction, some of the relations in the small quantum cohomology ring come from differential operators.

Proposition 3. *Whenever Proposition 1 holds, quantum differential operators of X may be lifted in $Y(\mathbb{E})$, while the quantum differential operators of the fiber Y may be extended to $Y(\mathbb{E})$. Both types of operators produce relations in the quantum cohomology $QH^*_sY(\mathbb{E})$.*

Proof. Recall that $D_i = \sum a_{ij}p_j$. Let

$$c_1(L_i) = \sum_{j=k+1}^{l} c_{ij}p_j, \ i = 0, 1, ..., n.$$

Recall that the nef basis $\{p_1, p_2, ..., p_k, p_{k+1}, ...p_l\}$ of $H^2(Y(\mathbb{E}), \mathbb{Z})$ is obtained by completing a nef basis $\{p_{k+1}, ..., p_l\}$ of X. Let

$$\mathcal{P}(\hbar, \hbar\partial/\partial t_{k+1}, ..., \hbar\partial/\partial t_l, q_2) = \sum_{\alpha \in \Lambda} q_2^\alpha \mathcal{P}_\alpha$$

be a polynomial differential operator with Λ a finite subset of the Mori cone of X. Suppose that

$$0 = \mathcal{P}J(X) = \sum_{\alpha\in\Lambda} q_2^\alpha \sum_\beta \mathcal{P}_\alpha \left(\exp(\frac{pt}{\hbar})q_2^\beta\right) J_\beta(X)$$

$$= \sum_{\alpha\in\Lambda} q_2^\alpha \sum_\beta c_{\alpha,\beta}\exp(\frac{pt}{\hbar})q_2^\beta J_\beta(X) = \exp(\frac{pt}{\hbar}) \sum_{\alpha\in\Lambda,\beta} q_2^{\alpha+\beta} c_{\alpha,\beta} J_\beta(X).$$

Let

$$\delta_\alpha = \prod_{i=1}^{n} \prod_{r_i=0}^{-L_i\cdot\alpha-1} (\sum_{j=1}^{k} a_{ij}\hbar\frac{\partial}{\partial t_j} + \sum_{j=k+1}^{l} c_{ij}\hbar\frac{\partial}{\partial t_j} - r_i\hbar),\ \tilde{\mathcal{P}} = \sum_{\alpha\in\Lambda} q_2^\alpha \delta_\alpha \mathcal{P}_\alpha,$$

with the convention that if

$$L_i(\alpha) = 0,$$

the factors of δ_α corresponding to L_i are missing. Notice that

$$L_{n+1}(\alpha) = ... = L_m(\alpha) = 0$$

since we have chosen L_i to be trivial for $i > n$. We compute

$$\tilde{\mathcal{P}}J(Y(\mathbb{E})) = \sum_{\alpha\in\Lambda} q_2^\alpha \delta_\alpha \sum_{\nu,\beta} \mathcal{P}_\alpha \left(q_2^\beta \exp(\frac{pt}{\hbar})\right) q_1^\nu \Omega_{\nu,\beta} J_\beta =$$

$$\sum_{\alpha\in\Lambda} q_2^\alpha \delta_\alpha \sum_{\nu,\beta} c_{\alpha,\beta}\exp(\frac{pt}{\hbar}) q_1^\nu q_2^\beta \Omega_{\nu,\beta} J_\beta.$$

One can easily show that

$$\delta_\alpha \left(\exp(\frac{pt}{\hbar}) q_1^\nu q_2^\beta \Omega_{\nu,\beta}\right) = \exp(\frac{pt}{\hbar}) q_1^\nu q_2^\beta \Omega_{\nu,\alpha+\beta}.$$

It follows that

$$\tilde{\mathcal{P}}J(Y(\mathbb{E})) = \exp(\frac{pt}{\hbar}) \sum_\nu q_1^\nu \sum_{\alpha\in\Lambda,\beta} c_{\alpha,\beta} q_2^{\alpha+\beta} \Omega_{\nu,\alpha+\beta} J_\beta(X) = 0.$$

Hence the relation $\mathcal{P}(0, p_{k+1}, ..., p_l, q_2) = 0$ in QH_s^*X lifts into the relation

$$\mathcal{P}(0, p_{k+1}, ..., p_l, q_2 \prod_{i=1}^{n} D_i) = 0$$

in $QH_s^*Y(\mathbb{E})$, where

$$\left(\prod_{i=1}^{n} D_i\right)^\alpha := \prod_{i=1}^{n} D_i^{-L_i(\alpha)}, \forall\alpha \in MX.$$

For a curve class ν in the fiber of π, consider the following differential operator

$$\Delta_\nu(\hbar\frac{\partial}{\partial t_1}, ..., \hbar\frac{\partial}{\partial t_l}, q_j) := \prod_{i:D_i(\nu)>0} \prod_{m=0}^{D_i(\nu)-1} (\sum_{j=1}^{k} a_{ij}\hbar\frac{\partial}{\partial t_j} - \sum_{j=k+1}^{l} c_{ij}\hbar\frac{\partial}{\partial t_j} + m\hbar)$$

$$-q^\nu \prod_{i:D_i(\nu)<0} \prod_{m=0}^{-D_i(\nu)-1} (\sum_{j=1}^{k} a_{ij}\hbar\frac{\partial}{\partial t_j} - \sum_{j=k+1}^{l} c_{ij}\hbar\frac{\partial}{\partial t_j} + m\hbar).$$

It is easy to show that it satisfies

$$\Delta_\nu J(Y(\mathbb{E})) = 0.$$

It follows that

$$\Delta_\nu(p_1, ..., p_l, q_j) = 0$$

in $QH_s^*Y(\mathbb{E})$, i.e.

$$\prod_{i=1}^{r} D_i^{D_i(\nu)} = q^\nu.$$

These are precisely the extensions to $Y(\mathbb{E})$ of the small quantum cohomolgy relations of the fiber Y.

□

Sometimes *all* the relations in QH_s^*X come from quantum differential operators, hence QH_s^*X pulls back to $QH_s^*Y(\mathbb{E})$. This is the case when X is a Fano toric variety. The results of this section yield a complete description of $QH^*Y(E)$ which generalizes previous results of Costa et al [4] and Qin et al [15] and Givental [9].

4. The General (Nontoric) Case

We believe that Proposition 1 holds for any X. On one end, the equality of the $d = 0$ terms in $J(Y(\mathbb{E})) = I(Y(\mathbb{E}))$ is easy to establish. Indeed, the relative Gromov-Witten theory of the Y-bundle over $B\mathbb{T}$ associated with the universal bundle $E\mathbb{T} \mapsto B\mathbb{T}$ is precisely the $\mathbb{T}$-equivariant GW theory of Y (Astashkevich and Sadov [1]). The latter pulls back under the classifying map $X \mapsto B\mathbb{T}$ to the relative GW theory of $Y(\mathbb{E})$ over X. It follows that the restriction of $J(Y(\mathbb{E}))$ to $\nu = 0$ is obtained by substituting $c_1(L_i)$ for λ_i in $J^{\mathbb{T}}(Y)$. Since Y is assumed to be Fano, the generator $J^{\mathbb{T}}(Y)$ is known (see for example [8]) and the substitution $c_1(L_i) \mapsto \lambda_i$ is easily seen to yield the desired equality. At the other end, the $\nu = 0$ equality follows as an application of the equivariant quantum Lefshetz principle for the action of a torus on the fibers of $Y(\mathbb{E})$. The fixed point component relevant for the equivariant and localization considerations ([12]) consists of the maps that land in the section $s(X)$. The top chern class of the virtual normal bundle for this component is that of the $\mathbb{H}^1$-bundle for $\oplus_{i=1}^m L_i$. Calculations are easy to carry out (see for example Elezi [7]).

References

[1] A. Astashkevich and V. Sadov, *Quantum Cohomology of Partial Flag manifolds*, Comm. Math. Phys. **170** (1995) no 3, 503-528.

[2] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23**, 1-28.

[3] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. 128(1): 45-88, 1997.

[4] L. Costa and R. M. Mir-Roig, *Quantum cohomology of projective bundles over* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, Internat. J. Math. 11 (2000), **6**, 761-797.

[5] D. Cox, *Homogeneous Coordinate Ring of a Toric Variety*, J. Algebraic Geom. **4** 1995, 17-50.

[6] A. Elezi, *A Mirror Conjecture for Projective Bundles*, Internat. Math. Res. Notices **55** 2005, 3445-3458.

[7] A. Elezi, *Mirror Symmetry and Quantum Cohomology for Projective Bundles*, Int. J. Pure Appl. Math. (2007) **36**, no. 1, 75-86.

[8] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, in *Proceedings of symposia in pure mathematics: Algebraic geometry Santa Cruz 1995*, **62** Part 2, 45-96.

[9] A. Givental, *A mirror theorem for toric complete intersections*, in *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Progr. Math., **160**, Birkhăuser, 1998 141-175.
[10] A. Givental, *Equivariant Gromov-Witten invariants*, Int. Math. Res. Notices **13** (1996), 613-663.
[11] A. Givental, *Homological Geometry and Mirror Symmetry*, in *Proceedings of the ICM*, Zürich 1994, Birkhäuser, 1995, v.1, 472-480.
[12] B. Lian, K. Liu, and S.-T.Yau, *Mirror principle II*, Asian J. Math **3** (1999), no. 1, 109-146.
[13] B. Lian, K. Liu, and S.-T.Yau, *Mirror principle III*, Asian J. Math, **3** (1999), no. 4, 771-800.
[14] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, Jour. AMS 11 (1998), no 1, 119-174.
[15] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, Berlin, 1998.
[16] Zh. Qin and Y. Ruan, *Quantum cohomology of projective bundles over* $\mathbf{P}^n$, Trans. Amer. Math. Soc. 350 (1998), no. 9, 3615–3638.
[17] P. Sankaran and V. Uma, *Cohomology of toric bundles*, Comm. Math. Helv. **78**, (2003), 540-554.

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