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LIMITS OF α-HARMONIC MAPS

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Abstract

Critical points of approximations of the Dirichlet energy à la Sacks-Uhlenbeck are known to converge to harmonic maps in a suitable sense. However, we show that not every harmonic map can be approximated by critical points of such perturbed energies. Indeed, we prove that constant maps and maps of the form $u^R(x) = Rx$, $R \in O(3)$, are the only critical points of E_α for maps from S^2 to S^2 whose α-energy lies below some threshold. In particular, nontrivial dilations (which are harmonic) cannot arise as strong limits of α-harmonic maps.

1. Introduction

Let (M^2, g) and (N^n, h) be smooth, compact Riemannian manifolds without boundary and let N be isometrically embedded into some $\mathbb{R}^k$. (The dimension of M is two and that of N is arbitrary.) For every $u \in W^{1,2}(M, N)$ the Dirichlet energy $E(u)$ is defined by

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2 \, dA_M = \int_M e(u)\, dA_M, \tag{1.1}$$

where $e(u) = \frac{1}{2}|\nabla u|^2$ is the energy density of u.

In a pioneering paper, [9], Sacks and Uhlenbeck introduced, for every $\alpha > 1$ and every $u \in W^{1,2\alpha}(M, N)$, the functional $E_\alpha(u) = \frac{1}{2}\int_M (1 + |\nabla u|^2)^\alpha \, dA_M$. For us, it shall be more convenient to define

$$E_\alpha(u) = \frac{1}{2}\int_M (2 + |\nabla u|^2)^\alpha \, dA_M. \tag{1.2}$$

Critical points of E_α are called α-harmonic maps and they solve the elliptic system

$$\operatorname{div}\Big((2 + |\nabla u|^2)^{\alpha-1}\nabla u\Big) + (2 + |\nabla u|^2)^{\alpha-1} A(u)(\nabla u, \nabla u) = 0, \tag{1.3}$$

where A is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^k$. Critical points of E_α are smooth (see [9]) and therefore we can carry out the differentiation in equation (1.3) to get

$$\Delta u + A(u)(\nabla u, \nabla u) = -2(\alpha - 1)(2 + |\nabla u|^2)^{-1}\langle \nabla^2 u, \nabla u\rangle \nabla u. \tag{1.4}$$

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By a remarkable result of Hélein, [7], critical points of E also turn out to be smooth and satisfy

$$\Delta u + A(u)(\nabla u, \nabla u) = 0.$$

In [9], Sacks and Uhlenbeck showed that, as $\alpha \downarrow 1$, a sequence of α-harmonic maps with uniformly bounded energy converges, away from a finite (possibly empty) set of points $p_1, \dots, p_\ell$, to a harmonic map from M to N. Furthermore, non-trivial bubbles (harmonic maps from the two-sphere S^2) develop at each of $p_1, \dots, p_\ell$. (This is far from a precise statement of the convergence that occurs but it suffices for our purposes.) It would be useful to associate a Morse index to a harmonic map with bubbles. An α-harmonic map has a well-defined Morse index (see e.g. [8], [12]) and so, it seems worthwhile to investigate whether every harmonic map from a surface can be captured by the Sacks-Uhlenbeck limiting process. We shall show that this is not the case, even when M and N are the round unit two-sphere $S^2 \subset \mathbb{R}^3$.

In this case the equation (1.4) simplifies to

$$\Delta u + u|\nabla u|^2 = -2(\alpha - 1)(2 + |\nabla u|^2)^{-1}\langle \nabla^2 u, \nabla u\rangle \nabla u. \tag{1.5}$$

For $u\colon S^2 \to S^2$ we can define the degree of u by

$$\deg(u) = \frac{1}{4\pi}\int_{S^2} J(u)\, dA_{S^2}, \tag{1.6}$$

where

$$J(u) = u \cdot e_1(u) \wedge e_2(u)$$

is the Jacobian of u, and (e_1, e_2) stands for a local oriented orthonormal frame of TS^2. For every $u \in W^{1,2\alpha}(S^2, S^2)$ with $\deg(u) = 1$ we can estimate

$$\begin{aligned} 8\pi &= \int_{S^2} (1 + J(u))\, dA_{S^2} \\ &\leqslant \int_{S^2} (1 + e(u))\, dA_{S^2} \\ &\leqslant (2^{1-\alpha} E_\alpha(u))^{\frac{1}{\alpha}} (4\pi)^{\frac{\alpha-1}{\alpha}}. \end{aligned} \tag{1.7}$$

Hence we get

$$E_\alpha(u) \geqslant 2^{2\alpha+1}\pi \tag{1.8}$$

for every u as above. On the other hand we have for every $R \in SO(3)$ that the map $u^R(x) = Rx$ satisfies

$$E_\alpha(u^R) = 2^{2\alpha+1}\pi. \tag{1.9}$$

From (1.7) it follows that equality in this estimate is attained only for conformal maps u with constant energy density equal to 1. Hence the rotations are the only minimisers of E_α among all maps with degree 1.

By contrast we have the following theorem due to Wood and Lemaire (see (11.5) in [5]).

Theorem 1.1. *The harmonic maps between 2-spheres are precisely the rational maps and their complex conjugates (i.e., rational in z or $\bar{z}$).*

In particular, a rational map u has energy given by $E(u) = 4\pi|\deg(u)|$, which is the least energy that a map of this degree can have. As we shall discuss more fully in a moment, the rational maps of degree one include nontrivial dilations which are not minimisers, indeed not even critical points, of the E_α energy for $\alpha > 1$.

Theorem 1.2. *There exists $\varepsilon > 0$ and $\overline{\alpha} - 1 > 0$ small such that the only critical points u_α of E_α which satisfy $E_\alpha(u_\alpha) \leqslant 2^{2\alpha+1}\pi + \varepsilon$ and $\alpha \leqslant \overline{\alpha}$ are the constant maps and maps of the form $u^R(x) = Rx$, $R \in O(3)$.*

Remark 1.3. An upper bound on the energy is necessary in order to deduce the conclusions of Theorem 1.2. In Section 8 we will construct critical points of E_α of degree one that have large energy and that are not rotations.

Our proof of Theorem 1.2 goes as follows. After recalling some basic formulas for the Möbius group in Section 2, we prove in Section 3 that maps with low enough E_α energy must stay close in $W^{1,2}$ to some Möbius map. We then improve this result in Section 4 for critical points of E_α (with low energy), where we show closeness (after a conformal pull-back) to the identity in $W^{2,p}$, $p \in [1, \frac{3}{2}]$.

In Section 5 we show that elements in the Möbius group that are close to u_α as in Theorem 1.2 lie in a compact set depending on $E_\alpha(u_\alpha)$. The techniques used in this section are similar to those used by Kazdan and Warner and also in the study of the semiclassical nonlinear Schrödinger equation; see for instance, Chapter 8.1 in [1]. We proceed in section 6 to further improve the $W^{2,p}$-closeness, and we finally prove our main theorem in Section 7.

In Section 8 we construct a rotationally symmetric α-harmonic map of degree one with large energy which is not a rotation. As a byproduct we obtain the existence of α-harmonic maps of degree one from the disk to S^2 which map the boundary circle to a point and we also obtain α-harmonic maps of degree one which map an annulus to the sphere in such a way that the two boundary circles are mapped to antipodal points. Note that there are no such harmonic maps.

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2. The action of the Möbius group

Let $\varphi\colon S^2 \to S^2$ be a holomorphic map of degree 1. Given an arbitrary map $u\colon S^2 \to S^2$, we shall be interested in how $e(u \circ \varphi)$ and $E_\alpha(u \circ \varphi)$ depend on φ. For this, it is convenient to identify $S^2 \subset \mathbb{R}^3$ with $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via the stereographic projection from the north pole. If we denote the domain $S^2 \subset \mathbb{R}^3$ as $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the target $S^2 \subset \mathbb{R}^3$ as $\{(u^1, u^2, u^3) \in \mathbb{R}^3 : (u^1)^2 + (u^2)^2 + (u^3)^2 = 1\}$, then the stereographic identifications with $\widehat{\mathbb{C}}$ are given by

$$x + iy = \frac{2\zeta}{1 + |\zeta|^2}, \quad z = \frac{|\zeta|^2 - 1}{|\zeta|^2 + 1}; \quad u^1 + iu^2 = \frac{2\eta}{1 + |\eta|^2}, \quad u^3 = \frac{|\eta|^2 - 1}{|\eta|^2 + 1}.$$

The inverse maps are

$$\zeta = \frac{x + iy}{1 - z}; \qquad \eta = \frac{u^1 + iu^2}{1 - u^3}.$$

2.1. The Möbius group. The holomorphic maps of degree one from $\widehat{\mathbb{C}}$ to itself are the so-called fractional linear transformations which are of the form

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d}, \quad ad - bc = 1.$$

They form a group, called the Möbius group, which is the projective special linear group $PSL(2, \mathbb{C})$. Given $M \in SL(2, \mathbb{C})$, let λ, λ^{-1}, $\lambda > 0$, be the eigenvalues of MM^*. The singular value decomposition of matrices (see, e.g., [11]) tells us that there exists $U, V \in SU(2)$ such that,

$$M = UDV^*, \text{ where } D = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-(1/2)} \end{pmatrix}. \tag{2.1}$$

Elements of the subgroup $SU(2)$ of $SL(2, \mathbb{C})$ represent a rotation; indeed, if I denotes the 2×2 identity matrix then, $SO(3)$ may be identified with $SU(2)/\{I, -I\}$, which establishes $SU(2)$ as the double cover of $SO(3)$. The diagonal matrices of the form $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-(1/2)} \end{pmatrix}$ represent the dilations m_λ which are defined by

$$m_\lambda(\zeta) := \lambda\zeta.$$

2.2. Energy density in stereographic coordinates. In the following a map $u\colon S^2 \to S^2$ shall also be denoted by $\eta\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. However, we shall still denote by u the map to S^2 that arises from identifying the domain S^2 with $\widehat{\mathbb{C}}$. We have:

- the energy density of u, $e(u)$, is given by:
$$e(u)(\zeta) = \frac{(1+|\zeta|^2)^2}{2(1+|\eta|^2)^2}\,|\nabla_0\eta|^2,$$
where $\nabla_0\eta$ is the Euclidean gradient of η as a map from $\mathbb{C}$ to $\mathbb{C}$ with the flat metrics on both domain and target.
- The area element dA_{S^2} on the domain S^2 is given by:
$$dA_{S^2} = \frac{4}{(1+|\zeta|^2)^2}\,dA_0,$$
where $dA_0 := \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta}$ is the Euclidean area element on $\mathbb{C}$.

2.3. Transformation of the energy density and the α-energy under composition by a Möbius transformation. Given $M \in SL(2,\mathbb{C})$ and a map $u\colon \widehat{\mathbb{C}} \to S^2$, let u_M be the map defined by

$u_M(\zeta) = u(M\zeta)$ where, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then, by $M\zeta$ we mean $\dfrac{a\zeta+b}{c\zeta+d}$.

We have

$$\begin{aligned} e(u_M)(\zeta) &= \frac{(1+|\zeta|^2)^2}{2(1+|\eta(M\zeta)|^2)^2}\left|\frac{d}{d\zeta}\left(\frac{a\zeta+b}{c\zeta+d}\right)\right|^2 |\nabla_0\eta|^2(M\zeta) \\ &= \frac{(1+|\zeta|^2)^2}{|c\zeta+d|^4(1+|M\zeta|^2)^2}\big(e(u)(M\zeta)\big). \end{aligned} \tag{2.2}$$

Now

$$\begin{aligned} |c\zeta+d|^2(1+|M\zeta|^2) &= |a\zeta+b|^2 + |c\zeta+d|^2 \\ &= \left|\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\zeta \\ 1\end{pmatrix}\right|^2 \\ &= \left|\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-(1/2)}\end{pmatrix}\begin{pmatrix}\zeta \\ 1\end{pmatrix}\right|^2 \quad \text{(by (2.1))} \\ &= \frac{\lambda^2|\zeta|^2+1}{\lambda}. \end{aligned} \tag{2.3}$$

Using (2.3) in (2.2) gives

$$e(u_M)(\zeta) = \frac{\lambda^2(1+|\zeta|^2)^2}{(1+\lambda^2|\zeta|^2)^2}\big(e(u)(M\zeta)\big). \tag{2.4}$$

The transformation relation (2.4) allows us to restrict our attention to the dilations m_λ. Set $u_\lambda = u \circ m_\lambda$, i.e., $u_\lambda(\zeta) = u(\lambda\zeta)$ and set

$$\chi_\lambda(\zeta) = \frac{(1+\lambda^2|\zeta|^2)^2}{\lambda^2(1+|\zeta|^2)^2}. \tag{2.5}$$

Then

$$e(u)(\lambda\zeta) = \chi_\lambda(\zeta)\big(e(u_\lambda)(\zeta)\big)$$

for every $\lambda > 0$ and therefore,

$$\begin{aligned} E_\alpha(u) &= 2^{\alpha-1}\int_{\mathbb{C}} \big(1+e(u)(\zeta)\big)^\alpha \frac{4}{(1+|\zeta|^2)^2}\, dA_0(\zeta) \\ &= 2^{\alpha-1}\int_{\mathbb{C}} \big(1+e(u)(\lambda\zeta)\big)^\alpha \frac{4\lambda^2}{(1+|\lambda\zeta|^2)^2}\, dA_0(\zeta) \\ &= 2^{\alpha-1}\int_{\mathbb{C}} \big(1+\chi_\lambda(\zeta)e(u_\lambda)(\zeta)\big)^\alpha \frac{4}{\chi_\lambda(\zeta)(1+|\zeta|^2)^2}\, dA_0(\zeta), \end{aligned}$$

that is,

$$E_\alpha(u) = E_{\alpha,\lambda}(u_\lambda) = E_{\alpha,\lambda^{-1}}(u_{\lambda^{-1}}), \tag{2.6}$$

where $E_{\alpha,\lambda}$ is the functional defined by

$$E_{\alpha,\lambda}(v) = \frac{1}{2}\int_{S^2} \big(2+\chi_\lambda|\nabla_{S^2}v|^2\big)^\alpha \frac{1}{\chi_\lambda} dA_{S^2}. \tag{2.7}$$

Clearly u is a critical point of E_α if, and only if, u_λ is a critical point of $E_{\alpha,\lambda}$. Moreover, due to the above symmetry of E_α in λ, λ^{-1}, we assume throughout the rest of the paper that $\lambda \geqslant 1$.

Proposition 2.1. *If χ_λ is as in (2.5), the Euler Lagrange equation satisfied by a critical point v of $E_{\alpha,\lambda}$ is*

$$\Delta v + |\nabla v|^2 v + f_1 + f_2 = 0,$$

where

$$f_1 := (\alpha-1)\left(\frac{\chi_\lambda \nabla(|\nabla v|^2)\cdot\nabla v}{2+\chi_\lambda|\nabla v|^2}\right) \tag{2.8}$$

and

$$f_2 := (\alpha-1)\left(\frac{\chi_\lambda |\nabla v|^2 \nabla \log\chi_\lambda \cdot\nabla v}{2+\chi_\lambda|\nabla v|^2}\right). \tag{2.9}$$

The proof of this proposition is just a straightforward computation.

3. Closeness to the Möbius group

The aim of this section is to prove the following proposition.

Proposition 3.1. *For any $\delta > 0$ there exists $\varepsilon > 0$ such that, if $1 \leqslant \alpha \leqslant 2$ and if $E_\alpha(u) \leqslant 2^{2\alpha+1}\pi + \varepsilon$, where u is of degree 1, then there exists $M \in PSL(2,\mathbb{C})$ such that*

$$\|\nabla(u_M - Id)\|_{L^2(S^2)} \leqslant \delta. \tag{3.1}$$

Furthermore, there is a $\delta^\star > 0$ so that for $0 < \delta < \delta^\star$ there is a constant C such that, if $\lambda \geqslant 1$ is the largest eigenvalue of MM^* (see (2.1)) then

$$(\alpha-1)(\log\lambda)\min\{\log\lambda, 1\} \leqslant C\delta. \tag{3.2}$$

Remark 3.2. It is worth noting at this stage that M is not unique and that an optimal choice of M will be made in Section 6.

The proof of the above proposition relies on the three lemmas below.

Lemma 3.3. *Given $\delta > 0$, there exists $\varepsilon > 0$, sufficiently small, with the following property: for all $\alpha \geqslant 1$, if $u \in W^{1,2\alpha}(S^2, S^2)$ is of degree 1 and $E_\alpha(u) \leqslant 2^{2\alpha+1}\pi + \varepsilon$, there exists $M \in PSL(2, \mathbb{C})$ such that*

$$\|\nabla(u_M - Id)\|_{L^2(S^2)} \leqslant \delta. \tag{3.3}$$

Proof. If $E_\alpha(u) \leqslant 2^{2\alpha+1}\pi + \varepsilon$ then by (1.7) we have

$$\begin{aligned} E_1(u) &= \int_{S^2} (1 + e(u))\, dA_{S^2} \\ &\leqslant \left(\frac{2^{1-\alpha} E_\alpha(u)}{4\pi}\right)^{\frac{1}{\alpha}} 4\pi \\ &\leqslant \left(1 + \frac{\varepsilon}{2^{2\alpha+1}\pi}\right)^{\frac{1}{\alpha}} 8\pi \\ &\leqslant 8\pi + \varepsilon. \end{aligned}$$

If, for a contradiction, the lemma were not true, we could find a sequence $\varepsilon_n \downarrow 0$, a sequence $u_n \in W^{1,2}(S^2, S^2)$ of degree one, with $E_1(u_n) \leqslant 8\pi + \varepsilon_n$ and $\delta > 0$ such that

$$\big\|\nabla\big((u_n)_M - Id\big)\big\|_{L^2(S^2)} > \delta \quad \text{for all } M \in PSL(2, \mathbb{C}). \tag{3.4}$$

But u_n would then be a minimising sequence for E_1 of degree one and therefore, by Theorem 1 in [4], there exists $M_n \in PSL(2, \mathbb{C})$ such that $(u_n)_{M_n}$ converges strongly in Dirichlet norm to a degree one minimiser u_∞ of E_1. (We remark that, by energetic reasons, multiple splitting into maps of different degrees is excluded.) By Theorem 1.1, u_∞ is of the form $\zeta \mapsto M_\infty \zeta$ for some $M_\infty \in PSL(2, \mathbb{C})$. By the conformal invariance of the Dirichlet integral we have that

$$\Big\|\nabla\big((u_n)_{M_n M_\infty^{-1}} - Id\big)\Big\|_{L^2(S^2)} \to 0.$$

This then contradicts (3.4) and concludes the proof. q.e.d.

We still need to establish a bound on the largest eigenvalue λ of MM^* in the previous lemma. The rough plan for doing this is that, because of the closeness in Dirichlet norm provided by (3.3), $E_{\alpha,\lambda}(u_M)$ should be close to $E_{\alpha,\lambda}(Id)$. We should then be able to explicitly describe how $E_{\alpha,\lambda}(Id)$ grows with λ. Recall that the relation between E_α and $E_{\alpha,\lambda}$ is given by (2.7). This plan is executed in the next two lemmas.

Lemma 3.4. *If $\lambda \geqslant 1$ and $1 \leqslant \alpha \leqslant 2$, we have*

$$E_{\alpha,\lambda}(v) - E_{\alpha,\lambda}(Id) \geqslant -\alpha 2^{\alpha-2}(1+\lambda^2)^{\alpha-1} \|\, |\nabla_{S^2} v|^2 - 2\, \|_{L^1(S^2)}. \tag{3.5}$$

Proof. By the mean value theorem, there is a positive function $g : S^2 \to \mathbb{R}_+$ whose value at p lies between $|\nabla_{S^2} v(p)|^2$ and $2 = |\nabla_{S^2} Id|^2$

such that

$$E_{\alpha,\lambda}(v) - E_{\alpha,\lambda}(Id) = \frac{\alpha}{2}\int_{S^2} (2+\chi_\lambda g)^{\alpha-1}\left(|\nabla_{S^2} v|^2 - 2\right) dA_{S^2}. \tag{3.6}$$

Let

$A_+ := \{p \in S^2 : |\nabla_{S^2} v(p)|^2 \geqslant 2\}$ and $A_- := \{p \in S^2 : |\nabla_{S^2} v(p)|^2 < 2\}$.

Then, on A_+ we have $g \geqslant 2$ and on A_- we have $g \leqslant 2$. Therefore,

$$\begin{aligned}\int_{A_+} &(2+\chi_\lambda g)^{\alpha-1}(|\nabla_{S^2} v|^2 - 2)\, dA_{S^2} \\ &\geqslant 2^{\alpha-1}\int_{A_+} (1+\chi_\lambda)^{\alpha-1}\left(|\nabla_{S^2} v|^2 - 2\right) dA_{S^2}\end{aligned}$$

and, since $(|\nabla_{S^2} v|^2 - 2)$ is negative on A_-,

$$\begin{aligned}\int_{A_-} &(2+\chi_\lambda g)^{\alpha-1}(|\nabla_{S^2} v|^2 - 2)\, dA_{S^2} \\ &\geqslant 2^{\alpha-1}\int_{A_-} (1+\chi_\lambda)^{\alpha-1}\left(|\nabla_{S^2} v|^2 - 2\right) dA_{S^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{S^2} &(2+\chi_\lambda g)^{\alpha-1}(|\nabla_{S^2} v|^2 - 2)\, dA_{S^2} \\ &\geqslant 2^{\alpha-1}\int_{S^2} (1+\chi_\lambda)^{\alpha-1}\left(|\nabla_{S^2} v|^2 - 2\right) dA_{S^2}.\end{aligned} \tag{3.7}$$

Now $\sup_{S^2} \chi_\lambda = \lambda^2$ and therefore,

$$\left|\int_{S^2} (1+\chi_\lambda)^{\alpha-1}\left(|\nabla_{S^2} v|^2 - 2\right) dA_{S^2}\right| \leqslant (1+\lambda^2)^{\alpha-1}\|\, |\nabla_{S^2} v|^2 - 2\,\|_{L^1(S^2)}. \tag{3.8}$$

Estimate (3.5) is established by putting together (3.6), (3.7) and (3.8). q.e.d.

The next lemma describes how $E_{\alpha,\lambda}(Id)$ grows with λ.

Lemma 3.5. *We have that*

$$E_{\alpha,\lambda}(Id) = E_\alpha(m_{\lambda^{-1}}) = E_\alpha(m_\lambda). \tag{3.9}$$

Moreover, by letting

$$\xi(\alpha,\lambda) := E_\alpha(m_\lambda) - 2^{2\alpha+1}\pi, \tag{3.10}$$

there exists a fixed constant C such that, for $1 < \alpha \leqslant 2$,

$$\xi(\alpha,\lambda) \geqslant \begin{cases} C\lambda^{2\alpha-2}, & \text{if } (\alpha-1)\log\lambda \geqslant 2, \\ C(\alpha-1)\log\lambda, & \text{if } (\alpha-1) \leqslant (\alpha-1)\log\lambda \leqslant 2 \\ C(\alpha-1)(\log\lambda)^2, & \text{if } 0 \leqslant \log\lambda \leqslant 1. \end{cases} \tag{3.11}$$

Additionally, the function $E_\alpha(m_\lambda)$ is increasing in λ and we have for $0 \leqslant (\alpha-1)\log\lambda \leqslant 2$ that

$$\frac{\partial}{\partial \log\lambda} E_\alpha(m_\lambda) = \frac{\partial}{\partial \log\lambda} E_{\alpha,\lambda}(Id) \geqslant C(\alpha-1)\frac{|\log\lambda|}{1+|\log\lambda|}. \tag{3.12}$$

Proof. We start by obtaining an explicit formula for $E_\alpha(m_\lambda)$: set $r := |\zeta|$ and then, as we saw in §2,

$$e(m_\lambda)(\zeta) = \lambda^2 \frac{(1+r^2)^2}{(1+\lambda^2 r^2)^2} = \frac{1}{\chi_\lambda(\zeta)}.$$

So,

$$E_\alpha(m_\lambda) = 2^{\alpha-1} 8\pi \int_0^\infty \left(1 + \frac{\lambda^2(1+r^2)^2}{(1+\lambda^2 r^2)^2}\right)^\alpha \frac{r}{(1+r^2)^2}\, dr.$$

We make the change of variable

$$w := \lambda \frac{1+r^2}{1+\lambda^2 r^2}$$

for which

$$dw = 2\lambda r \frac{1-\lambda^2}{(1+\lambda^2 r^2)^2} dr$$

and obtain

$$E_\alpha(m_\lambda) = 2^{\alpha+1}\pi \frac{\lambda}{\lambda^2-1} \int_{1/\lambda}^{\lambda} (1+w^2)^\alpha w^{-2}\, dw.$$

Setting $\lambda := e^\tau$ and $w := e^t$ yields:

$$\begin{aligned} E_\alpha(m_{e^\tau}) &= 2^{\alpha+1}\pi \frac{e^\tau}{e^{2\tau}-1} \int_{-\tau}^{\tau} (1+e^{2t})^\alpha e^{-t}\, dt \\ &= \frac{2^\alpha \pi}{\sinh\tau} \int_{-\tau}^{\tau} (e^{-t}+e^t)^\alpha e^{(\alpha-1)t}\, dt \\ &= \frac{2^{2\alpha+1}\pi}{\sinh\tau} \int_0^\tau (\cosh t)^\alpha \cosh((\alpha-1)t)\, dt, \end{aligned} \tag{3.13}$$

where we have used

$$\int_{-\tau}^{0} (e^{-t}+e^t)^\alpha e^{(\alpha-1)t}\, dt = \int_0^\tau (e^{-t}+e^t)^\alpha e^{-(\alpha-1)t}\, dt\,.$$

It is immediate from this expression for $E_\alpha(m_\lambda)$ that we have $E_\alpha(m_\lambda) = E_\alpha(m_{\lambda^{-1}})$ and the relation (3.9) then follows by taking (2.6) into account.

As expected we have $E_1(m_{e^\tau}) = 8\pi\ \forall\, \tau \in \mathbb{R}$ and $E_\alpha(m_1) = 2^{2\alpha+1}\pi$.

It will be convenient to set

$$\beta := (\alpha - 1),$$

to make the change of variables

$$s := \beta t, \quad \sigma := \beta\tau = (\alpha - 1)\log\lambda$$

and to introduce the functions

$$g(s) := (\cosh(s/\beta))^{\beta}\cosh s$$

(3.14) and

$$G(\sigma) := \frac{1}{\beta\sinh(\sigma/\beta)}\int_0^{\sigma}(\cosh(s/\beta))g(s)\,ds.$$

Then (3.13) becomes

(3.15)

$$E_{\alpha}(m_{e(\sigma/\beta)}) = \frac{2^{2\alpha+1}\pi}{\beta\sinh(\sigma/\beta)}\int_0^{\sigma}(\cosh(s/\beta))g(s)\,ds = 2^{2\alpha+1}\pi G(\sigma).$$

The lower bound $\cosh t > \frac{1}{2}e^t$ yields

$$g(s) > \left(\frac{e^{s/\beta}}{2}\right)^{\beta}\frac{e^s}{2} = \frac{e^{2s}}{2^{\alpha}}.$$

We shall now prove the first inequality in (3.11). So, we assume that $\sigma \geqslant 2$ and $1 < \alpha \leqslant 2$ and estimate G from below as follows:

$$\begin{aligned} G(\sigma) &> \frac{1}{\beta\sinh(\sigma/\beta)}\int_{\sigma-1}^{\sigma}(\cosh(s/\beta))g(s)\,ds \\ &> \frac{1}{2^{\alpha}\beta\sinh(\sigma/\beta)}\int_{\sigma-1}^{\sigma}(\cosh(s/\beta))e^{2s}\,ds \\ &> \frac{e^{(2\sigma-2)}}{2^{\alpha}}\;\frac{1}{\beta\sinh(\sigma/\beta)}\int_{\sigma-1}^{\sigma}(\cosh(s/\beta))\,ds \\ &> \frac{e^{2\sigma}}{2e^2}\;\frac{\sinh(\sigma/\beta)-\sinh((\sigma-1)/\beta)}{\sinh(\sigma/\beta)}. \end{aligned}$$

Keeping in mind that $0 \leqslant \beta \leqslant 1$, we have,

$$\sinh(\sigma/\beta) - \sinh((\sigma-1)/\beta) > \frac{e^{\sigma/\beta}}{2}(1 - e^{-1/\beta}) > \sinh(\sigma/\beta)\left(\frac{e-1}{e}\right).$$

It follows that

$$G(\sigma) - 1 > e^{2\sigma}\left(\frac{e-1}{2e^3} - \frac{1}{e^4}\right),$$

i.e., if $(\alpha-1)\log\lambda \geqslant 2$ and $1 < \alpha \leqslant 2$ then

$$\xi(\alpha,\lambda) \geqslant 2^{2\alpha+1}\pi\left(\frac{e^2-e-2}{2e^4}\right)\lambda^{2\alpha-2}$$

as claimed.

To estimate $G(\sigma)-1$ from below for $\sigma\in[0,2]$, we calculate $G'(\sigma)$ from (3.14):

$$G'(\sigma)=\frac{\cosh(\sigma/\beta)}{\beta\sinh(\sigma/\beta)}g(\sigma)-\frac{\cosh(\sigma/\beta)}{\beta^2\sinh^2(\sigma/\beta)}\int_0^\sigma(\cosh(s/\beta))g(s)\,ds.$$

Now

$$\begin{aligned}g(\sigma)=&\frac{1}{\beta\sinh(\sigma/\beta)}\int_0^\sigma(\cosh(s/\beta))g(s)\,ds\\&+\frac{1}{\sinh(\sigma/\beta)}\int_0^\sigma(\sinh(s/\beta))g'(s)\,ds.\end{aligned}$$

Differentiating the expression for g from (3.14) gives

$$\begin{aligned}g'(s)&=(\cosh(s/\beta))^{\beta-1}(\sinh(s/\beta)\cosh s+\cosh(s/\beta)\sinh s)\\&=(\cosh(s/\beta))^{\beta-1}\sinh(\alpha s/\beta).\end{aligned}$$

Therefore, we obtain:

(3.16)
$$G'(\sigma)=\frac{\cosh(\sigma/\beta)}{\beta\sinh^2(\sigma/\beta)}\int_0^\sigma(\sinh(s/\beta))(\cosh(s/\beta))^{\beta-1}\sinh(\alpha s/\beta)\,ds.$$

We shall estimate G' from below differently in the two regimes $0\leqslant\sigma\leqslant\beta$ and $0<\beta\leqslant\sigma\leqslant 2$. We start with the latter case for which we shall show that G' is bounded below by a positive constant, independent of β.

Using $\dfrac{\cosh(\sigma/\beta)}{\sinh(\sigma/\beta)}>1$ and $\dfrac{\sinh(\alpha s/\beta)}{\cosh(s/\beta)}\geqslant\tanh(\alpha s/\beta)$ in (3.16), we obtain, for $\theta\in(0,1)$ and $\beta\leqslant\sigma$,

$$\begin{aligned}G'(\sigma)&>\frac{1}{\sinh(\sigma/\beta)}\int_{\theta\beta}^\sigma(\tfrac{1}{\beta}\sinh(s/\beta))(\cosh(s/\beta))^\beta\tanh(\alpha s/\beta)\,ds\\&\geqslant\tanh\theta\,\frac{\cosh(\sigma/\beta)-\cosh\theta}{\sinh(\sigma/\beta)}\\&\geqslant\tanh\theta\,\left(1-\frac{\cosh\theta}{\sinh 1}\right),\end{aligned}$$

where we also used that $\tanh(\alpha\theta)\geqslant\tanh\theta$ and $\cosh(s/\beta)\geqslant 1$ in the second estimate.

We now choose $\theta>0$ so that $\cosh\theta\leqslant\frac{1}{2}\sinh 1$ and deduce that there exists $C>0$, independent of anything, such that if $\alpha>1$ and $\lambda\geqslant e$, i.e., $\tau\geqslant 1$ and $0<\beta\leqslant\sigma$ then

$$G'(\sigma)\geqslant C>0. \tag{3.17}$$

It follows that for $0<\beta\leqslant\sigma$ we get

$$G(\sigma)\geqslant G(\beta)+C(\sigma-\beta). \tag{3.18}$$

The lower bound on G' for $\sigma \in (0,\beta]$ is straightforward. First use the inequality $\cosh(\sigma/\beta)(\cosh(s/\beta))^{\beta-1} \geqslant (\cosh(s/\beta))^{\beta} \geqslant 1$ for every $s \in [0,\sigma]$ in (3.16) to get

$$G'(\sigma) \geqslant \frac{1}{\beta \sinh^2(\sigma/\beta)} \int_0^{\sigma} (\sinh(s/\beta)) \sinh(\alpha s/\beta)\, ds.$$

Next, use $(\sinh(s/\beta)) \sinh(\alpha s/\beta) \geqslant \frac{s^2}{\beta^2}$ and the inequality $\sinh x \leqslant x(\cosh x)$ for $x \geqslant 0$ to get

$$\begin{aligned} G'(\sigma) &\geqslant \frac{1}{\beta(\cosh(\sigma/\beta))^2\sigma^2} \int_0^{\sigma} s^2\, ds \\ &\geqslant \frac{\sigma}{3\beta(\cosh 1)^2}; \qquad \text{we have used } 0 \leqslant \sigma/\beta \leqslant 1. \end{aligned} \tag{3.19}$$

It follows that,

$$\text{for } 0 \leqslant \sigma \leqslant \beta, \quad G(\sigma) - G(0) \geqslant \frac{\sigma^2}{6\beta(\cosh 1)^2} \geqslant \frac{(\alpha-1)(\log\lambda)^2}{6(\cosh 1)^2}. \tag{3.20}$$

We can now establish the last two estimates in (3.11). If $\alpha - 1 \leqslant (\alpha-1)\log\lambda \leqslant 2$ then, by (3.18) and (3.20) we have that

$$\begin{aligned} \xi(\alpha,\lambda) &\geqslant 2^{2\alpha+1}\pi \big(\big(G(\alpha-1) - 1\big) + C(\alpha-1)(\log\lambda - 1)\big) \\ &\geqslant C(\alpha-1)\log\lambda. \end{aligned}$$

If $\log\lambda \leqslant 1$ then, we obtain again from (3.20) that

$$\xi(\alpha,\lambda) \geqslant \frac{2^{2\alpha+1}\pi}{6(\cosh 1)^2}(\alpha-1)(\log\lambda)^2.$$

Finally, $E_\alpha(m_\lambda)$ increases with λ because, from (3.16), G' is evidently positive. Moreover, in order to show (3.12) we note that it follows from (3.15) that

$$\frac{\partial}{\partial \log\lambda} E_{\alpha,\lambda}(Id) = (\alpha-1)2^{2\alpha+1}\pi G'((\alpha-1)\log\lambda).$$

For $1 \leqslant \log\lambda \leqslant 2(\alpha-1)^{-1}$ we use (3.17) in order to get

$$\frac{\partial}{\partial \log\lambda} E_{\alpha,\lambda}(Id) \geqslant C(\alpha-1) \geqslant C(\alpha-1)\frac{|\log\lambda|}{1+|\log\lambda|}.$$

For $0 < \log\lambda \leqslant 1$ we use (3.19) to conclude

$$\frac{\partial}{\partial \log\lambda} E_{\alpha,\lambda}(Id) \geqslant C(\alpha-1)\log\lambda \geqslant C(\alpha-1)\frac{|\log\lambda|}{1+|\log\lambda|}.$$

The proof of Lemma 3.5 is complete. q.e.d.

We can now prove the main result of this section.

Proof of Proposition 3.1. Having proved Lemma 3.3, it only remains to establish (3.2). We do this by applying Lemma 3.4 with $v = u_M$, M as provided by (3.3) and $\lambda \geqslant 1$ equal to the largest eigenvalue of MM^*. Then, with δ as in (3.3), we have

$$2^{2\alpha+1}\pi + \varepsilon \geqslant E_\alpha(u) = E_{\alpha,\lambda}(u_M) \geqslant E_{\alpha,\lambda}(Id) - \alpha\pi 2^{2\alpha+1}\lambda^{2\alpha-2}\delta, \tag{3.21}$$

where we used that

$$\begin{aligned}\| |\nabla_{S^2} u_M|^2 - 2 \|_{L^1(S^2)} &\leqslant \|\nabla(u_M - Id)\|_{L^2(S^2)} \|\nabla(u_M + Id)\|_{L^2(S^2)} \\ &\leqslant \delta\sqrt{(8\pi+\varepsilon)(8\pi)} \leqslant \delta(16\pi).\end{aligned}$$

Recall that

$$E_{\alpha,\lambda}(Id) = E_\alpha(m_\lambda) = 2^{2\alpha+1}\pi + \xi(\alpha,\lambda)$$

and observe that ε in Lemma 3.3 can be chosen no larger than δ. Therefore, (3.21) can be rewritten as

$$\delta(1 + C'\lambda^{2\alpha-2}) \geqslant \xi(\alpha,\lambda). \tag{3.22}$$

If $(\alpha-1)\log\lambda \geqslant 2$, i.e. $\lambda^{2\alpha-2} \geqslant e^4$, then (3.11) provides the lower bound $\xi(\alpha,\lambda) \geqslant C\lambda^{2\alpha-2}$. So, (3.22) cannot hold if $0 \leqslant \delta < \delta^* := \min\{\frac{C}{2C'}, \frac{C}{2}e^4\}$. Therefore, $\lambda^{2\alpha-2}$ must be less than e^4 and so, from (3.11) and (3.22), we deduce that

$$\delta(1 + C'e^4) \geqslant C(\alpha-1)(\log\lambda)\min\{\log\lambda, 1\}. \qquad \text{q.e.d.}$$

4. Closeness in the $W^{2,p}$-norm

In this section we prove a refinement of Proposition 3.1, showing closeness between u_M and the identity in $W^{2,p}$, $p \in [1, \frac{3}{2}]$. In the proof of Proposition 5.1, we will need to restrict the range of p to lie in $(\frac{4}{3}, \frac{3}{2}]$.

Proposition 4.1. *There exist $1 < \alpha_0$, $\delta_0 > 0$ and a constant C depending only on α_0 and δ_0 such that, for every $1 < \alpha \leqslant \alpha_0$, every $0 < \delta \leqslant \delta_0$ and every critical point $v \in W^{1,2\alpha}(S^2, S^2)$ of $E_{\alpha,\lambda}$ satisfying (3.1) and (3.2) we have, for any $p \in [1, \frac{3}{2}]$,*

$$\|v - Id\|_{L^\infty(S^2)} + \|\nabla(v - Id)\|_{W^{1,p}(S^2)} \leqslant C(\delta + \alpha - 1). \tag{4.1}$$

Proof. We define a map $\psi\colon S^2 \to \mathbb{R}^3$ by

$$v = Id + \psi$$

and we obtain from Proposition 3.1 that

$$\|\nabla\psi\|_{L^2(S^2)} \leqslant \delta.$$

By Proposition 2.1, ψ satisfies

$$\begin{aligned}\Delta\psi = &-2\psi - 2\langle\nabla\psi, \nabla Id\rangle Id - |\nabla\psi|^2\psi - 2\langle\nabla\psi, \nabla Id\rangle\psi \\ &- |\nabla\psi|^2 Id - f_1 - f_2.\end{aligned} \tag{4.2}$$

We shall first estimate the average of ψ by integrating this equation and observing from (2.8) that

$$|f_1(\zeta)| \leqslant C(\alpha-1)|\nabla^2 v(\zeta)| \leqslant C(\alpha-1)(1+|\nabla^2\psi(\zeta)|) \tag{4.3}$$

and that

$$|f_2(\zeta)| \leqslant C(\alpha-1)|(\nabla \log \chi_\lambda)(\zeta)|\,|\nabla v(\zeta)|. \tag{4.4}$$

When integrating (4.2), keep also in mind that $\|\psi\|_{L^\infty(S^2)} \leqslant 2$ and make use of Proposition 3.1 and Lemma A.1 to conclude that

$$\begin{aligned} |\fint_{S^2} \psi\, dA_{S^2}| &\leqslant C\delta + C(\alpha-1)\|\nabla^2 v\|_{L^1(S^2)} \\ &\qquad + C(\alpha-1)\|\nabla v\|_{L^2(S^2)}\|\nabla \log \chi_\lambda\|_{L^2(S^2)} \\ &\leqslant C(\delta+\alpha-1) + C(\alpha-1)\|\nabla^2\psi\|_{L^1(S^2)}. \end{aligned} \tag{4.5}$$

This estimate on the average of ψ allows us to use standard L^p-estimates for the Laplacian and the Sobolev-Poincaré inequality to conclude that, for every $p \in (\frac{4}{3}, \frac{3}{2}]$,

$$\begin{aligned} \|\nabla\psi\|_{W^{1,p}(S^2)} &\leqslant C\left(\|\Delta\psi\|_{L^p(S^2)} + \|\psi\|_{L^p(S^2)}\right) \\ &\leqslant C\left(\|\Delta\psi\|_{L^p(S^2)} + \|\nabla\psi\|_{L^2(S^2)} + |\fint_{S^2}\psi\, dA_{S^2}|\right) \\ &\leqslant C\left(\|\Delta\psi\|_{L^p(S^2)} + \delta + \alpha - 1 + (\alpha-1)\|\nabla^2\psi\|_{L^p(S^2)}\right). \end{aligned}$$

By picking $\alpha_0 > 1$ sufficiently close to 1 so that $C(\alpha_0 - 1) \leqslant \frac{1}{2}$ we get

$$\|\nabla\psi\|_{W^{1,p}(S^2)} \leqslant C\left(\|\Delta\psi\|_{L^p(S^2)} + \delta + \alpha - 1\right). \tag{4.6}$$

The plan now is to estimate $\|\Delta\psi\|_{L^p(S^2)}$, by using (4.2). The L^p norm of the right hand side of (4.2) requires us to estimate the L^{2p}-norm of $\nabla\psi$ which we do by means of the Gagliardo-Nirenberg interpolation inequality:

$$\|\nabla\psi\|^2_{L^{2p}(S^2)} \leqslant C\|\nabla\psi\|_{L^2(S^2)}\big(\|\nabla^2\psi\|_{L^p(S^2)} + \|\nabla\psi\|_{L^2(S^2)}\big).$$

Using (4.2), (4.5), a Poincaré-type inequality, Hölder's inequality, the Gagliardo-Nirenberg estimate from above, (4.3), (4.4) and Lemma A.1, we get

$$\begin{aligned} \|\Delta\psi\|_{L^p(S^2)} &\leqslant C\big(\|\psi - \fint_{S^2}\psi\, dA_{S^2}\|_{L^p(S^2)} + |\fint_{S^2}\psi\, dA_{S^2}| \\ &\qquad + \|\nabla\psi\|_{L^2(S^2)} + \|\nabla\psi\|^2_{L^{2p}(S^2)} + \|f_1\|_{L^p(S^2)} + \|f_2\|_{L^2(S^2)}\big) \\ &\leqslant C(\delta+\alpha-1)(1+\|\nabla^2\psi\|_{L^p(S^2)}). \end{aligned}$$

We can insert this estimate into (4.6) and then choose $\alpha_0 - 1$ and δ_0 small in order to get

$$\|\nabla\psi\|_{W^{1,p}(S^2)} \leqslant C(\delta+\alpha-1).$$

Using once more (4.5) and the Sobolev embedding theorem, we get, for any $p \in (\frac{4}{3}, \frac{3}{2}]$,

$$\begin{aligned}\|\psi\|_{L^\infty(S^2)} &\leqslant C\|\psi - \fint_{S^2} \psi \, dA_{S^2}\|_{W^{2,p}(S^2)} + C\left|\fint_{S^2} \psi \, dA_{S^2}\right| \\ &\leqslant C(\delta + \alpha - 1).\end{aligned}$$

This concludes the proof in the case $p \in (\frac{4}{3}, \frac{3}{2}]$ and the estimate for $p \in [1, \frac{4}{3}]$ follows from Hölder's inequality. q.e.d.

5. A bound on λ

In this section we shall show how the estimates (4.1) and (3.2) imply a very slow growth on $\frac{\partial}{\partial \log \lambda} E_{\alpha,\lambda}(Id)$ which, when coupled with (3.12), implies a bound on λ, independent of how close α is to 1. We start by computing $\frac{d}{d\lambda} E_{\alpha,\lambda}(v)$ directly from (2.7) and (2.5):

$$\begin{aligned}\log(\chi_\lambda(\zeta)) &= 2\log(1 + \lambda^2|\zeta|^2) - 2\log\lambda - 2\log(1 + |\zeta|^2) \\ \frac{d}{d\lambda}\log(\chi_\lambda(\zeta)) &= \frac{4\lambda|\zeta|^2}{1 + \lambda^2|\zeta|^2} - \frac{2}{\lambda} \\ \frac{d}{d\log\lambda}\log(\chi_\lambda(\zeta)) &= \frac{2(\lambda^2|\zeta|^2 - 1)}{\lambda^2|\zeta|^2 + 1}\end{aligned}$$

and thus

$$\begin{aligned}&\frac{d}{d\log\lambda} E_{\alpha,\lambda}(v) \\ &\quad = \frac{1}{2}\frac{d}{d\log\lambda}\int_{S^2} \left(2 + \chi_\lambda|\nabla_{S^2} v|^2\right)^\alpha \frac{1}{\chi_\lambda} \, dA_{S^2} \\ &\quad = \int_{S^2} (2 + \chi_\lambda|\nabla_{S^2} v|^2)^{\alpha-1} \left((\alpha - 1)|\nabla_{S^2} v|^2 - \frac{2}{\chi_\lambda})\right) z(\lambda\zeta) \, dA_{S^2},\end{aligned}$$

where, as in section 2, $z(\zeta) := \dfrac{|\zeta|^2 - 1}{|\zeta|^2 + 1} \in [-1, 1)$.

We wish to estimate $\dfrac{d}{d\log\lambda} E_{\alpha,\lambda}(Id) - \dfrac{d}{d\log\lambda} E_{\alpha,\lambda}(v)$ in terms of a suitable norm of the difference between Id and v.

$$\frac{d}{d\log\lambda} E_{\alpha,\lambda}(Id) - \frac{d}{d\log\lambda} E_{\alpha,\lambda}(v)$$

(5.1)

$$\begin{aligned}&= -\int_{S^2} \left((2 + 2\chi_\lambda)^{\alpha-1} - (2 + \chi_\lambda|\nabla_{S^2} v|^2)^{\alpha-1}\right) \frac{2z(\lambda\zeta)}{\chi_\lambda} \, dA_{S^2} \quad + \\ &(\alpha - 1)\int_{S^2} \left(2\,(2 + 2\chi_\lambda)^{\alpha-1} - |\nabla_{S^2} v|^2\,(2 + \chi_\lambda|\nabla_{S^2} v|^2)^{\alpha-1}\right) z(\lambda\zeta) \, dA_{S^2}\,.\end{aligned}$$

As in the proof of Lemma 3.4, there is a positive function $g\colon S^2 \to \mathbb{R}_+$ whose value at p lies between $|\nabla_{S^2} v(p)|^2$ and $2 = |\nabla_{S^2} Id|^2$ such that
$$\big((2+2\chi_\lambda)^{\alpha-1} - (2+\chi_\lambda|\nabla_{S^2}v|^2)^{\alpha-1}\big) = (\alpha-1)(2+g\chi_\lambda)^{\alpha-2}\chi_\lambda(2-|\nabla_{S^2}v|^2).$$
Similarly,
$$\begin{aligned} 2\,(2+2\chi_\lambda)^{\alpha-1} - |\nabla_{S^2}v|^2\,(2+\chi_\lambda|\nabla_{S^2}v|^2)^{\alpha-1} &= (2+2\chi_\lambda)^{\alpha-1}(2-|\nabla_{S^2}v|^2) \\ &\quad + (\alpha-1)(2+g\chi_\lambda)^{\alpha-2}\chi_\lambda(2-|\nabla_{S^2}v|^2)|\nabla_{S^2}v|^2\,. \end{aligned}$$
If $\alpha \leqslant 2$,
$$(2+g\chi_\lambda)^{\alpha-2} \leqslant 1.$$
Moreover,
$$\begin{aligned} \frac{\chi_\lambda|\nabla_{S^2}v|^2}{2+g\chi_\lambda} &\leqslant \begin{cases} \frac{1}{2}|\nabla_{S^2}v|^2, & \text{if } |\nabla_{S^2}v|^2 \geqslant 2 \\ 1, & \text{if } |\nabla_{S^2}v|^2 \leqslant 2, \end{cases} \\ &\leqslant 1 + |\nabla_{S^2}v|^2 \end{aligned}$$
and
$$(2+2\chi_\lambda)^{\alpha-1} \leqslant 4^{\alpha-1}\lambda^{2\alpha-2}, \quad (2+g\chi_\lambda)^{\alpha-1} \leqslant 4^{\alpha-1}\lambda^{2\alpha-2}(1+|\nabla_{S^2}v|^{2\alpha-2}).$$
Therefore, using that $|z| \leqslant 1$,
$$\left|\big((2+2\chi_\lambda)^{\alpha-1} - (2+\chi_\lambda|\nabla_{S^2}v|^2)^{\alpha-1}\big)\frac{2z(\lambda\zeta)}{\chi_\lambda}\right| \leqslant 2(\alpha-1)|2-|\nabla_{S^2}v|^2| \tag{5.2}$$
and
$$\begin{aligned} &\big|\big(2\,(2+2\chi_\lambda)^{\alpha-1} - |\nabla_{S^2}v|^2\,(2+\chi_\lambda|\nabla_{S^2}v|^2)^{\alpha-1}\big)z(\lambda\zeta)\big| \\ &\qquad \leqslant C\lambda^{2\alpha-2}\,\big|2-|\nabla_{S^2}v|^2\big|\,(1+(\alpha-1)|\nabla_{S^2}v|^{2\alpha}). \end{aligned} \tag{5.3}$$
Using (5.2) and (5.3) in (5.1) we can finally estimate
$$\begin{aligned} &\frac{d}{d\log\lambda}E_{\alpha,\lambda}(Id) - \frac{d}{d\log\lambda}E_{\alpha,\lambda}(v) \\ &\leqslant C(\alpha-1)(1+\lambda^{2\alpha-2})\int_{S^2}\big|2-|\nabla_{S^2}v|^2\big|\,(1+(\alpha-1)|\nabla_{S^2}v|^{2\alpha})\,dA_{S^2} \\ &\leqslant C(\alpha-1)(1+\lambda^{2\alpha-2})\|\nabla(v-Id)\|_{L^2(S^2)}(\,\|\nabla Id\|_{L^2(S^2)} + \|\nabla v\|_{L^2(S^2)}) \\ &\quad + C(\alpha-1)^2(1+\lambda^{2\alpha-2})\|\nabla(v-Id)\|_{L^{2\alpha+2}(S^2)} \\ &\qquad \cdot(\,\|\nabla Id\|_{L^{2\alpha+2}(S^2)} + \|\nabla v\|_{L^{2\alpha+2}(S^2)})\|\nabla v\|^{2\alpha}_{L^{2\alpha+2}(S^2)}. \end{aligned} \tag{5.4}$$

Proposition 5.1. *There exist $1 < \alpha_0$, $\delta_0 > 0$, possibly smaller than those in Proposition 4.1, such that if $v \in W^{1,2\alpha}(S^2,S^2)$ is a critical point of $E_{\alpha,\lambda}$ satisfying (3.1) and (3.2), $1 < \alpha \leqslant \alpha_0$, $0 < \delta \leqslant \delta_0$, then*
$$\log\lambda \leqslant C(\delta+\alpha-1). \tag{5.5}$$

Proof. As in Proposition 4.1, we set $\psi := v - Id$. By the Sobolev embedding,

$$\|\nabla\psi\|_{L^{2\alpha+2}(S^2)} \leqslant C(\alpha)\|\nabla\psi\|_{W^{1,p}(S^2)}, \quad p := \frac{2\alpha+2}{\alpha+2}.$$

Note that, since we may assume $\alpha_0 \leqslant 2$, we have that $p \in (\frac{4}{3}, \frac{3}{2}]$, as already indicated just before the statement of Proposition 4.1. Moreover, $C(\alpha)$ can then be chosen independent of α. So, taking α_0 and δ_0 as in Proposition 4.1, we get, from (4.1),

$$\|\nabla\psi\|_{L^{2\alpha+2}(S^2)} \leqslant C(\delta + \alpha - 1). \tag{5.6}$$

In particular, $\|\nabla v\|_{L^{2\alpha+2}(S^2)} \leqslant \|\nabla\psi\|_{L^{2\alpha+2}(S^2)} + \|\nabla Id\|_{L^{2\alpha+2}(S^2)} \leqslant C$.

By (3.2) we have

$$\lambda^{2\alpha-2} < \max\{e^{2C\delta}, e^{2\alpha_0-2}\}. \tag{5.7}$$

We now note that

$$E_{\alpha,\tau}(v) = E_{\alpha,\lambda}(v_{\lambda\tau^{-1}}),$$

which gives

$$\begin{aligned}\frac{d}{d\log\tau}E_{\alpha,\tau}(v)|_{\tau=\lambda} &= \left(\tau\frac{d}{d\tau}E_{\alpha,\tau}(v)\right)|_{\tau=\lambda}\\ &= \lambda\left(\frac{d}{d\tau}E_{\alpha,\lambda}(v_{\lambda\tau^{-1}})\right)|_{\tau=\lambda} = 0,\end{aligned}$$

where the last equality holds because v is a critical point of $E_{\alpha,\lambda}$.

It then follows from (3.12), (5.4), (5.6) and (5.7) that

$$C'^{-1}(\alpha-1)\frac{\log\lambda}{1+\log\lambda} \leqslant \frac{d}{d\log\lambda}E_{\alpha,\lambda}(Id) \leqslant C(\alpha-1)(\delta+\alpha-1). \tag{5.8}$$

The estimate (5.5) now follows by taking $\alpha_0 - 1$ and δ_0 sufficiently small. q.e.d.

6. Optimal λ and better closeness in the $W^{2,p}$-norm

Of course, we wish to prove that $\lambda = 1$. However, the choice of λ provided by Proposition 3.1 has some flexibility and therefore, at the moment, we cannot hope to do better than (5.5). So we have to choose λ optimally, which we do as follows.

Proposition 3.1 suggests that we should choose M so as to minimise $\|\nabla(u_M - Id)\|^2_{L^2(S^2)} = \|\nabla(u - M^{-1})\|^2_{L^2(S^2)}$. This minimisation is possible because, as $M \to \infty$ in the Möbius group $PSL(2,\mathbb{C})$, $\|\nabla(u - M^{-1})\|^2_{L^2(S^2)} \to \|\nabla u\|^2_{L^2(S^2)} + \|\nabla Id\|^2_{L^2(S^2)} \geqslant 16\pi$ and therefore, we only need to minimise $\|\nabla(u_M - Id)\|^2_{L^2(S^2)}$ over a compact subset of $PSL(2,\mathbb{C})$. In order to see this we note that up to rotations, M can only go to infinity if it approaches a dilation from the south pole towards the north pole by a huge factor λ, so that the energy of m_λ is concentrated on a small disk D centred at the south pole. Take D so small that the energy of u on D is less than ε and the energy of m_λ outside of D is less than ε. By breaking up the integral for

$$\|\nabla(u - M^{-1})\|^2_{L^2(S^2)} = \|\nabla u\|^2_{L^2(S^2)} + 2\langle\nabla u, \nabla M^{-1}\rangle_{L^2(S^2)} + \|\nabla M^{-1}\|^2_{L^2(S^2)}$$

into the contributions from D and its complement, we see that

$$\langle\nabla u, \nabla M^{-1}\rangle_{L^2(S^2)}$$

is small and noting that by conformal invariance $\|\nabla M^{-1}\|_{L^2(S^2)} = \|\nabla Id\|_{L^2(S^2)}$, the claim follows.

From now on, we shall assume that M does minimise $\|\nabla(u_M - Id)\|_{L^2(S^2)}$. Of course, all the estimates proved so far still hold.

Proposition 6.1. *With the optimal M chosen as above, set, as usual, $v := u_M$ and assume that v satisfies the hypotheses of Proposition 5.1. Then, for $p \in [1, \frac{3}{2}]$, we have*

$$\|(v - Id)\|_{W^{2,p}} \leqslant C(\alpha - 1)(\log\lambda). \tag{6.1}$$

Remark 6.2. Note that, in light of (5.5), the estimate (6.1) is an improvement of (4.1).

Proof. By Hölder's inequality, it suffices to prove (6.1) for $p \in (\frac{4}{3}, \frac{3}{2}]$ and we shall restrict p in this range throughout this proof so that we may use all the estimates established so far.

We notice that, by (4.1), v approaches the identity map pointwise as δ and $(\alpha - 1)$ tend to zero. So we may write

$$v = Id + \psi = \exp_{Id}\hat{\psi} \quad (= Id + \hat{\psi} + O(|\hat{\psi}|^2)); \qquad \hat{\psi} \in T_{Id}W^{1,2\alpha}(S^2, S^2).$$

More explicitly, if $\mathbf{x} = (x, y, z) \in S^2 \subset \mathbb{R}^3$, then

$$v(\mathbf{x}) = \mathbf{x}\sqrt{1 - |\hat{\psi}(\mathbf{x})|^2} + \hat{\psi}(\mathbf{x}), \qquad \hat{\psi}(\mathbf{x}) \cdot \mathbf{x} \equiv 0.$$

So ψ and $\hat{\psi}$ are related as follows:

$$\hat{\psi}(\mathbf{x}) = \psi(\mathbf{x}) + \tfrac{1}{2}|\psi(\mathbf{x})|^2\mathbf{x}, \qquad \psi(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \left(1 - \sqrt{1 - |\hat{\psi}(\mathbf{x})|^2}\right)\mathbf{x},$$
$$|\hat{\psi}|^2 = |\psi|^2(1 - \tfrac{1}{4}|\psi|^2) \leqslant |\psi|^2 = 2(1 - \sqrt{1 - |\hat{\psi}|^2}). \tag{6.2}$$

Differentiating these relations we see that

$$|\nabla\psi - \nabla\hat{\psi}| = O(|\hat{\psi}|\,|\nabla\hat{\psi}|) + O(|\hat{\psi}|^2) = O(|\psi|\,|\nabla\psi|) + O(|\psi|^2),$$
$$|\nabla^2\psi - \nabla^2\hat{\psi}| = O(|\hat{\psi}||\nabla^2\hat{\psi}|) + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}|^2)$$
$$= O(|\psi||\nabla^2\psi|) + O(|\nabla\psi|^2) + O(|\psi|^2) \tag{6.3}$$

and therefore, we derive the following equation for $\hat{\psi}$ by taking the component of (4.2) orthogonal to the identity:

(6.4)
$$(\Delta\hat{\psi})^T + 2\hat{\psi} = -\,2\langle\nabla\hat{\psi}, \nabla Id\rangle\hat{\psi} - f_1^T - f_2^T + O(|\nabla\hat{\psi}|^2) + O(|\hat{\psi}|^2),$$

where T denotes orthogonal projection of a vector at $\mathbf{x} \in S^2$ onto $T_{\mathbf{x}}S^2$, i.e. onto the orthogonal complement of $\mathbf{x}$, and f_1 and f_2 are given by (2.8) and (2.9).

Next, we let e_1, e_2 be an orthonormal basis for $T_{\mathbf{x}}S^2$ with $D_{e_i}e_j(\mathbf{x}) = 0$, where D is the covariant derivative on TS^2. We calculate at $\mathbf{x}$:

$$D_{e_i}\hat{\psi}(\mathbf{x}) = e_i(\hat{\psi})(\mathbf{x}) - ((e_i(\hat{\psi}) \cdot \mathbf{x})\mathbf{x} = e_i(\hat{\psi})(\mathbf{x}) + (\hat{\psi}(\mathbf{x}) \cdot e_i(\mathbf{x}))\mathbf{x}$$

and, since $\hat{\psi}(\mathbf{x}) = \sum_{i=1}^2 (\hat{\psi}(\mathbf{x}) \cdot e_i)e_i$, we conclude that

$$(\Delta\hat{\psi})^T + \hat{\psi} = \Delta_{TS^2}\hat{\psi},$$

where Δ_{TS^2} is the (rough) connection Laplacian on vector fields on S^2. Next it follows from [**3**], Proposition A3, that

$$\Delta_H\hat{\psi} = \Delta_{TS^2}\hat{\psi} - \hat{\psi},$$

where Δ_H is the (negative semi-definite) Hodge Laplacian. Furthermore, it was calculated in [**10**] that

$$-\Delta_{TS^2}\hat{\psi} - \hat{\psi} = -(\Delta\hat{\psi})^T - 2\hat{\psi} = J\hat{\psi},$$

where J is the Jacobi operator of the energy functional at the identity on S^2. By standard Hodge theory, the spectrum of Δ_{TS^2} is the same as the spectrum of Δ on functions shifted up by 1, i.e., the spectrum of Δ_{TS^2} is $\{-1, -5, \dots\}$. Indeed, if $\Delta\phi + c\phi = 0$ then $\Delta_{TS^2}(\nabla\phi) + (c-1)\nabla\phi = 0$ and $\Delta_{TS^2}(*\nabla\phi) + (c-1)(*\nabla\phi) = 0$ where $*$ is rotation by 90° in TS^2. These two equations follow from the above relation between Δ_H and Δ_{TS^2} and the facts that the exterior derivative d and $*$ both commute with Δ_H; the second equation follows from the first and the conformal invariance of the Dirichlet integral in two dimensions. So, the kernel of J consists precisely of the span of the gradient of the linear functions on S^2 and their 90° rotations. But this is precisely the tangent space Z of the Möbius group at the identity; the flow of the gradient of a linear function is a dilation and the flow of a 90° rotation of the gradient of a linear function is a rotation.

We shall be making use of the elliptic estimate

$$\|\hat{\psi}\|_{W^{2,p}} \leqslant C(\|J\hat{\psi}\|_{L^p} + \|\hat{\psi}_0\|_{L^p}),$$

where $\hat{\psi}_0$ is the orthogonal projection of $\hat{\psi}$ onto the kernel of J with respect to the inner product on $L^2(S^2)$. We start by estimating $\hat{\psi}_0$. From the minimising property of $\|\nabla(v - Id)\|^2_{L^2(S^2)}$ it follows that

$$-\int_{S^2} \nabla v \cdot \nabla\xi \, dA_{S^2} + \int_{S^2} \nabla Id \cdot \nabla\xi \, dA_{S^2} = 0 \quad \forall\, \xi \in Z.$$

Now $\nabla Id \cdot \nabla\xi = \operatorname{div}\xi$ and $\int_{S^2}(\operatorname{div}\xi)\, dA_{S^2} = 0$. Therefore

$$\int_{S^2} v \cdot \Delta\xi \, dA_{S^2} = 0 \quad \forall\, \xi \in Z. \tag{6.5}$$

We have

$$\Delta\xi(\mathbf{x}) = (\Delta\xi)^T(\mathbf{x}) + (\Delta\xi\cdot\mathbf{x})\mathbf{x}$$

and, since $\xi \in Z$, $(\Delta\xi)^T = -2\xi$. If, as before, e_1, e_2 is an orthonormal basis for $T_{\mathbf{x}}S^2$ so that $D_{e_i}e_j(\mathbf{x}) = 0$, then

$$\begin{aligned}
\Delta\xi\cdot\mathbf{x} &= \sum_{i=1}^{2}\Big(e_i\big(e_i(\xi)\cdot\mathbf{x}\big) - \big(e_i(\xi)\cdot e_i\big)(\mathbf{x})\Big)\\
&= -\sum_{i=1}^{2}\Big(e_i(\xi\cdot e_i)(\mathbf{x}) + \big(e_i(\xi)\cdot e_i\big)(\mathbf{x})\Big)\\
&= -\sum_{i=1}^{2}\Big(\big(e_i(\xi)\cdot e_i\big)(\mathbf{x}) + \big(e_i(\xi)\cdot e_i\big)(\mathbf{x})\Big)\\
&= -2\operatorname{div}\xi(\mathbf{x}),
\end{aligned}$$

where we used $\xi\cdot\mathbf{x} = 0$ in the second line and $\xi\cdot e_i(e_i) = \xi\cdot D_{e_i}e_i = 0$ in the third line. Using these calculations of $\Delta\xi$ in (6.5) yields

$$\int_{S^2} v\cdot\xi\, dA_{S^2} + \int_{S^2}(v\cdot\mathbf{x})(\operatorname{div}\xi)\, dA_{S^2} = 0,$$

and, taking into account (6.2), the fact that ξ is tangent to S^2 and $\int_{S^2}(\operatorname{div}\xi)\, dA_{S^2} = 0$, we obtain

$$\begin{aligned}
\int_{S^2}\hat\psi\cdot\xi\, dA_{S^2} &= -\int_{S^2}\sqrt{1-|\hat\psi|^2}(\operatorname{div}\xi)\, dA_{S^2}\\
&= \int_{S^2}\big(1-\sqrt{1-|\hat\psi|^2}\big)(\operatorname{div}\xi)\, dA_{S^2}.
\end{aligned}$$

We now choose $\xi = \hat\psi_0$ and get

$$\|\hat\psi_0\|^2_{L^2(S^2)} \leqslant \|\hat\psi\|^2_{L^\infty(S^2)}\int_{S^2}|\nabla\hat\psi_0|\, dA_{S^2}.$$

But $(\Delta\hat\psi_0)^T = -2\hat\psi_0$ because $\hat\psi_0 \in Z$ and therefore

$$\begin{aligned}
\int_{S^2}|\nabla\hat\psi_0|\, dA_{S^2} &\leqslant C\left(\int_{S^2}|\nabla\hat\psi_0|^2\, dA_{S^2}\right)^{1/2}\\
&= 2C\left(\int_{S^2}-\Delta\hat\psi_0\cdot\hat\psi_0\, dA_{S^2}\right)^{1/2}\\
&= 2C\|\hat\psi_0\|_{L^2(S^2)}.
\end{aligned}$$

We have proved that, for $p \in [\frac{4}{3}, \frac{3}{2}]$,

$$\|\hat{\psi}_0\|_{L^p(S^2)} \leqslant C\|\hat{\psi}_0\|_{L^2(S^2)} \leqslant C\|\hat{\psi}\|^2_{L^\infty(S^2)} \leqslant C\|\hat{\psi}\|_{L^\infty(S^2)}\|\hat{\psi}\|_{W^{2,p}}. \tag{6.6}$$

We next estimate $\|J\hat{\psi}\|_{L^p}$ by estimating the L^p norm of the right hand side of (6.4).

From (4.4), (A.2) and (5.5) we have,

$$|f_2| \leqslant C(\alpha-1)\left(\sup|\nabla \log \chi_\lambda|\right)|\nabla v| \leqslant C(\alpha-1)(\log\lambda)|\nabla v|,$$

where we have used $(\lambda-1) \leqslant C(\log\lambda)$ which holds because of the bound (5.5) on λ. Therefore,

$$\|f_2^T\|_{L^p(S^2)} \leqslant C(\alpha-1)(\log\lambda)\|\nabla v\|_{L^p(S^2)}. \tag{6.7}$$

To estimate $\|f_1\|_{L^p(S^2)}$ we recall that

$$|\nabla v|^2 = |\nabla Id|^2 + 2\langle\nabla Id, \nabla\psi\rangle + |\nabla\psi|^2 = 2 + 2\operatorname{div}\psi + |\nabla\psi|^2$$

and therefore,

$$\left|\nabla(|\nabla v|^2)\right| \leqslant C\,|\nabla^2\psi|\,(1+|\nabla v|).$$

It follows from (2.8) and the estimate

$$\frac{\chi_\lambda|\nabla v|(1+|\nabla v|)}{2+\chi_\lambda|\nabla v|^2} \leqslant \frac{1}{2}\sqrt{\chi_\lambda} + 1 \leqslant 1+\lambda \leqslant C$$

that

$$|f_1| \leqslant C(\alpha-1)|\nabla^2\psi|\left(\frac{\chi_\lambda|\nabla v|(1+|\nabla v|)}{2+\chi_\lambda|\nabla v|^2}\right) \leqslant C(\alpha-1)|\nabla^2\psi|, \tag{6.8}$$

where we have used $\chi_\lambda < \lambda^2$ and the bound (5.5) on λ.

Using these bounds on f_1 and f_2 and (6.3) in (6.4), keeping in mind that $\|\nabla v\|_{L^p(S^2)}$ is bounded by the energy of v, we see, also using (6.6), that

$$\begin{aligned}\|\hat{\psi}\|_{W^{2,p}} &\leqslant C(\|J\hat{\psi}\|_{L^p} + \|\hat{\psi}_0\|_{L^p}) \\ &\leqslant C\|\hat{\psi}\|_{L^\infty(S^2)}\|\nabla\hat{\psi}\|_{L^p(S^2)} + C(\alpha-1)\big(\|\nabla^2\hat{\psi}\|_{L^p(S^2)} + (\log\lambda)\big) \\ &\qquad + C\|\nabla\hat{\psi}\|^2_{L^{2p}(S^2)} + C\|\hat{\psi}\|_{L^\infty(S^2)}\|\hat{\psi}\|_{W^{2,p}}.\end{aligned}$$

We now appeal to the Gagliardo-Nirenberg interpolation inequality

$$\|\nabla\hat{\psi}\|^2_{L^{2p}(S^2)} \leqslant C\|\nabla\hat{\psi}\|_{L^2(S^2)}\|\nabla\hat{\psi}\|_{W^{1,p}(S^2)}$$

and use (4.1) with δ_0 and $\alpha_0 - 1$ sufficiently small, to conclude that

$$\|\hat{\psi}\|_{W^{2,p}} \leqslant C(\alpha-1)(\log\lambda). \tag{6.9}$$

The estimate (6.1) follows from (6.9) on taking (6.3) into account. q.e.d.

7. Proof of Theorem 1.2

We start with a classification result for α-harmonic maps of degree 0 with "small" energy.

Proposition 7.1. *Fix $\eta > 0$. Then there exists $\overline{\alpha} - 1 > 0$ small, $\overline{\alpha}$ depending only on η, such that if $1 < \alpha \leqslant \overline{\alpha}$ and $u\colon S^2 \to S^2$ is α-harmonic, of degree zero and $E(u) \leqslant 8\pi - \eta$, then u is constant.*

Proof. If the proposition is not true, then we can find a sequence $\alpha_j \searrow 1$ and a sequence of non-constant maps $u_j\colon S^2 \to S^2$ such that $\deg(u_j) = 0$, u_j is α_j-harmonic and $E(u_j) \leqslant 8\pi - \eta \; \forall\, j \in \mathbb{N}$. By the results of Sacks-Uhlenbeck [9] we know that two possibilities can occur:

(i) u_j converges smoothly to a harmonic map $u^*\colon S^2 \to S^2$ of degree zero which is therefore constant, or

(ii) there exist two harmonic maps $u^*\colon S^2 \to S^2$ and $u^B\colon S^2 \to S^2$ and a point $p \in S^2$ such that, a subsequence of u_j (still denoted by u_j) converges smoothly on compact subsets of $S^2 \setminus \{p\}$ to u^* and a nontrivial bubble u^B develops at p. Since $E(u^B) < 8\pi$ we have $|\deg(u^B)| = 1$. By choosing the orientation of the domain S^2 relative to that of the image S^2 appropriately, we may, and we will, assume that $4\pi \deg(u^B) = E(u^B) = 4\pi$. (It follows that u^* is constant, but this is not of direct importance to us.)

In case (i), $E(u_j) \to 0$ as $j \to \infty$. But then, by Theorem 3.3 in Sacks-Uhlenbeck [9], there exists $\varepsilon > 0$ and $\alpha_0 > 1$ such that, if v is α-harmonic, $1 \leqslant \alpha < \alpha_0$ and $E(v) < \varepsilon$ then v is constant. In particular, u_j is constant for large enough j, contrary to our assumption.

In case (ii), we can find a sequence D_j of discs centred at p, whose radii r_j decrease to 0 and a sequence $\sigma_j \searrow 0$ such that $\sigma_j / r_j \uparrow +\infty$ and, if

$$v_j(z) := u_j(r_j z), \quad |z| < \sigma_j / r_j,$$

then

$$\sup_{|z| < \sigma_j / r_j} \left(|v_j(z) - u^B(z)| + |\nabla v_j(z) - \nabla u^B(z)| \right) \to 0 \;\text{ as }\; j \to \infty.$$

In particular,

$$\int_{D_j} J(u_j)\, dA_{S^2} \to 4\pi \deg(u^B) = 4\pi \;\text{ as }\; j \to \infty$$

and

$$\int_{D_j} |\nabla u_j|^2 \, dA_{S^2} \to \int_{S^2} |\nabla u^B|^2 \, dA_{S^2} = 8\pi \;\text{ as }\; j \to \infty.$$

But then, for large enough j,

$$\begin{aligned}\int_{S^2} J(u_j)\,dA_{S^2} &= \int_{D_j} J(u_j)\,dA_{S^2} + \int_{S^2\setminus D_j} J(u_j)\,dA_{S^2}\\ &\geqslant (4\pi - \tfrac{1}{4}\eta) - \frac{1}{2}\int_{S^2\setminus D_j} |\nabla u_j|^2\,dA_{S^2}\\ &\geqslant (4\pi - \tfrac{1}{4}\eta) - \big((8\pi-\eta) - (4\pi - \tfrac{1}{4}\eta)\big)\\ &= \tfrac{1}{2}\eta > 0.\end{aligned}$$

Therefore, for large enough j, u_j has nonzero degree, which is again contrary to our assumption. q.e.d.

Proof of Theorem 1.2. Since we have Proposition 7.1 at our disposal, we only need (modulo orientation) to classify the α-harmonic maps of degree 1 which satisfy the assumptions of Theorem 1.2.

In order to do this, we go back to the proof of Proposition 5.1, using our improved estimate (6.1) to obtain

$$\|\nabla\psi\|_{L^{2\alpha+2}(S^2)} \leqslant C(\alpha-1)(\log\lambda).$$

The string of inequalities in (5.8) now becomes

$$C'^{-1}(\alpha-1)\frac{\log\lambda}{1+\log\lambda} \leqslant \frac{d}{d\log\lambda}E_{\alpha,\lambda}(Id) \leqslant C(\alpha-1)^2(\log\lambda).$$

By demanding that α be sufficiently close, but not equal, to 1, we conclude that $\lambda = 1$. But by (6.1), this implies that v is the identity and the Möbius transformation M which minimises $\|\nabla(u_M - Id)\|^2_{L^2(S^2)}$ must be a rotation. So u is a rotation, as claimed. q.e.d.

8. Other α-harmonic maps of degree 1

In this section we shall construct rotationally symmetric α-harmonic maps of degree 1 that are not rotations. Of course, their α-energy will be strictly bigger than $2^{2\alpha+1}\pi$. We shall also construct α-harmonic maps of degree 1 from the disk to the sphere which map the boundary circle to a point. This was proved to not be possible for a harmonic map by Lemaire (see, for instance, (12.6) in [**5**]). We shall further construct a map of degree 1 from the annulus to the sphere which is α-harmonic and which maps the boundary circles to antipodal points.

8.1. Rotationally symmetric maps. For $n \in \mathbb{N}$, $r \in [n\pi, (n+1)\pi]$ and $\theta \in [0, 2\pi]$, we consider a parameterisation of S^2 given by

$$(r,\theta) \mapsto (\sin r\,\cos\theta,\ \sin r\,\sin\theta,\ \cos r).$$

This parameterisation is orientation preserving if n is even and orientation reversing if n is odd. In these coordinates, the metric on S^2 is given by

$$dr^2 + (\sin r)^2 d\theta^2.$$

We shall be interested in maps u_f from S^2 to itself which are of the form

$$(r,\theta) \mapsto (\sin(f(r))\cos\theta,\ \sin(f(r))\sin\theta,\ \cos(f(r)))$$

with

$$f\colon [0,\pi] \to \mathbb{R},\ f(0)=0,\ f(\pi)=n\pi.$$

These maps are rotationally symmetric and, for $n>1$, wrap over S^2 more than once; the degree is zero if n is even and one if n is odd. The energy density $e(u_f)$ of such a map is given by

$$e(u_f) = \frac{1}{2}\left((f')^2 + \frac{(\sin f)^2}{(\sin r)^2}\right)$$

and, in order to express the α-harmonic map operator (1.5) at u_f, we compute:

$$\begin{aligned}
\frac{\partial u_f}{\partial r} &= f'(r)\big(\cos(f(r))\cos\theta,\ \cos(f(r))\sin\theta,\ -\sin(f(r))\big),\\
\frac{\partial u_f}{\partial \theta} &= \big(-\sin(f(r))\sin\theta,\ \sin(f(r))\cos\theta,\ 0\big),\\
\frac{\partial^2 u_f}{\partial r^2} &= \frac{f''(r)}{f'(r)}\frac{\partial u_f}{\partial r} - (f'(r))^2 u_f,\\
\frac{\partial^2 u_f}{\partial \theta^2} &= -\sin(f(r))(\cos\theta,\ \sin\theta,\ 0)\\
&= -\sin(f(r))\left(\sin(f(r))u_f + \frac{\cos(f(r))}{f'(r)}\frac{\partial u_f}{\partial r}\right).
\end{aligned}$$

The Laplacian writes as $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\cos r}{\sin r}\frac{\partial}{\partial r} + \frac{1}{(\sin r)^2}\frac{\partial^2}{\partial \theta^2}$ and so,

$$\begin{aligned}
&\Delta u_f + |\nabla u_f|^2 u_f + (\alpha-1)(2+|\nabla u_f|^2)^{-1}\nabla(|\nabla u_f|^2)\cdot\nabla u_f\\
&\quad= \frac{f''(r)}{f'(r)}\frac{\partial u_f}{\partial r} - (f'(r))^2 u_f + \frac{\cos r}{\sin r}\frac{\partial u_f}{\partial r}\\
&\qquad - \frac{\sin(f(r))}{(\sin r)^2}\left(\sin(f(r))u_f + \frac{\cos(f(r))}{f'(r)}\frac{\partial u_f}{\partial r}\right)\\
&\qquad + \left((f')^2 + \frac{(\sin f)^2}{(\sin r)^2}\right)u_f + \frac{(\alpha-1)}{(2+|\nabla u_f|^2)}\frac{\partial|\nabla u_f|^2}{\partial r}\frac{\partial u_f}{\partial r}\\
&\quad= \frac{1}{f'(r)}\frac{\partial u_f}{\partial r}(f''(r) + \frac{\cos r}{\sin r}f'(r) - \frac{(\cos f(r))(\sin f(r))}{(\sin r)^2}\\
&\qquad + \frac{(\alpha-1)}{(2+|\nabla u_f|^2)}\frac{\partial|\nabla u_f|^2}{\partial r}).
\end{aligned}$$

Thus u_f is α-harmonic if

$$f''(r) + \frac{\cos r}{\sin r} f'(r) - \frac{(\cos f(r))(\sin f(r))}{(\sin r)^2} + \frac{(\alpha - 1)}{(2 + |\nabla u_f|^2)} \frac{\partial |\nabla u_f|^2}{\partial r} = 0. \tag{8.1}$$

8.2. Construction of rotationally symmetric α-harmonic maps. We shall specialise to the case $n = 3$ (though our arguments will work for any other integer value of n) and we define

$$X := \{f \colon [0, \pi] \to \mathbb{R} : u_f \in W^{1,2\alpha}(S^2, \mathbb{R}^3), \quad f(0) = 0, \ f(\pi) = 3\pi\}.$$

Let $\Lambda := \inf_{f \in X} I(f)$ where

$$I(f) := E_\alpha(u_f) = \pi \int_0^\pi \left(2 + (f')^2 + \frac{(\sin f)^2}{(\sin r)^2}\right)^\alpha \sin r \, dr.$$

A direct calculation shows that $f \in X$ is a critical point of I if, and only if, u_f is an α-harmonic map, i.e., if, and only if, f satisfies (8.1). This is a manifestation of the principle of symmetric criticality of Palais; see, for example, Remark 11.4(a) in [**2**]. The symmetry group in question here is the group $O(2)$ of the rotations about the axis $(0, 0, z)$ and reflections in planes containing the line $(0, 0, z)$.

If f_j is a sequence in X, we shall write u_j instead of u_{f_j}. Let f_j be a sequence in X such that $I(f_j) \downarrow \Lambda$. Then u_j is a bounded sequence in $W^{1,2\alpha}(S^2, \mathbb{R}^3)$ and therefore, a subsequence, still denoted by u_j, converges weakly in $W^{1,2\alpha}(S^2, \mathbb{R}^3)$ and uniformly in $C^0(S^2, \mathbb{R}^3)$ to $u^* := u_{f^*}$ for some $f^* \in X$.[1] By the lower semi-continuity of E_α with respect to weak convergence in $W^{1,2\alpha}(S^2, \mathbb{R}^3)$, we have that $I(f^*) = E_\alpha(u^*) = \Lambda$. Thus u^* is an α-harmonic map of degree 1 which is not a rotation. We get a lower bound on $E_\alpha(u^*)$ by arguing as in (1.7) and (1.8):

$$\begin{aligned} E_\alpha(u^*) &= \pi \int_0^\pi \left(2 + (f^{*\prime})^2 + \frac{(\sin f^*)^2}{(\sin r)^2}\right)^\alpha \sin r \, dr \\ &\geqslant \pi \left(\int_0^\pi \left(2 + (f^{*\prime})^2 + \frac{(\sin f^*)^2}{(\sin r)^2}\right) \sin r \, dr\right)^\alpha \left(\int_0^\pi \sin r \, dr\right)^{1-\alpha} \\ &\geqslant 2^{1-\alpha}\pi \left(\int_0^\pi \left(2 \sin r + 2|f^{*\prime}(\sin f^*)|\right) \, dr\right)^\alpha. \end{aligned}$$

[1]This uniform convergence in C^0 fails when $\alpha = 1$ and this is precisely why this construction does not yield harmonic maps of the type considered in this section.

There exist r_1, $r_2 \in (0,\pi)$ such that $f^*(r_1)=\pi$ and $f^*(r_2)=2\pi$. Then

$$\begin{aligned}\int_0^{\pi} |f^{*\prime}(\sin f^*)|\,dr &\geqslant \int_0^{r_1} f^{*\prime}(\sin f^*)\,dr - \int_{r_1}^{r_2} f^{*\prime}(\sin f^*)\,dr \\ &\quad + \int_{r_2}^{\pi} f^{*\prime}(\sin f^*)\,dr \\ &= -\cos f^*(r)|_0^{r_1} + \cos f^*(r)|_{r_1}^{r_2} - \cos f^*(r)|_{r_2}^{\pi} \\ &= 6.\end{aligned}$$

It follows that

$$E_\alpha(u^*) \geqslant 2^{3\alpha+1}\pi.$$

Let D_1 be the geodesic disc in S^2 of radius r_1 and centred at $(0,0,1)$, let D_2 be the geodesic disc in S^2 of radius r_2 and centred at $(0,0,-1)$ and let A be the annulus between D_1 and D_2. Then the restriction of u^* to D_1 is an α-harmonic map of degree 1 onto all of S^2 which maps all of the boundary of D_1 to $(0,0,-1)$. Similarly, the restriction of u^* to A is an α-harmonic map of degree 1 onto all of S^2 which maps the two boundaries of A to antipodal points of S^2.

A more explicit construction of the above maps, via finite-dimensional reductions, is being worked out in [**6**].

Appendix A. An estimate for the function χ_λ

Lemma A.1. *There is a constant $C>0$, independent of $\lambda \geqslant 1$, such that*

$$\|\nabla \log \chi_\lambda\|_{L^2(S^2)} \leqslant \begin{cases} C(\log\lambda) & \text{for } 0 \leqslant \log\lambda \leqslant 1; \\ C(\log\lambda)^{\frac12} & \text{for } \log\lambda \geqslant 1.\end{cases} \tag{A.1}$$

Proof. First of all we note that

$$\frac{d}{dr}\log\chi_\lambda(r) = 4\left(\frac{\lambda^2 r}{1+\lambda^2 r^2} - \frac{r}{1+r^2}\right) = \frac{4r(\lambda^2-1)}{(1+r^2)(1+\lambda^2r^2)}, \tag{A.2}$$

and hence we estimate

$$\begin{aligned}\|\nabla \log\chi_\lambda\|_{L^2(S^2)} &= 4(\lambda^2-1)\left(8\pi\int_0^\infty \frac{r^3}{(1+\lambda^2r^2)^2(1+r^2)^4}\,dr\right)^{1/2} \\ &\leqslant 4(\lambda^2-1)(8\pi)^{1/2} \\ &\qquad \left(\int_0^{1/\lambda} r^3dr + \frac{1}{\lambda^4}\int_{1/\lambda}^1 \frac{1}{r}\,dr + \frac{1}{\lambda^4}\int_1^\infty \frac{1}{r^9}\,dr\right)^{1/2}.\end{aligned}$$

So,

$$\|\nabla\log\chi_\lambda\|_{L^2(S^2)} \leqslant 4(8\pi)^{1/2}\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda-1}{\lambda}\right)\left(\frac14+\frac18+\log\lambda\right)^{1/2}.$$

Now, for $1 \leqslant \lambda \leqslant e$, we have

$$\frac{\lambda - 1}{\lambda} \leqslant \log \lambda \quad \text{and} \quad (\tfrac{1}{4} + \tfrac{1}{8} + \log \lambda)^{1/2} \leqslant \sqrt{2}$$

and, for $\log \lambda \geqslant 1$, we have

$$\frac{\lambda - 1}{\lambda}(\tfrac{1}{4} + \tfrac{1}{8} + \log \lambda)^{1/2} \leqslant \sqrt{2}(\log \lambda)^{1/2}$$

which yield the desired estimate (A.1). q.e.d.

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