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LIMITING BEHAVIOR OF SEQUENCES OF PROPERLY EMBEDDED MINIMAL DISKS

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Abstract

We develop a theory of "minimal θ-graphs" and characterize the behavior of limit laminations of such surfaces, including an understanding of their limit leaves and their curvature blow-up sets. We use this to prove that it is possible to realize families of catenoids in euclidean space as limit leaves of sequences of embedded minimal disks, even when there is no curvature blow-up. Our methods work in a more general Riemannian setting, including hyperbolic space. This allows us to establish the existence of a complete, simply connected, minimal surface in hyperbolic space that is not properly embedded.

1. Introduction

Let D_n be a sequence of properly embedded minimal disks in an open subset W of a Riemannian 3-manifold. Then there is a subsequence $D_{n(i)}$ such that the curvatures of the $D_{n(i)}$ blow up at the points of closed subset $K \subset W$ (possibly empty), and such that the $D_{n(i)}$ converge smoothly away from K to a minimal lamination $\mathcal{L}$ of $W \setminus K$. One would like to know what closed sets K and what laminations $\mathcal{L}$ can arise in this way. Colding and Minicozzi proved very strong theorems about such K and $\mathcal{L}$. In particular, they showed (under mild hypotheses on the ambient metric) that K is contained in a rectifiable curve, and that for each point p in K, there is a unique leaf L of the lamination such that $p \in \overline{L}$ and such that $L \cup \{p\}$ is smooth. (See [6, Section I.1]. See also [6, Theorem 0.1] for a closely related result.) Later it was shown that K is contained in a C^1 curve, and that $L \cup \{p\}$ is perpendicular to that curve. See [17] and [24].

In this paper, we give a more detailed description of the lamination and of the singular set for a certain rich class of minimal disks. In particular, we prove

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Theorem 1.1. *Let* $\mathbf{B} \subset \mathbf{R}^3$ *be the unit ball and let* $Z \subset \mathbf{R}^3$ *be the vertical coordinate axis. Suppose* D_n *is a sequence of properly embedded minimal disks in the ball* $\mathbf{B}$ *with the property that each disk* $D = D_n$ *satisfies*

(1) $\mathbf{B} \cap Z \subset D$, *and the images of* $D \setminus Z$ *under rotations about* Z *foliate* $\mathbf{B} \setminus Z$.

Then there is a subsequence $D_{n(i)}$, *a relatively closed subset* K *of* $\mathbf{B} \cap Z$, *and a minimal lamination* $\mathcal{L}$ *of* $\mathbf{B} \setminus K$ *such that*

1. *The curvatures of the* $D_{n(i)}$ *blow-up precisely at the points of* K.
2. *The* $D_{n(i)}$ *converge smoothly away from* K *to the lamination* $\mathcal{L}$.
3. *The limit leaves of* $\mathcal{L}$ *are catenoids and rotationally invariant disks.*
4. *The curvature blow-up set* K *is precisely the set of centers of the disks in statement 3.*
5. *If* L *is a non-limit leaf of* $\mathcal{L}$, *then* $L \setminus Z$ *and its rotations around* Z *foliate an open subset of* $\mathbf{B} \setminus Z$. *In fact, each component of the complement of the limit leaves of* $\mathcal{L}$ *in* $\mathbf{B} \setminus Z$ *is foliated by the rotations of such a non-limit leaf.*

It is not hard to produce examples of disks D_n satisfying condition (1) of Theorem 1.1. In particular, let $C_n \subset \partial\mathbf{B}$ be a smooth, simple closed curve that intersects each horizontal circle in $\partial\mathbf{B}$ in exactly two diametrically opposite points. Then there is unique embedded minimal disk D_n such that $\partial D_n = C_n$ and such that $Z \cap \mathbf{B} \subset D_n$. Furthermore, it is easy to show that the disk satisfies condition (1) of Theorem 1.1. See Section 3 below.

By choosing suitable curves C_n and taking the corresponding disks D_n, we can produce interesting examples of blow-up sets K and limit laminations $\mathcal{L}$. For example, let $\mathcal{M}$ be the lamination of $\mathbf{B}$ consisting of all the area-minimizing catenoids in $\mathbf{B}$ with axis Z that are symmetric about the xy-plane, together with all horizontal disks that are disjoint from those catenoids (See Figure 1.) We show that there is a sequence D_n of properly embedded minimal disks in $\mathbf{B}$ with a limit lamination $\mathcal{L}$ (from Theorem 1.1) whose rotationally invariant leaves are precisely the surfaces in $\mathcal{M}$ and that has exactly one leaf that is not rotationally invariant. (That additional leaf contains a segment of Z.) More generally, if $\mathcal{M}^*$ is essentially any symmetric sublamination of $\mathcal{M}$, we show that there is a sequence D_n such that the rotationally invariant leaves of the limit lamination are precisely the surfaces in $\mathcal{M}^*$. (See Theorem 6.3.) Of course by Statement 4 of Theorem 1.1, the curvatures of the $D_{n(i)}$ blow-up precisely at the centers of the disks in $\mathcal{M}^*$.

The results stated above remain true if the Euclidean metric on $\mathbf{B}$ is replaced by any suitable rotationally symmetric Riemannian metric. In particular, they remain true for the Poincaré metric on $\mathbf{B}$. We show that

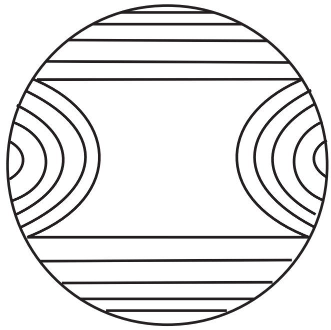

Figure 1. Limit Leaves $\mathcal{M}$ in the unit ball B. Depicted here in cross section is the lamination $\mathcal{M}$ of **B** consisting of all area-minimizing catenoids with axis Z and symmetry plane $\{z = 0\}$, together with all horizontal disks that are disjoint from the catenoids. Essentially any symmetric sublamination $\mathcal{M}^*$ of $\mathcal{M}$ can be realized as the set of limit leaves of a limit lamination of a sequence of properly embedded minimal disks in **B**. This is proved in Theorem 6.3.

many kinds of limit laminations and blow-up sets occur for sequences of disks that are properly embedded in all of hyperbolic space. This is in very sharp contrast to the situation in $\mathbf{R}^3$. Consider a sequence $\mathbf{B}_1 \subset \mathbf{B}_2 \subset \dots$ of balls that exhaust $\mathbf{R}^3$ and properly embedded minimal disks $D_n \subset \mathbf{B}_n$. By work of Colding and Minicozzi [5], with extensions by Meeks-Rosenberg [18] and Meeks [17], there are only three possible behaviors (after passing to a subsequence):

- The D_n converge smoothly to a helicoid.
- The D_n converge smoothly to a lamination of $\mathbf{R}^3$ by parallel planes.
- The curvature blow-up set K is a straight line, and disks D_n converge smoothly in $\mathbf{R}^3 \setminus K$ to the foliation consisting of all planes perpendicular to K.

Note that if D_n is the portion in the ball **B** of a helicoid with axis Z and if the curvatures of the D_n tend to infinity, then the curvature blow-up set is $Z \cap \mathbf{B}$. Colding and Minicozzi [3] constructed an example in which the blow-up set is $Z^- \cap \mathbf{B}$ (where Z^- is the set of points $(0, 0, z)$ with $z \leq 0$.) Khan [13] then showed that K can be any finite subset of $Z \cap \mathbf{B}$. The authors of this paper proved that K can be any relatively closed subset of $Z \cap \mathbf{B}$ [12]. In particular, sets with non-integral Hausdorff dimension can arise as blow-up sets. (Subsequently, Kleene [14] gave another proof of this theorem.) In all of those examples, the limit leaves of the limit lamination are precisely the horizontal disks centered at points of K. (Indeed, in all of those examples, the disks

D_n satisfy condition (1) of Theorem 1.1, and they have an additional property: the tangent plane to D_n is not vertical except at points on Z.) In Section 2 we will develop the theory of embedded minimal disks, satisfying condition (1).

1.2. The mathematical advances in this paper.

1. We prove that it is possible to realize families of catenoids (as well as horizontal disks) as limit leaves of a limit lamination of embedded minimal disks, even when there is no curvature blow-up. This result raises the question of whether it is possible to produce limit leaves (of a limit lamination of a sequence of embedded minimal disks) that are neither disks nor annuli. Under the assumption that W is mean convex and contains no closed minimal surfaces, Bernstein and Tinaglia [2] have recently proved that the answer is no.
2. The constructions to produce these examples work for more general Riemannian metrics (such as the Poincaré metric) on $\mathbf{B}$.
3. Colding and Minicozzi [4] proved a general Calabi-Yau conjecture for complete embedded minimal surfaces in $\mathbf{R}^3$ of finite topology: such a surface must be properly embedded. We use our limit lamination theory to prove that such a theorem fails in hyperbolic three-space, even for simply connected minimal surfaces. This was originally proved by Baris Coskunuzer [8] by entirely different methods. (More recently, Coskunuzer, Meeks, and Tinaglia [9] proved existence of complete, non-proper examples with constant mean curvature h for all $h \in [0, 1)$.) Our approach yields a minimal example on either side of any area-minimizing catenoid in hyperbolic space. See Theorem 9.3.

1.3. An outline of the sections of this paper. In Section 2, minimal θ-graphs are introduced and their limiting behavior is analyzed. They are essentially the surfaces satisfying condition (1) of Theorem 1.1, but in a more general Riemannian setting.

In Section 3 we prove the existence of minimal θ-graphs with prescribed boundary. In Section 4, we prove (under suitable hypotheses) smooth convergence at the boundary for sequences of minimal θ-graphs. In Section 5 we use a standard calibration-type argument to establish a necessary area-minimization property for laminations consisting of planes and catenoids to appear as the limit leaves of a limit lamination of minimal θ-graphs. We conjecture that it is a sufficient condition.

In Section 6, we use this existence results of the previous two sections to show that we can, under certain conditions, specify the limit leaves of a limit laminations coming from a sequence of minimal θ-disks. In particular, we construct sequences of embedded minimal disks whose limit laminations have prescribed limit-leaf sublaminations containing catenoids.

In Sections 7-9, we extend the results of Sections 3-6 to hyperbolic three-space. Handling the infinite-area minimal surfaces that arise there requires an additional argument. That argument (in Section 8) was inspired by the work of Collin and Rosenberg [7] on minimal graphs in $H^2 \times \mathbf{R}$. In Section 9, we prove (Theorem 9.3) that there exists a complete and simply connected embedded minimal surface in hyperbolic space that is not properly embedded.

2. θ-graphs

In this section we will denote by W a connected open set in $\mathbf{R}^3$ that is rotationally symmetric about the x_3-axis Z.

Definition 2.1. (θ-graph, spanning θ-graph) Let M be a smooth surface in $W \setminus Z$. Then M is a **θ-graph** if it can be written in the form

$$\{(r\cos\theta(r,z), r\sin\theta(r,z), z) : (r,z) \in V\}, \tag{2}$$

where $\theta(r,z) : V \to \mathbf{R}$ is a smooth, real-valued function, and V is an open subset of

$$\{(r,z) : \ r > 0, \ (r,0,z) \in W\}.$$

A θ-graph $M \subset W \setminus Z$ intersects each rotationally invariant circle at most once. We say that M is a **spanning θ-graph** if it intersects every rotationally invariant circle in $W \setminus Z$ exactly once. This is equivalent to the assertion that the domain of definition of $\theta(r,z)$ equals $\{(r,z) : r > 0, \ (r,0,z) \in W\}$, and also equivalent to the requirement that M and its rotated images foliate $W \setminus Z$.

Remark 2.2 (Simple examples of θ-graphs)**.** Let $W = \mathbf{R}^3$ and $V = \{(r,z) : \ r > 0\}$. If we let $\theta(r,z) = c$ in (2), then the surface is a vertical halfplane with boundary Z. If we let $\theta(r,z) = z/\alpha$, for any $\alpha \neq 0$, then the surface is a half-helicoid with pitch $2\pi\alpha$ and axis Z.

Lemma 2.3. *Let $W \subset \mathbf{R}^3$ be a rotationally invariant domain. Suppose M is a smooth embedded surface in $W \setminus Z$. Then the following two conditions are equivalent:*

1. *a) Given any rotationally invariant circle S, either M is disjoint from S or intersects S precisely once, and the intersection is transverse.*
   b) Any closed curve in M has winding number 0 about Z.
2. *M is a θ-graph.*

Proof. Statement $1(a)$ is equivalent to a weakened form of Statement 2, produced by replacing the function $\theta : V \to \mathbf{R}$ in Definition 2.1 by a smooth function taking values in $\mathbf{R}$ modulo 2π. The function θ lifts to a single-valued function into $\mathbf{R}$ if and only if assertion $1(b)$ holds. q.e.d.

2.4. The relationship between spanning θ-graphs and the surfaces of Theorem 1.1. We are interested in properly embedded minimal surfaces $M \subset W$ with $\partial M \subset \partial W$ that satisfy the following property:

$$W \cap Z \subset M, \text{ and the rotations of } M \setminus Z \text{ foliate } W \setminus Z. \tag{3}$$

This is condition (1) of Theorem 1.1 stated for the domain W. As indicated in the introduction, all the existence theorems for sequences of minimal disks are proved by producing surfaces of this kind. Their intimate relationship with spanning θ-graphs is given by the following lemma.

Lemma 2.5. *Let $W \subset \mathbf{R}^3$ be a simply connected domain that is rotationally symmetric around Z. Let M be a smooth, properly embedded surface in W. Then M satisfies (3) if and only if $M \setminus Z$ consists of two components, each of which is a spanning θ-graph, and the components are related by ρ_Z, $180°$ rotation about Z by π.*

Proof. The lemma follows immediately from the definitions. q.e.d.

We will focus on spanning θ-graphs in this paper, mindful that Lemma 2.5 provides the link between these graphs and their doubles, the surfaces of Theorem 1.1 and its generalization, Theorem 2.10 below.

2.6. θ-graphs considered as graphs in the simply connected covering of $W \setminus Z$. We will have occasion in Section 4 and in the Appendix to view θ-graphs as surfaces lying in the simply connected covering of $W \setminus Z$. Suppose we have a domain

$$V \subset \{(r, z) : r > 0,\ (r, 0, z) \in W\}. \tag{4}$$

For $p = (r, z) \in V$, and $\theta \in \mathbf{R}$, let $\pi : V \times \mathbf{R} \to W \setminus Z$ be the mapping $\pi(p, \theta) = (r\cos\theta, r\sin\theta, z)$. Note that a rotation around Z in W corresponds to a vertical translation in $V \times \mathbf{R}$. In this setting, the definition of a θ-graph M (Definition 2.1) is equivalent to the following:

> *The surface M can be lifted to $V \times \mathbf{R}$ as a graph of a smooth function $\theta : V \to \mathbf{R}$.*

Suppose now that W is endowed with a rotationally invariant metric g. Pulling back g to $V \times \mathbf{R}$ produces a metric g^* on $V \times \mathbf{R}$ in which vertical translations are isometries (corresponding to rotations in W). Note that the metric g^* on $V \times \mathbf{R}$ is not the product metric. A surface M is g-minimal in $W \setminus Z$ if and only if its lift M^* is g^*-minimal in $V \times \mathbf{R}$.

The simple examples in Remark 2.2 with $W = \mathbf{R}^3$ and $V = \{(r, z) : r > 0, \}$ are minimal surfaces. They lift to minimal surfaces $V \times \mathbf{R}$: The vertical halfplane in $\mathbf{R}^3$ bounded by Z lifts to a horizontal planar slice $\theta(r, z) = c$; the half-helicoid in $\mathbf{R}^3$ with axis Z lifts to the graph of $\theta(r, z) = z/\alpha$, $\alpha \neq 0$, a halfplane that is neither vertical nor horizontal.

Theorem 2.7 (Boundary regularity theorem for minimal θ-graphs)**.** *Suppose that $M \subset W \setminus Z$ is a spanning θ-graph that is minimal for a smooth, rotationally invariant metric on W. Then $M \cup (W \cap Z) = \overline{M} \cap W$ is a smooth manifold-with-boundary, the boundary being $Z \cap W$.*

Now suppose that W is bounded and simply connected and that the metric extends smoothly to $\overline{W}$. Let $\Gamma = \overline{M} \cap \partial W$. If $\Gamma \cup \rho_Z \Gamma$ is a smooth, simple closed curve, then $\overline{M \cup \rho_Z M}$ is a smooth, embedded manifold-with-boundary.

The first assertion is local, so it suffices to consider the case when W is a simply connected, which implies that M is a disk. (Otherwise, replace W and M by $\mathbf{B}(p,r) \subset W$ and $M \cap \mathbf{B}(p,r)$, where $p \in Z \cap W$.)

Thus Theorem 2.7 is an immediate consequence of the following more general boundary regularity theorem:

Theorem 2.8. [27]*. Suppose that U is an open subset of a smooth Riemannian 3-manifold, that C is a smooth, properly embedded curve in U, that D is a properly embedded minimal surface in $U \setminus C$, and that $D \cup C$ is topologically a manifold with boundary, the boundary being C. Then $D \cup C$ is a smooth manifold-with-boundary.*

2.9. Properties of limit laminations of sequences of minimal spanning θ-graphs. We now state and prove the main theorem of this section.

Theorem 2.10. *Suppose that the open unit ball $\mathbf{B}$ in $\mathbf{R}^3$ is endowed with a smooth Riemannian metric that is rotationally invariant around Z. Suppose that D_n is a sequence of minimal spanning θ-graphs in $\mathbf{B} \setminus Z$. Then, after passing to a subsequence, the D_n converge smoothly on compact subsets of $\mathbf{B} \setminus Z$ to a minimal lamination $\mathcal{L}$ of $\mathbf{B}$ with the following properties:*

1. *Each leaf of $\mathcal{L}$ is either rotationally symmetric about Z or is a θ-graph.*
2. *Each rotationally invariant circle in $\mathbf{B}$ either is contained in a rotationally invariant leaf of $\mathcal{L}$ or else intersects $\mathcal{L}$ transversely in a single point.*
3. *The limit leaves of $\mathcal{L}$ are precisely the leaves that are rotationally invariant about Z.*

Let $\mathcal{L}'$ be the set of rotationally invariant leaves in $\mathcal{L}$, and let K be the set of points in $\mathbf{B} \cap Z \cap \overline{\cup \mathcal{L}'}$.

4. *Each connected component $\mathcal{O}$ of $\mathbf{B} \setminus \overline{\cup \mathcal{L}'}$ contains a unique leaf L of $\mathcal{L}$. That leaf is a spanning θ-graph in $\mathcal{O}$, and $\mathcal{O}$ contains no other points of $\mathcal{L}$. Furthermore, $\overline{L} \cap \mathcal{O}$ is a smooth manifold-with-boundary (the boundary being $Z \cap \mathcal{O}$), and $\overline{D_n} \cap \mathcal{O}$ converges to $\overline{L} \cap \mathcal{O}$ smoothly on compact subsets of $\mathcal{O}$.*

5. *Each component I of $(\mathbf{B}\cap Z)\setminus K$ lies on the boundary of a non-limit leaf $L\in\mathcal{L}$. The leaf L can be extended smoothly by Schwarz reflection across Z, and no point in I lies in the closure of $\mathcal{L}\setminus L$.*
6. *The lamination $\mathcal{L}'$ extends smoothly to a lamination of $\mathbf{B}$. If $p\in K$, then there is a unique leaf $L(p)\in\mathcal{L}'$ whose closure contains p, and $L(p)\cup\{p\}$ is a smooth surface that meets Z orthogonally.*
7. *The curvature blowup of the D_n occurs precisely at the points of K.*

In the following corollary (and throughout the paper), ρ_Z denotes $180°$ rotation about Z.

Corollary 2.11. *The doubled disks $D_n\cup(\mathbf{B}\cap Z)\cup\rho_Z D_n$ converge smoothly in $\mathbf{B}\setminus K$ to the lamination $\mathcal{L}^*$ obtained from $\mathcal{L}$ as follows: for each connected component $\mathcal{O}$ of $\mathbf{B}\setminus\overline{\cup\mathcal{L}'}$, we replace the leaf L in $\mathcal{O}$ (see Statement 4) by $\overline{L}\cup\rho_Z\overline{L}\cap\mathcal{O}$. In particular, $\mathcal{L}$ and $\mathcal{L}^*$ have the same rotationally invariant leaves.*

Proof of theorem. The rotational Killing field $\partial/\partial\theta$ defines a Jacobi field J_n on each D_n. Note that D_n is stable because J_n never vanishes. (In fact D_n has a certain area-minimizing property: see Corollary 3.2.) Thus the curvature is uniformly bounded on compact subsets of $\mathbf{B}\setminus Z$, so a subsequence converges smoothly to a lamination $\mathcal{L}$. In particular, there is no curvature blowup in $\mathbf{B}\setminus Z$. By relabeling, we may assume that the subsequence is the original sequence.

Let L be a leaf of $\mathcal{L}$. As above, there is a Jacobi field J on L, defined by the rotational vector field $\partial/\partial\theta$. This Jacobi field does not change sign on L since J_n does not vanish on D_n. Thus by the maximum principle, it either vanishes nowhere on L or it vanishes everywhere on L. In the first case, L is transverse to every circle S that is rotationally invariant about Z. In the second case, L is rotationally invariant about Z.

Let S be a rotationally invariant circle in $\mathbf{B}$. Since S is compact and since it intersects each D_n, it must also intersect $\mathcal{L}$. Using the previous paragraph, we conclude that S either intersects $\mathcal{L}$ transversally, or it lies entirely in a rotationally invariant leaf of $\mathcal{L}$. If the circle S intersects $\mathcal{L}$ transversely, then it intersects $\mathcal{L}$ in a single point since it intersects each D_n in a single point. Thus the leaf L through that point is not a limit leaf. Let U be the union of L and its rotated images. The convergence of $D_n\cap U$ to $L\cap U$ is smooth and single-sheeted, so any closed curve $\alpha\subset L$ is a limit of closed curves α_n in D_n. By Lemma 2.3 (Statement 1(b)), the winding number of α_n about Z is 0. Thus the winding number of α about Z is also 0. By Lemma 2.3, L is a θ-graph.

We have proved Statements 1 and 2, and we have established that limit leaves are rotationally invariant. To prove Statement 3, we must establish that rotationally invariant leaves of $\mathcal{L}$ are limit leaves. Suppose that L is a rotationally invariant leaf of $\mathcal{L}$, and let p_n be a sequence of points in $\mathbf{B} \setminus (Z \cup L)$ converging to a point in L. The rotationally invariant circle though p_n contains a point q_n of the lamination $\mathcal{L}$. Since p_n is not in L, neither is q_n. By passing to subsequence, we may assume that the q_n converge to a point $q \in L$. We have shown that L contains a point q that is a limit of points q_n in $\mathcal{L} \setminus L$. Thus L is a limit leaf.

To prove Statement 4, let $\mathcal{O}$ be a connected component of $\mathbf{B} \setminus \overline{\cup \mathcal{L}'}$. By Statements 1, 2, and 3, for each point (x, z) in

$$U := \{(x, z) : x > 0,\ (x, 0, z) \in \mathcal{O}\},$$

the rotationally invariant circle through $(x, 0, z)$ intersects the lamination in a single point $F(x, z)$, and F defines a smooth embedding of U into $\mathcal{O}$. Since $\mathcal{O}$ is connected, U is connected, and therefore $L = F(U)$ is connected. In particular, L is a leaf of $\mathcal{L}$ rather than a union of leaves. By Statement 1, L is a θ-graph. We have already seen that it intersects each rotationally invariant circle in $\mathcal{O}$. Thus L is a spanning θ-graph in $\mathcal{O}$. By Theorem 2.7, $\overline{D_n} \cap \mathcal{O}$ and $\overline{L} \cap \mathcal{O}$ are smooth manifolds-with-boundary, the boundary being $Z \cap \mathcal{O}$. This proves Statement 4, except for the assertion about smooth convergence.

We already know smooth convergence away from Z, so to prove the smooth convergence in Statement 4, it suffices to consider the case when $\mathbf{B} \cap Z$ is nonempty. In that case, the smooth convergence is an immediate consequence of the following general theorem (which is true in arbitrary dimensions and codimensions):

Theorem 2.12 ([26, Theorem 6.1]). *Suppose that M is a smooth, connected manifold-with-boundary properly embedded in an open subset $\mathcal{O}$ of a smooth Riemannian manifold, and suppose that $\mathcal{O} \cap \partial M$ is nonempty. Suppose that M_n is a sequence of smooth minimal manifolds-with-boundary that are properly embedded in $\mathcal{O}$ and suppose that $\mathcal{O} \cap \partial M_n$ converges smoothly to $\mathcal{O} \cap \partial M$. Suppose also that*

$$\{p \in \mathcal{O} : \liminf \operatorname{dist}(p, M_n) = 0\} \subset M. \tag{5}$$

Then M_n converges smoothly to M on compact subsets of $\mathcal{O}$.

To apply Theorem 2.12, we let $M := \overline{L} \cap \mathcal{O}$ and $M_n := \overline{D_n} \cap \mathcal{O}$. Then $\partial M = \partial M_n = Z \cap \mathcal{O}$, and (5) holds because (in our situation) M_n converges smoothly to M on compact subsets of $\mathcal{O} \setminus Z$. Thus the smooth convergence asserted by Theorem 2.12 holds. This completes the proof of Statement 4.

Statement 5 follows immediately from Statement 4 by letting $\mathcal{O}$ be the connected component of $\mathbf{B} \setminus \overline{\cup \mathcal{L}'}$ containing the interval I.

We now prove Statement 6. Let $p \in K$. By definition of K, there is a sequence $p_n \in \cup\mathcal{L}'$ converging to p. Let α_n be the angle that the tangent plane to $\mathcal{L}'$ at p_n makes with the horizontal. To prove Statement 6, it suffices to show that $\alpha_n \to 0$. Let q_n be the point in Z nearest to p_n. Translate the limit leaf through p_n by $-q_n$ and dilate by $1/|p_n - q_n|$ to get a surface Σ_n. Note that Σ_n is rotationally invariant and stable. Since it is stable, the norm of the second fundamental form times distance to Z is uniformly bounded. Thus (after passing to a subsequence) the Σ_n converge smoothly on compact subsets of $\mathbf{R}^3 \setminus Z$ to a stable, rotationally invariant minimal surface Σ. The only rotationally invariant minimal surfaces in $\mathbf{R}^3$ are catenoids and horizontal planes. Since catenoids are unstable, Σ must be a horizontal plane—in fact, the plane $z = 0$. Since this limit is independent of choice of subsequence, in fact the sequence Σ_n converges to the plane $z = 0$. Hence $\alpha(p_n) \to 0$, proving Statement 6, except for uniqueness.

If uniqueness failed, we would have two rotationally invariant disks tangent to each other at a point p on Z. The intersection set would consist of p together with a collection of rotationally invariant circles. But near a common point of two distinct minimal surfaces in a 3-manifold, the intersection set consists of two or more curves that meet at the point. This proves uniqueness.

Remark 2.13. The proof of Statement 6 shows that if $L \subset \mathbf{B}$ is a stable, rotationally invariant, embedded minimal surface that contains $p \in \mathbf{B} \cap Z$ in its closure, then $L \cup \{p\}$ is a smooth minimal surface.

We now prove Statement 7. By the smooth convergence $D_n \to \mathcal{L}$ in $\mathbf{B} \setminus Z$ and by Statement 4, we already know that the curvatures of the D_n are uniformly bounded on compact subsets of $\mathbf{B} \setminus K$. Thus we need only show if $p \in K$, then the curvatures of the D_n blow up at p. Suppose not. Then (by passing to a subsequence) we can assume that the curvatures of the $\overline{D_i}$ are uniformly bounded in some neighborhood of p. Since the tangent plane to $\overline{D_i}$ at p is vertical, it follows that for a sufficiently small ball $\mathbf{B}(p, r) \subset \mathbf{B}$, the slopes of the tangent planes to the surfaces $D_i \cap \mathbf{B}(p, r)$ are all ≥ 1. Hence if L is leaf of $\mathcal{L}$, then the slope of the tangent planes to $L \cap \mathbf{B}(p, r)$ are all ≥ 1. But by Statement 6, since $p \in K$, there is a rotationally invariant leaf $L(p)$ such that $L(p) \cup p$ is a smooth manifold. In particular, the tangent plane at p is horizontal, so $L(p)$ contains points arbitrarily close to p with slopes arbitrarily close to 0. The contradiction proves Statement 7, and thereby completes the proof of the Theorem 2.10. q.e.d.

Proposition 2.14. *Each leaf of $\mathcal{L}$ lifts to a properly embedded surface in the universal cover U of $\mathbf{B} \setminus Z$.*

Proof. Let $V = \{(r, z) : (r, 0, z) \in \mathbf{B}\}$. Then we can regard $U = V \times \mathbf{R}$ as the universal cover of $\mathbf{B} \setminus Z$, the covering map being

$$\pi : V \times \mathbf{R} \to \mathbf{B} \setminus Z,$$
$$\pi(r, z, \theta) = (r \cos\theta, r \sin\theta, z).$$

Let L be a leaf of $\mathcal{L}$ and let $p \in L$. Let $p_n \in D_n$ converge to p. Let $\tilde{D}_n$ be a lift of D_n to the universal cover of $\mathbf{B} \setminus Z$, and let $\tilde{p}_n$ be the point in $\tilde{D}_n$ that projects to p_n. By making suitable vertical translations, we can assume that the points $\tilde{p}_n$ converge to a point $\tilde{p}$ that projects to p.

Since $\tilde{D}_n$ is a minimal graph, it satisfies the following bound: if C is any compact region with smooth boundary in U, then

$$\text{area}(\tilde{D}_n \cap C) \le \frac{1}{2} \text{area}(\partial C). \tag{6}$$

Since the $\tilde{D}_n$ are stable minimal surfaces, a subsequence converges smoothly to a limit $\tilde{D}$. By (6), the limit $\tilde{D}$ is properly embedded. Note that $\tilde{D}$ is a lift of L. q.e.d.

Corollary 2.15. *If Σ is a rotationally invariant leaf of $\mathcal{L}$, then Σ is properly embedded in $\mathbf{B} \setminus Z$.*

Proof. Let

$$\sigma = \{((x^2 + y^2)^{1/2}, z) : (x, y, z) \in \Sigma\}.$$

Then $\sigma \times \mathbf{R}$ is the lift of Σ to the universal cover.

Since $\sigma \times \mathbf{R}$ is a properly embedded surface in $V \times \mathbf{R}$ (by Proposition 2.14), σ is a properly embedded curve in V. The result follows immediately. q.e.d.

Next we prove that each rotationally invariant leaf in Theorem 2.10 is either a punctured disk or an annulus, and that the corresponding disk or annulus is properly embedded in $\mathbf{B}$. The reader may wish to skip the proof, since the theorem is obviously true in the cases we are most interested in (namely, when the Riemannian metric on $\mathbf{B}$ is the Euclidean metric or the Poincaré metric).

Proposition 2.16. *Let Σ be a rotationally invariant leaf in the lamination $\mathcal{L}$. Then either Σ is a punctured disk such that $\overline{\Sigma} \cap \mathbf{B}$ properly embedded in $\mathbf{B}$, or Σ is an annulus that is properly embedded in $\mathbf{B}$.*

Now suppose that $\overline{\mathbf{B}}$ is compact with smooth boundary, that the metric extends smoothly to $\overline{\mathbf{B}}$, and that $\overline{\mathbf{B}}$ is strictly mean convex with respect to the metric. Then Σ is smooth at the boundary: $\overline{\Sigma}$ is either a smoothly embedded closed disk or a smoothly embedded closed annulus.

Proof. Let $\mathcal{D}$ be the planar domain

$$\mathcal{D} = \{(r, 0, z) \in \mathbf{B} : r > 0\}$$

and let σ be the curve in $\mathcal{D}$ given by

$$\sigma = \Sigma \cap \mathcal{D}.$$

Thus Σ is the surface of revolution obtained by rotating σ around Z.

In Corollary 5.2, we show that

$$\overline{\Sigma} \text{ cannot be a smooth closed surface in } \mathbf{B}. \tag{7}$$

Thus σ is not a closed curve, so it has two ends.

If one end of σ contained a point p of $Z \cap \mathbf{B}$ and if the other end contained a point q of $Z \cap \mathbf{B}$ in its closure, then $\Sigma \cup \{p, q\}$ would be a smooth embedded surface in $\mathbf{B}$ (by Statement 6 of Theorem 2.10), contradicting (7) above.

Thus either σ contains no points of $Z \cap \mathbf{B}$ in its closure, or exactly one end of σ contains a point p of $Z \cap \mathbf{B}$ in its closure. In the first case, Σ is a properly embedded annulus in $\mathbf{B}$. In the second case, $\overline{\Sigma} \cap \mathbf{B} = \Sigma \cup \{p\}$ is a properly embedded disk in $\mathbf{B}$. (The properness follows from Corollary 2.15 above.)

Now suppose that $\partial \mathbf{B}$ is smooth and that the metric extends smoothly to $\overline{\mathbf{B}}$. If σ contained an endpoint p of $Z \cap \mathbf{B}$ in its closure, then $\Sigma \cup \{p\}$ would be a smooth minimal surface (by Remark 2.13), contradicting the mean convexity of $\partial \mathbf{B}$ at p. Thus σ cannot contain an endpoint of $Z \cap \mathbf{B}$ in its closure. It follows that at least one end of σ contains a point q of $(\partial \mathcal{D}) \setminus Z$ in its closure. By the strict mean convexity, that end of σ must converge to q. Thus the union of Σ and the circle corresponding to q is a smooth manifold with boundary. The two ends of σ cannot converge to the same point in $(\partial \mathcal{D}) \setminus Z$, since then $\overline{\Sigma}$ would be a closed surface in $\overline{\mathbf{B}}$, which is impossible by Corollary 5.3.

We have shown that either σ has one endpoint in $(\partial \mathcal{D}) \setminus Z$ and the other endpoint in $Z \cap \mathbf{B}$, in which case $\overline{\Sigma}$ is a disk, or $\overline{\Sigma}$ has both endpoints in $(\partial \mathcal{D}) \setminus Z$, in which case $\overline{\Sigma}$ is an annulus. q.e.d.

3. Existence of minimal θ-graphs with prescribed boundary

In this section, we prove existence and uniqueness of spanning minimal θ-graphs for a large family of prescribed boundary curves.

Theorem 3.1. *Let $\mathbf{B}$ be the open unit ball in $\mathbf{R}^3$, and suppose that $\overline{\mathbf{B}}$ is mean convex with respect to a smooth Riemannian metric g that is rotationally invariant about Z.*

Let γ be a smooth curve in $\partial \mathbf{B}$ joining $p^+ = (0,0,1)$ to $p^- = (0,0,-1)$ such that γ intersects each horizontal circle in $\partial \mathbf{B}$ exactly once, and such that the curve $\gamma \cup \rho_Z \gamma$ is smooth. Let Γ be the union of γ with $Z \cap \mathbf{B}$. Then among all oriented surfaces (of arbitrary genus) with boundary Γ, there is a unique surface D of least area. The surface D is a θ-graph, and $\overline{D \cup \rho_Z D}$ is a smoothly embedded disk with boundary $\gamma \cup \rho_Z \gamma$.

Furthermore, if $M \subset \mathbf{B}$ is any oriented, embedded minimal surface with finite area, finite genus, and with boundary Γ, then $M = D$.

Concerning the hypothesis that $\gamma \cup \rho_Z\gamma$ is smooth, note that smoothness of γ implies smoothness of $\gamma \cup \rho_Z\gamma$ except possibly at the endpoints of γ. For $\gamma \cup \rho_Z\gamma$ to be smooth at an endpoint of γ, the necessary and sufficient condition is the vanishing of curvature and all even order derivatives of curvature at that endpoint.

Corollary 3.2. *Suppose that D is a spanning θ-graph in $\mathbf{B}$ that is minimal with respect to a smooth, rotationally invariant Riemannian metric on $\mathbf{B}$.*

If $\overline{\mathbf{B}'} \subset \mathbf{B}$ is rotationally invariant about Z, mean convex, and smoothly diffeomorphic to a closed ball, then $D \cap \overline{\mathbf{B}'}$ is the unique least-area integral current among all integral currents in $\overline{\mathbf{B}'}$ having boundary $\partial(D \cap \mathbf{B}')$.

If $\mathbf{B}$ can be exhausted by such subdomains $\overline{\mathbf{B}'_n}$, then D is an area-minimizing integral current.

Proof of corollary. Apply the theorem to $D \cap U$. q.e.d.

Proof of Theorem 3.1. Let D be an oriented area-minimizing surface (i.e., integral current) in $\mathbf{B}$ bounded by Γ. (To be precise, we let D be the set of points in $\mathbf{B}\backslash Z$ in the support of that integral current.) Note we are not restricting the genus of D. By the Hardt-Simon boundary regularity theorem [11], $\overline{D}$ is a smooth, embedded manifold-with-boundary except at the corners p^+ and p^- of Γ. Let C be a tangent cone to D at p^-. Then C lies in the upper halfspace $\{x_3 \geq 0\}$, and the boundary of C consists of the positive x_3-axis together with a horizontal ray, both with multiplicity 1. The only such cone is the corresponding quarter-plane with multiplicity one. Now $\overline{D \cup \rho_Z D}$ is a minimal surface with boundary $\gamma \cup \rho_Z\gamma$, and it is smoothly immersed everywhere except possibly at p^+ and at p^-. We have just shown that the tangent cone to $\overline{D \cup \rho_Z D}$ at p^- is a halfplane with multiplicity one. By Allard's Boundary Regularity Theorem [1], $\overline{D \cup \rho_Z D}$ is a smoothly embedded manifold-with-boundary near p^-. Likewise, it is a smoothly embedded manifold-with-boundary near p^+.

Let σ be a closed curve in D. By pushing σ slightly in the direction of the unit normal to D, we get a closed curve σ' that is homotopic to σ in $W \setminus Z$. Note that σ' is disjoint from D. Thus its algebraic intersection number with D is 0. By elementary topology, the winding number about Z of a closed curve in $W \setminus Z$ is equal to its linking number with Γ, which is equal to its intersection number with D. Thus the winding number of σ' about Z is 0. Since σ and σ' are homotopic in $W \setminus Z$, the same is true of σ.

We have shown: every closed curve in D has winding number 0 about Z. Thus D lifts to the universal cover of $W \setminus Z$. Equivalently, there is an angle function $\theta_D : D \setminus Z \to \mathbf{R}$ such that

$$p = (r(p)\cos\theta_D(p), r(p)\sin\theta_D(p), z)$$

for all $p = (x, y, z) \in D \setminus Z$, where $r(p) = \sqrt{x^2 + y^2}$. The smoothness of $\overline{D \cup \rho_Z D}$ implies that $\theta_D(\cdot)$ extends continuously to $\overline{D}$.

Now define

$$\omega : \overline{D} \to \mathbf{R}$$

by letting $\omega(p)$ be the maximum of $\theta_D(q) - \theta_D(p)$ among all $q \in \overline{D}$ such that q and p lie on the same rotationally invariant circle. Note that ω is upper semicontinuous and that $\omega = 0$ on $\partial D = \Gamma$ (by the smoothness of D at the boundary). Thus if ω did not vanish everywhere, it would attain its maximum at some interior point $p \in D$. But at that point, the strong maximum principle would be violated. (Note that the surface D and the surface obtained by rotating D through angle $-\omega(p)$ would touch each other at p.) Thus $\omega(\cdot) \equiv 0$, which implies that D is a θ-graph. Every rotationally invariant circle in W links Γ and therefore must intersect D. Thus D is a spanning θ-graph.

To prove the uniqueness assertion, suppose that M is a finite-genus, finite-area, orientable, embedded minimal surface in W with boundary Γ. By classical boundary regularity theory, $\overline{M \cup \rho_Z M}$ is a minimal immersed surface, possibly with branch points. Since the boundary of $\overline{M \cup \rho_Z M}$ lies on ∂W, it cannot have any boundary branch points. Also, it cannot have interior branch points in $W \setminus Z$ since M is embedded. Finally, it cannot have a branch point on Z, since then M would have a boundary branch point on Z, which implies that M is not embedded near that point, a contradiction. We have shown that $\overline{M \cup \rho_Z M}$ is a smoothly immersed surface-with-boundary.

Just as for D, it follows that there is a continuous angle function

$$\theta_M : \overline{M} \to \mathbf{R}$$

such that

$$p = (r(p)\cos\theta_M(p), r(p)\sin\theta_M(p), z) \tag{8}$$

for $p = (x, y, z) \in \overline{M}$, where $r(p) = \sqrt{x^2 + y^2}$.

Note that on $\Gamma \cap \partial W$, θ_M and θ_D differ by a constant multiple of 2π. Note also that adding a multiple of 2π to $\theta_M(\cdot)$ does not affect (8). Thus we can assume that $\theta_D \equiv \theta_M$ on $\Gamma \cap \partial W$.

Now define a continuous function

$$\phi : \overline{M} \to \mathbf{R},$$
$$\phi(p) = \theta_D(q) - \theta_M(p),$$

where q is the unique point of intersection of D with the rotationally invariant circle containing p. (Here we allow circles of radius 0, so if that $p \in Z$, then $q(p) = p$.)

Now $\phi \equiv 0$ on $\overline{M} \cap \partial W$, so if it were not everywhere 0, then $|\phi(\cdot)|$ would attain a strictly positive maximum at some point $p \in \overline{M} \cap W$. But that would violate the strong maximum principle (if $p \in M$) or the strong boundary maximum principle (if $p \in Z$).

(If this is not clear, consider $\overline{M}$ and the surface obtained by rotating $\overline{D}$ by angle $-\phi(p)$. The two surfaces are tangent at p, and there is a neighborhood of p in which the two surfaces have no transverse intersections.) q.e.d.

4. Smooth convergence at the boundary

In this section, we will assume that

(i) D_n is a sequence of spanning minimal θ-graphs in $\mathbf{B} \setminus Z$ with boundaries of the form

$$\partial D_n = \gamma_n \cup \overline{I},$$

where $I = \mathbf{B} \cap Z$ and γ_n is an embedded curve in $\partial \mathbf{B}$ connecting the endpoints $p^+ = (0,0,1)$ and $p^- = (0,0,-1)$ of I.

(ii) the Riemannian metric on $\mathbf{B}$ extends smoothly to $\overline{\mathbf{B}}$, and $\overline{\mathbf{B}}$ is strictly mean convex.

In Theorem 2.10 of Section 2, we proved that, away from a closed subset $K \subset I$, a subsequence of the D_n converge smoothly to a limit lamination $\mathcal{L}$. The set K is precisely the set on which the curvature of the surfaces D_n blow up. In this section we provide conditions under which the convergence is smooth up to the boundary in $\partial\mathbf{B} \setminus \{p^+, p^-\}$. This involves establishing uniform curvature estimates in a neighborhood of points on the boundary of $\mathbf{B}$.

In Theorem B.2 in Appendix B, we prove the following curvature estimate.

Theorem 4.1. *Suppose in addition to* (i) *and* (ii) *that the curvature and the first derivative of curvature of* γ_n *are bounded independently of* n*. Then the curvatures* $B(D_n, \cdot)$ *are uniformly bounded away from* $\overline{I}$*.*

This uniform curvature estimate is enough to conclude that the boundaries of the leaves of a limit lamination $\mathcal{L}$ are regular at the points of their boundary in $\partial\mathbf{B}\setminus\{p^+, p^-\}$. We already know from Theorem 2.10 that they are regular at the points of $I \subset \partial D_n$ that are not in the curvature blowup set K.

Theorem B.2 of Appendix B is stated in terms of minimal graphs in $V \times \mathbf{R}$, where $V = \{(x, 0, z) : x > 0, (x, 0, z) \in \mathbf{B}\}$. As explained in that Appendix and in Section 2.6, this is equivalent to the situation considered in Theorem 4.1 above.

Note that if the curves γ_n converge smoothly to a lamination of $\partial\mathbf{B} \setminus \{p^+, p^-\}$, then (away from I) we have uniform bounds on the curvature of γ_n and the first derivative of curvature. Therefore we can use Theorem 4.1 to conclude smooth convergence up to the boundary:

Theorem 4.2. *Suppose, in addition to* (i) *and* (ii)*, that the* D_n *converge smoothly in* $\mathbf{B} \setminus Z$ *to a lamination* $\mathcal{L}$*, and that the curves* γ_n *converge smoothly to a lamination* $\mathcal{G}$ *of* $\partial\mathbf{B} \setminus \{p^+, p^-\}$*. Then the convergence* $D_n \to \mathcal{L}$ *is smooth up to* $\partial\mathbf{B} \setminus \{p^+, p^-\}$*.*

In particular, if L *is a leaf of* $\mathcal{L}$*, and if* L^* *is a lift of it to the universal cover* $\mathcal{U}$ *of* $\overline{\mathbf{B}} \setminus Z$*, then the closure* $\overline{L^*}$ *of* L^* *in* $\mathcal{U}$ *is a smooth embedded manifold-with-boundary, and each component of* $\partial\overline{L^*}$ *projects to a leaf* γ *of* $\mathcal{G}$*. Furthermore, every leaf* γ *of* $\mathcal{G}$ *arises in this way: if* $\gamma \in \mathcal{G}$*, there is a lift* L^* *of a leaf of* $\mathcal{L}$ *and a component of* $\partial\overline{L^*}$ *that projects to* γ*.*

Corollary 4.3. *If* γ *is a rotationally invariant leaf of* $\mathcal{G}$*, then* $\mathcal{L}$ *contains a rotationally invariant leaf with* γ *as one of its boundary components.*

Proof. Let L be a leaf of $\mathcal{L}$ associated to γ as in Theorem 4.2. (That is, suppose L and γ have lifts L^* and γ^* to the universal cover of $\overline{\mathbf{B}} \setminus Z$ such that $L^* \cup \gamma^*$ is a smooth manifold-with-boundary.) If L is rotationally invariant, we are done. If not, L and its images under rotations about Z foliate a rotationally invariant region Ω in $\mathbf{B}$. Note that Ω is bounded by rotationally invariant leaves of $\mathcal{L}$. Two of those leaves must each have γ as a boundary component. q.e.d.

5. Necessary conditions for a lamination to appear as the limit leaves of the limit lamination of a sequence of minimal θ-graphs

As in Theorem 2.10, let D_n be a sequence of oriented spanning minimal θ-graphs in $\mathbf{B}$ that converge smoothly in $\mathbf{B} \setminus Z$ to a lamination $\mathcal{L}$ of $\mathbf{B} \setminus Z$. Let $\mathcal{R}_n$ be the oriented foliation of $\mathbf{B} \setminus Z$ consisting of D_n and its rotated images. Let ν_n be the unit normal vectorfield to $\mathcal{D}_n$ compatible with the orientation. Note that $\mathcal{R}_n$ will converge to an oriented rotationally invariant foliation $\mathcal{R}$ of $\mathbf{B} \setminus Z$. In particular, the vectorfields ν_n converge uniformly on compact subsets of $\mathbf{B} \setminus Z$ to the unit normal vectorfield ν to $\mathcal{R}$ compatible with the orientation of $\mathcal{R}$.

The rotationally invariant leaves of $\mathcal{R}$ are precisely the rotationally invariant leaves of $\mathcal{L}$, that is the leaves of $\mathcal{L}'$.

In this section, we will prove some additional properties of the collection of rotationally invariant leaves $\mathcal{L}'$.

Proposition 5.1. *Suppose that* $\mathcal{R}$ *is an oriented, minimal foliation of an open subset of a Riemannian manifold. Then*

$$\operatorname{Div} \nu = 0,$$

where ν *is the unit normal vectorfield to* $\mathcal{R}$ *given by the orientation.*

Proof. Let $\overline{\nabla}$ denote covariant differentiation with respect to the metric, and let div denote the divergence operator on a fixed leaf of $\mathcal{R}$. We have for any vectorfield X:

$$\operatorname{Div} X = \operatorname{div} X + \overline{\nabla}_\nu X \cdot \nu,$$

so

$$\operatorname{Div} \nu = \operatorname{div} \nu + \overline{\nabla}_\nu \nu \cdot \nu = H + \frac{1}{2}\nu(\nu \cdot \nu),$$

where H is the mean curvature of the fixed leaf of $\mathcal{R}$. Since all leaves of $\mathcal{R}$ are minimal and since ν has unit length, Div $\nu = 0$. q.e.d.

Corollary 5.2. *Suppose that $\mathcal{R}$ is an oriented foliation of $\mathbf{B} \setminus Z$ by surfaces that are minimal with respect to a smooth Riemannian metric on $\mathbf{B}$, where $\mathbf{B}$ is the open unit ball in $\mathbf{R}^3$.*

If Σ is a closed, connected, embedded surface in $\mathbf{B}$, then $\Sigma \setminus Z$ cannot be a leaf of $\mathcal{R}$.

Proof. Let U be the region in $\mathbf{B}$ bounded by Σ. Now ν is not defined on Z, but Z is a closed set with 2-dimensional Hausdorff measure 0, so even if $U \cap Z$ is not empty, we can apply the Divergence Theorem A.1 on U to get:

$$(*) \qquad \int_\Sigma \nu \cdot n \, dA = \int_U \operatorname{Div} \nu = 0,$$

where n is the unit normal to Σ that points out of U. If Σ were a leaf of $\mathcal{L}$, then either $n = \nu$ on Σ or $n = -\nu$ on Σ, so that the left side of (*) would be equal to plus or minus the area of Σ, and thus the area of Σ would be 0, which is impossible. q.e.d.

Corollary 5.3. *Suppose that the Riemannian metric in Corollary 5.2 extends smoothly to $\overline{\mathbf{B}}$. Let Σ be an annulus in $\mathbf{B}$ such that the two boundary components of Σ are the same smooth, simple closed curve in $\partial\mathbf{B}$. Then Σ cannot be a leaf of $\mathcal{L}$.*

The proof is almost identical to the proof of Corollary 5.2.

Theorem 5.4. *Let $\mathcal{R}$ be an oriented, minimal foliation of $\mathbf{B} \setminus Z$ that is rotationally invariant about Z (with respect to a smooth, rotationally invariant metric on $\mathbf{B}$), and let ν be the associated unit normal vectorfield compatible with the orientation. Let $\mathcal{L}'$ be the sublamination consisting of the rotationally invariant leaves of $\mathcal{R}$. Let U be a regular open subset of $\mathbf{B}$ such that $M := (\partial U) \cap \mathbf{B}$ consists of leaves of $\mathcal{L}'$ on which the normal ν points out of U. Then M is area minimizing.*

Furthermore, if the metric extends smoothly to $\overline{\mathbf{B}}$ and if M' is another area-minimizing surface with $\partial M = \partial M'$ (as oriented surfaces in $\overline{\mathbf{B}}$), then M' is also made up of oriented leaves of $\mathcal{L}'$.

Recall that a *regular* open set is an open set U such that $U =$ interior$(\overline{U})$.

Proof. Case 1: Assume that the metric extends smoothly to $\overline{\mathbf{B}}$, and that $\overline{M}$ is a smooth, embedded manifold with boundary in $\overline{\mathbf{B}}$.

Let M' be a smoothly embedded, oriented surface in $\mathbf{B}$ with $\partial M' = \partial M$. By elementary topology, $M' = (\partial U') \cap \mathbf{B}$ for some regular open set U' of $\mathbf{B}$ with

$$\overline{U'} \cap \partial\mathbf{B} = \overline{U} \cap \partial\mathbf{B}. \tag{9}$$

Let

$$\Sigma = \overline{U'} \cap \partial\mathbf{B} = \overline{U} \cap \partial\mathbf{B}.$$

Let n and n' be the outward-pointing unit normal vectorfields on $\partial U = M \cup \Sigma$ and on $\partial U' = M' \cup \Sigma$, respectively. Note that $n = n'$ on Σ. Note also that $n|M = \nu|M$ is the unit normal vectorfield compatible with the orientation of M, and that $n'|M'$ is compatible with orientation of M'.

Now ν is not defined on Z. However, Z is a closed set with 2-dimensional Hausdorff measure 0, so we can apply the divergence theorem (see Theorem A.1) to ν on U to get

$$\begin{aligned} 0 &= \int_{\tilde{U}} \operatorname{Div} \nu \, dV \\ &= \int_M \nu \cdot n \, dA + \int_\Sigma \nu \cdot n \, dA. \end{aligned} \tag{10}$$

Likewise, applying the divergence theorem to ν on $\tilde{U}'$ gives

$$0 = \int_{M'} \nu \cdot n' \, dA + \int_\Sigma \nu \cdot n' \, dA. \tag{11}$$

Since $n = n'$ on Σ, combining (10) and (11) gives:

$$\int_M \nu \cdot n \, dA = \int_{M'} \nu \cdot n' \, dA.$$

The left side equals the area of M since, by hypothesis, $\nu \equiv n$ on M. Thus

$$\operatorname{area}(M) = \int_{M'} \nu \cdot n' \, dA \leq \operatorname{area}(M'), \tag{12}$$

with equality if and only if $\nu \equiv n'$, i.e., if and only if M' is also a leaf of $\mathcal{L}'$. This proves that M is area-minimizing, and it also proves the last assertion ("furthermore...") of the theorem.

Case 2: The general case. Let $\mathbf{B}_1 \subset \mathbf{B}_2 \subset \dots$ be an exhaustion of $\mathbf{B}$ by open balls centered at the origin such that for each i, $\partial\mathbf{B}_i$ is transverse to M. Then (by Case 1), $M \cap \mathbf{B}_i$ is area minimizing in $\mathbf{B}_i$ (i.e., it has area less than or equal to the area of any other surface in $\mathbf{B}_i$ with the same boundary.) Thus (by definition), M is area minimizing in $\mathbf{B}$. q.e.d.

We give two simple applications of Theorem 5.4 that will be used in the next section. For these corollaries, we assume that the metric extends smoothly to $\overline{\mathbf{B}}$.

Corollary 5.5. *Suppose $\mathcal{L}'$ contains two disks D and D' such that either ν points out of the region Ω between D and D' on $D \cup D'$, or it points into that region on $D \cup D'$. Then the two disks are area minimizing as an integral current. If there is an area-minimizing annulus with the same boundary as the disks, then it must also be a leaf of $\mathcal{L}'$.*

Proof. If ν on $D \cup D'$ points out of Ω, let $U = \Omega$. If it points into Ω, let $U = \mathbf{B} \setminus \overline{\Omega}$. Now apply Theorem 5.4. q.e.d.

Corollary 5.6. *Suppose that $\mathcal{L}'$ contains an annulus M. Then M is area minimizing as an integral current. If ∂M bounds another area-minimizing surface, then it must also be a leaf or union of leaves of $\mathcal{L}'$.*

Proof. Note that M divides $\mathbf{B}$ into two components: we let U be the component that such that ν on M points out of U. Now apply Theorem 5.4. q.e.d.

Remark 5.7. We conjecture the following partial converse to Theorem 5.4. Suppose that one has a finite collection of area-minimizing, rotationally invariant minimal surfaces. Let $\mathcal{M}$ be the augmentation of this collection to include all area-minimizing, rotationally invariant minimal surfaces with the same boundary. Then $\mathcal{M}$ can be realized as the rotationally invariant leaves $\mathcal{L}'$ of a lamination $\mathcal{L}$ that is a limit lamination of a sequence of spanning minimal θ-graphs in $\mathbf{B} \setminus Z$.

6. Specifying the rotationally invariant leaves of a limit lamination

In this section, we work with the open unit ball $\mathbf{B}$ in $\mathbf{R}^3$ and with a smooth Riemannian metric g on $\overline{\mathbf{B}}$ such that

- the mean curvature of $\partial\mathbf{B}$ is nonzero and points into $\mathbf{B}$.
- The metric is rotationally invariant about Z, and also invariant under $\mu(x, y, z) = (x, y, -z)$.

Definition 6.1. For $0 < a < 1$, let

$$c(a) = c^+(a) \cup c^-(a),$$

where $c^\pm(a)$ are the circles $(\partial\mathbf{B}) \cap \{z = \pm a\}$. We orient $c^-(a)$ by $d\theta$ and $c^+(a)$ by $-d\theta$. Let

$$\mathcal{M}(a)$$

be the set of rotationally invariant area-minimizing surfaces bounded by $c(a)$.

The hypotheses imply that $\mathcal{M}(a)$ is nonempty for every $a \in (0,1)$. Each surface in $\mathcal{M}(a)$ (indeed, any rotationally invariant surface bounded by $c(a)$) is either a pair of disks or an annulus.

If $M \in \mathcal{M}(a)$, then the area of M is less than the area of the annular component of $\partial\mathbf{B} \setminus c(a)$. It follows that if $a \in (0,1)$ is close to 0, then M is an annulus. (By the strict mean convexity of $\overline{\mathbf{B}}$, $M \setminus \partial M \subset \mathbf{B}$.)

Likewise, the area of a surface $M \in \mathcal{M}(a)$ is less than the area of the union of the two simply connected components of $\partial\mathbf{B} \setminus c(a)$. In particular, if $a \in (0,1)$ is close to 1, then the area of M is nearly 0. It follows that if a is close to 1, then M is a pair of disks. (For if a is close to 1, then any minimal annulus bounded by $c(a)$ would contain points from far $c(a)$, and thus by monotonicity would have area bounded away from 0.)

By a standard cut-and-paste argument, the surfaces in $\mathcal{M}(a)$ are disjoint from each other, except at their common boundary. By similar reasoning, if $a \neq a'$, the surfaces in $\mathcal{M}(a)$ are disjoint from the surfaces in $\mathcal{M}(a')$. Thus the collection of surfaces $\mathcal{M}(a)$, $0 < a < 1$, forms a minimal lamination of $\mathbf{B}$. Figure 1 shows that lamination for the Euclidean metric.

Note that if $0 < a < b < 1$ and if $\mathcal{M}(b)$ contains an annulus, then $\mathcal{M}(a)$ contains *only* annuli. For otherwise $\mathcal{M}(a)$ would contain a pair of disks, and those disks would intersect the annulus in $\mathcal{M}(b)$, which is impossible.

Consequently, there is an $a_{\text{crit}} \in (0,1)$ such that

1) if $a_{\text{crit}} < a < 1$, then $\mathcal{M}(a)$ contains at least one pair of disks, but no annuli.
2) if $0 < a < a_{\text{crit}}$, then $\mathcal{M}(a)$ contains at least one annulus, but no pairs of disks.
3) $\mathcal{M}(a_{\text{crit}})$ contains at least one pair of disks, and it contains at least one annulus.

(Note that (3) follows from (1) and (2), since the limit of area-minimizing annuli is also an area-minimizing annulus, and similarly for pairs of disks.)

For the Euclidean metric, for each $a \le a_{\text{crit}}$, $\mathcal{M}(a)$ contains exactly one minimal annulus, and for each $a \ge a_{\text{crit}}$, $\mathcal{M}(a)$ contains exactly one pair of minimal disks. But for general metrics, a given $\mathcal{M}(a)$ might contain multiple minimal annuli and/or multiple pairs of disks.

Definition 6.2. If T is a relatively closed subset of $(0,1)$, let $c(T)$ be the collection of circles in $\partial\mathbf{B}$ given by

$$c(T) = \bigcup_{a \in T} c(a),$$

and let $\mathcal{M}(T)$ be the lamination of $\mathbf{B}$ given by

$$\mathcal{M}(T) = \bigcup_{a \in T} \mathcal{M}(a).$$

We let

$$\mathcal{M} = \mathcal{M}(0,1).$$

Theorem 6.3. *Consider a smooth Riemannian metric on $\overline{\mathbf{B}}$ such that*

1) *the mean curvature of ∂B is nonzero and points into $\mathbf{B}$.*
2) *the metric is invariant under $(x,y,z) \mapsto (x,y,-z)$ and under rotations about Z.*

Let T be a relatively closed subset of $(0,1)$. Then there exists a sequence of spanning minimal θ-graphs in $\mathbf{B} \setminus Z$ that converge to a limit lamination $\mathcal{L}$ whose rotationally invariant leaves are given by $\mathcal{M}(T)$.

Remark 6.4. More precisely, the rotationally invariant leaves of $\mathcal{L}$ are the annuli in $\mathcal{M}(T)$ together with the disks in $\mathcal{M}(T)$ with their centers removed.

Proof. First suppose that $0 \notin \overline{T}$. Consider the collection $\mathcal{G}$ of θ-graphs γ in $\partial\mathbf{B}$ with the following properties:

- γ is invariant under the reflection $\mu(x,y,z) = (x,y,-z)$.
- $\frac{d\theta}{dz}$ is positive on $\gamma \cap \{z < 0\}$ (and therefore negative on $\gamma \cap \{z > 0\}$).

Then there is a sequence of curves γ_i, $i = 1, 2, \dots$, in $\mathcal{G}$ converging smoothly to a lamination $\mathcal{C}$ of $\partial\mathbf{B} \setminus Z$ such that the rotationally invariant leaves are precisely the circles in $c(T)$.

By Theorem 3.1 (applicable because we are assuming that $\overline{B}$ is mean convex), for each γ_i there exists a unique, smooth, embedded minimal θ-graph D_i with boundary $\gamma_i \cup (Z \cap \overline{\mathbf{B}})$. Because this boundary is μ-invariant, uniqueness implies that D_i is also μ-invariant. By passing to a subsequence, we can assume that the D_i converge smoothly to a lamination $\mathcal{L}$ of $\mathbf{B} \setminus Z$. Of course $\mathcal{L}$ must be μ-invariant.

To prove the theorem, we must prove that every rotationally invariant leaf of $\mathcal{L}$ is in $\mathcal{M}(T)$, and, conversely, that each surface in $\mathcal{M}(T)$ is a leaf of $\mathcal{L}$.

Step 1: Proof that every rotationally invariant leaf of $\mathcal{L}$ is in $\mathcal{M}(T)$. Suppose that L is a rotationally invariant leaf of $\mathcal{L}$. Then L must be a punctured disk or an annulus.

Case 1: L is a punctured disk. By Theorem 4.2, the boundary circle of L must be a leaf of $\mathcal{C}$, so it must be one of the two circles in $c(a)$ for some $a \in T$. By symmetry, $\mu(L)$ is also a leaf of $\mathcal{L}$. The two boundary circles of $L \cup \mu(L)$ are $c(a)$. By Corollary 5.5, $L \cup \mu(L)$ is area minimizing. Thus $L \cup \mu(L) \in \mathcal{M}(a)$.

Case 2: L is an annulus. By Theorem 4.2, the two boundary circles of L must both be circles in the family $c(T)$. Note the circles must be oppositely oriented. Therefore one boundary circle is $c^+(a)$ for some $a \in T$, and the other is $c^-(b)$ for some $b \in T$.

We claim that $a = b$. For otherwise, L and $\mu(L)$ would be two leaves of $\mathcal{L}$ that intersect along a circle at height 0, which is impossible. Thus $\partial L = c(a)$ for some $a \in T$. By Corollary 5.6, L is area-minimizing. Therefore $L \in \mathcal{M}(a)$.

This completes the proof that each rotationally invariant leaf in $\mathcal{L}$ is in $\mathcal{M}(a)$ for some $a \in T$.

Step 2: Proof that every surface in $\mathcal{M}(T)$ is a rotationally invariant leaf in $\mathcal{L}$. Suppose that $a \in T$. By Corollary 4.3, there is some rotationally invariant leaf L of $\mathcal{L}$ such that $c^+(a)$ is a component of ∂L. If L is an annulus, then (as we have proved above), $\partial L = c(a)$; in this case, let $\Sigma = L$. If L is a disk, then $\mu(L)$ is also in $\mathcal{L}$ (by μ-symmetry); In this case, we let $\Sigma = L \cup \mu(L)$.

We have shown: if $a \in T$, then $c(a)$ bounds a rotationally invariant surface Σ consisting of one leaf (an annulus) or two leaves (both disks) in $\mathcal{L}$. By Corollaries 5.5 and 5.6, Σ is area minimizing, so $\Sigma \in \mathcal{M}(a)$. If $\mathcal{M}(a)$ contains another surface Σ', then Σ together with Σ' bound a region Ω. By Theorem 5.4, since Σ is a rotationally invariant leaf (or pair of leaves) in $\mathcal{L}$, Σ' must also be in $\mathcal{L}$. Thus every surface Σ' in $\mathcal{M}(a)$ belongs to $\mathcal{L}$. This completes the proof assuming that $0 \notin \overline{T}$.

Now suppose that $0 \in \overline{T}$. In this case, no sequence $\gamma_i \in \mathcal{G}$ can converge smoothly to a lamination that includes the circles $c(T)$. For if γ_i in $\mathcal{G}$ converges smoothly to a lamination $\mathcal{C}$ of $\partial \mathbf{B} \setminus \{p^+, p^-\}$, then $\mathcal{C}$ contains a leaf that crosses the equator perpendicularly, which implies that $\mathcal{C}$ cannot contain circles arbitrarily near the equator.

However, even if $0 \in \overline{T}$, we can find a sequence of curves γ_i in $\mathcal{G}$ that converge smoothly in

$$\partial \mathbf{B} \setminus (\{p^+, p^-\} \cup \{z = 0\}) \tag{*}$$

to a lamination of (*) whose rotationally invariant leaves are precisely the circles in $c(T)$. The rest of the proof is almost exactly the same as the proof when $0 \notin \overline{T}$. q.e.d.

Remark 6.5. Let W be the open cylinder $\{(x, y, z) : x^2 + y^2 < 1\}$ with the Euclidean metric. In [12], the authors prove that given any closed subset T of Z, there is a sequence of spanning θ-graphs in $W \setminus Z$ that converge to a limit lamination whose rotationally invariant leaves are precisely the disks $W \cap \{z = c\}$, $c \in T$.

7. The hyperbolic case I. Existence of θ-graphs with prescribed boundary at infinity: Theorem 3.1 in the hyperbolic case

We will extend the existence result, Theorem 3.1 of Section 3 (and Theorem 2 of [12]), to hyperbolic space $\mathbf{H}^3$. In this section, $\mathbf{B}$ will denote the open unit ball centered at the origin in $\mathbf{R}^3$. We will be interested in surfaces in $\mathbf{B}$ that are **hyperbolically minimal**, i.e. minimal with respect to the hyperbolic (Poincaré) metric

$$ds^2 = \frac{4(dx_1^2 + dx_2^2 + dx_3^2)}{(1-|x|^2)^2}$$

on $\mathbf{B}$. This metric is clearly rotationally symmetric around any axis of the ball, in particular the x_3-axis Z. Note that Theorem 3.1 does not directly apply here because the metric does not extend smoothly to $\overline{\mathbf{B}}$, and the boundary (the unit sphere—at infinite distance from any point of $\mathbf{B}$) is not mean convex in the ordinary sense.

In what follows, for any subset S of $\overline{\mathbf{B}}$, the sets $\overline{S}$ and ∂S will continue to denote the closure of S and the boundary of S in $\overline{\mathbf{B}}$ with respect the Euclidean metric. We will refer to $\partial S \cap \partial \mathbf{B}$ as the **ideal boundary** of S. We will write $I = Z \cap \mathbf{B}$ and observe that the ideal boundary of I is equal to $Z \cap \partial \mathbf{B} = \{(0, 0, \pm 1)\}$.

Theorem 7.1. *Let γ be a smooth curve in $\partial \mathbf{B}$ joining p^- to p^+ such that γ intersects each rotationally invariant curve in $\partial \mathbf{B}$ exactly once, and such that the curve $C := \gamma \cup \rho_Z \gamma$ is smooth. Let Γ be the union of γ with Z. Then Γ bounds a spanning hyperbolically minimal θ-graph D such that $\overline{D \cup \rho_Z D}$ is a smoothly embedded disk with boundary C.*

Proof. Let $\mathcal{B}_n$ be a sequence of nested open balls centered at the origin such that $\overline{\mathcal{B}_n} \subset \mathbf{B}$ and such that $\bigcup_n \mathcal{B}_n = \mathbf{B}$. Let Γ_n be the image of Γ under the Euclidean homothety that takes $\mathbf{B}$ to $\mathcal{B}_n$.

By Theorem 3.1, the curve Γ_n bounds a unique spanning θ-graph D_n that is minimal with respect to the Poincaré metric. Its rotated images about Z foliate $\mathcal{B}_n \setminus Z$.

By Theorem 2.10, we can assume (by passing to a subsequence) that the D_n converge smoothly on compact subsets of $\mathbf{B} \setminus Z$ to a minimal lamination $\mathcal{L}$ of $\mathbf{B} \setminus Z$. (Theorem 2.10 assumes that all the D_n lie in the same domain $\mathbf{B} \setminus Z$. Here we have expanding domains $\mathcal{B}_n \setminus Z$. The proof is the same, requiring only the choice of subsequences at each stage.)

Let $\partial \mathcal{L} = \overline{\mathcal{L}} \setminus \mathcal{L}$. Note that $\partial \mathcal{L}$ is a subset of $(\partial \mathbf{B}) \cup I$.

Claim 1. *$\partial \mathcal{L}$ is contained in Γ.*

Proof. Every point in $p \in (\partial \mathbf{B}) \setminus \Gamma$ is contained in an open Euclidean ball U that is disjoint from Γ and that meets $\partial \mathbf{B}$ orthogonally. Note that U is disjoint from each Γ_n. Note also that $U \cap \mathbf{B}$ can be foliated by nested totally geodesic surfaces (the boundaries of smaller balls) that meet $\partial\mathbf{B}$ orthogonally and converge in the Euclidean metric to p. By the maximum principle, U is disjoint from D_n. This proves the claim. q.e.d.

Because Γ contains no circles it follows from Claim 1 that $\mathcal{L}$ contains no leaves that are rotationally invariant. We now use the properties of $\mathcal{L}$ that were proved in Theorem 2.10:

- By Property (3), $\mathcal{L}$ contains no limit leaves;
- By Property (7), there is no curvature blowup in $\mathbf{B} \setminus I$ and, by Property (6), there is no curvature blowup on I;
- Consequently, by Property (5), there is a single leaf D of $\mathcal{L}$ that contains I in its closure, and that leaf is a spanning θ-graph.

It follows from Property (4) that D is the only leaf of $\mathcal{L}$.

Because there is no curvature blowup on I, the local boundedness of the curvatures of the D_n implies that $D \cup I$ is a smooth manifold with boundary.

Claim 2. *Let $B(D,p)$ denote the norm of the second fundamental form of D with respect to the hyperbolic metric. Then $B(D,\cdot)$ is bounded above on D.*

Proof of Claim 2. If Claim 2 is false, then there is a sequence of points $p_n \in D$ such that $B(D,p_n) \to \infty$. By passing to a subsequence, we can assume that p_n converges (in the Euclidean sense) to a point $p \in \overline{\mathbf{B}}$. Since $D \cup I$ is a smooth manifold with boundary, $p \in \partial\mathbf{B}$. Since the $R_\theta D$ foliate $\mathbf{B}\setminus Z$, D is stable, and stability yields the following estimate:

$$B(D,p_n)\ \min\{1, \operatorname{dist}(p_n, \partial D)\} \le c, \tag{13}$$

where c is a constant independent of n, and dist denotes distance in the hyperbolic metric. (See [21].) Since the ideal boundary $\partial\mathbf{B}$ of $\mathbf{B}$ is infinitely far from p_n (in hyperbolic distance),

$$\operatorname{dist}(p_n, \partial D) = \operatorname{dist}(p_n, (\partial D) \cap \mathbf{B}) = \operatorname{dist}(p_n, I),$$

so (13) becomes

$$B(D,p_n)\ \min\{1, \operatorname{dist}(p_n, I)\} \le c.$$

Since we are assuming that $B(D,p_n) \to \infty$, this implies that $\operatorname{dist}(p_n, I) \to 0$. Hence p is one of the points of I. However, we have established that $D \cup I$ is a smooth manifold with boundary, so p must lie in ∂I. That is, $p = (0,0,1)$ or $p = (0,0,-1)$. Passing to a subsequence, we may assume without loss of generality that $p = (0,0,1)$, the North Pole, and $p_n \to p$.

Let $f_n : \overline{\mathbf{B}} \to \overline{\mathbf{B}}$ be a Mobius transformation (i.e., a hyperbolic isometry) with the property that $p_n' := f_n(p_n)$ lies on the plane $z = 0$ and that $f_n(Z) = Z$. Let D_n and γ_n be the images of D and γ, respectively, under f_n. Since

$$\mathrm{dist}(p'_n, Z) = \mathrm{dist}(f_n(p_n), f_n(Z)) = \mathrm{dist}(p_n, Z),$$

and p'_n lies on the plane $\{z = 0\}$ the p'_n converge to the origin $O = (0, 0, 0)$. Note that γ_n converges smoothly (except at the South Pole) to a great semicircle S joining the North and South Poles.

By passing to a subsequence, we may assume that the D_n converge to a minimal lamination $\mathcal{L}'$ of $\mathbf{B} \setminus Z$. As before, the ideal boundary of the lamination is contained in S and therefore does not contain any horizontal circles. Thus $\mathcal{L}'$ does not contain any rotationally symmetric leaves. That is, there are no limit leaves. Thus the curvatures of the D_n are uniformly bounded on compact subsets of $\mathbf{B}$, contradicting the fact that $B(D_n, p'_n) \to \infty$ and that $p'_n \to O$. This completes the proof the claim. q.e.d.

We now complete the proof of Theorem 7.1. Let

$$\mathcal{D} = D \cup \rho_Z D \cup I.$$

Then $\mathcal{D}$ is an embedded minimal disk whose ideal boundary is the smooth, simple closed curve $\gamma \cup \rho_Z(\gamma)$, and whose principal curvatures are uniformly bounded. It follows from the work of Hardt and Lin [10] that $\overline{\mathcal{D}}$ is a C^1-manifold with boundary and must meet the ideal boundary orthogonally. Based on this work, Tonegawa [22] was able to prove that in fact $\overline{\mathcal{D}}$ is a smooth manifold with boundary. (This assertion requires some explanation. First, Hardt and Lin assume that $\mathcal{D}$ is a hyperbolic-area-minimizing rectifiable current. Their proof works equally well if instead one assumes that $\mathcal{D}$ is a smooth minimal surface whose principal curvatures are bounded, and we have established these bounds in Claim 2. Such boundedness easily implies Lemma 2.1 of [10], which establishes the essential property of surfaces necessary for their proof of their result. Second, the main theorem of [10] states that near the boundary, $\overline{\mathcal{D}}$ is a union of sheets, each of which is a smooth manifold with boundary. But in our case there is clearly only one sheet since $\mathcal{D}$ intersects each horizontal circle centered on Z exactly once.) q.e.d.

Proposition 7.2. *Let D be a spanning hyperbolically minimal θ-graph as in Theorem 7.1. Let M be a hyperbolically minimal surface embedded in $\mathbf{B} \setminus Z$ such that $\partial M = \partial D$ and such that $\overline{M \cup \rho_Z M}$ is a C^1 manifold with boundary. Then $M = D$.*

The proof of Proposition 7.2 is exactly the same as the proof of the uniqueness assertion in Theorem 3.1.

8. The Hyperbolic case II. Necessary Conditions for a lamination to appear as limit leaves of a limit lamination: Section 5 in the hyperbolic case

The statements and proofs in Section 5 involved comparing areas of rotationally invariant surfaces in $\mathbf{B}$ with boundaries in $\partial\mathbf{B}$. If we endow $\mathbf{B}$ with the Poincaré metric, then the areas of such surfaces are infinite, so comparing them becomes problematic. We get around this problem by working with suitable compact exhaustions of the surfaces. Let $\mathrm{Cyl}(s)$ denote the points in $\mathbf{B}$ that are at (hyperbolic) distance at most s from $Z \cap \mathbf{B}$. Inspired by [7], where horocycles are used to cut off ends of divergent geodesics in order to define a Jenkins-Serrin-like condition for minimal graphs in $H^2 \times \mathbf{R}$ with infinite boundary values, we will make regions and surfaces finite by clipping them with the cylinders $\mathrm{Cyl}(s)$.

We will use the following fact about catenoids in hyperbolic space.

Theorem 8.2. *Let C be a half-catenoid with axis Z, and let D be another half-catenoid or a totally geodesic disk such that C and D have the same ideal boundary circle. For s large, let $\Sigma(s)$ be the portion of $\partial\,\mathrm{Cyl}(s)$ between C and D. Then*

$$\lim_{s\to\infty} \mathrm{area}(\Sigma(s)) = 0.$$

(A half catenoid is, by definition, one of the two components obtained from a catenoid by removing its waist, i.e., its unique closed geodesic.) See Appendix C, specifically Corollary C.2 and Remark C.3 for a proof of Theorem 8.2.

Theorem 8.3. *Consider the open unit ball $\mathbf{B} \subset \mathbf{R}^3$ with the Poincaré metric. Let $\mathcal{R}$ be an oriented, minimal foliation of $\mathbf{B} \setminus Z$ that is rotationally invariant about Z, and let $\nu(\cdot)$ be the associated unit normal vectorfield compatible with the orientation. Let $\mathcal{L}'$ be the sublamination consisting of rotationally invariant leaves of $\mathcal{R}$.*

Let U be a regular open subset of $\mathbf{B}$ such that $M := (\partial U) \cap \mathbf{B}$ consists of leaves of $\mathcal{L}'$ on which the normal ν points out of U. Then M is area-minimizing.

Furthermore, if M consists of finitely many leaves, and if M' is another rotationally invariant, area-minimizing surface with $\partial M = \partial M'$ (as oriented surfaces in $\overline{\mathbf{B}}$), then M' is also made up of oriented leaves of $\mathcal{L}'$.

Of course M has infinite area. Recall that such a surface is said to be area-minimizing provided every compact portion of it is area-minimizing.

Proof. Let $\mathbf{B}_r = \mathbf{B}(0, r)$ be the ball of Euclidean radius r centered at 0. Thus the hyperbolic radius of $\mathbf{B}_r$ tends to ∞ as $r \to 1$. Note that

$$\mathcal{R}_r := \{L \cap \mathbf{B}_r : L \in \mathcal{R}\}$$

is a rotationally invariant foliation of $\mathbf{B}_r \setminus Z$. Applying Theorem 5.4 to $\mathcal{R}_r$, $M \cap \mathbf{B}_r$, and $U \cap \mathbf{B}_r$, we see that $M \cap \mathbf{B}_r$ is area minimizing. Since this is true for each $r < 1$, the surface M is area minimizing.

To prove the "furthermore" assertion, let M' be a rotationally invariant area-minimizing surface with $\partial M' = \partial M$. By elementary topology, there is regular open set U' such that $M' = (\partial U') \cap \mathbf{B}$ and such that $\overline{U'} \cap \partial\mathbf{B} = \overline{U} \cap \partial\mathbf{B}$.

Note that $M' \cap \mathrm{Cyl}(s)$ has the same boundary as the surface consisting of $M \cap \mathrm{Cyl}(s)$, $(U \setminus U') \cap \partial\,\mathrm{Cyl}(s)$, and $(U' \setminus U) \cap \partial\,\mathrm{Cyl}(s)$ (provided the latter two surfaces are oriented suitably). Thus, since M' is area-minimizing,

$$\text{area}(M' \cap \mathrm{Cyl}(s)) \le \text{area}(M \cap \mathrm{Cyl}(s)) + \text{area}((U \Delta U') \cap \partial\,\mathrm{Cyl}(s)), \tag{14}$$

where $U \Delta U' = (U \setminus U') \cup (U' \setminus U)$ denotes the symmetric difference of U and U'. By Theorem 8.2,

$$\text{area}((U \Delta U') \cap \partial\,\mathrm{Cyl}(s)) \to 0 \tag{15}$$

as $s \to \infty$, so by (14),

$$\text{area}(M' \cap \mathrm{Cyl}(s)) \le \text{area}(M \cap \mathrm{Cyl}(s)) + o(1), \tag{16}$$

where $o(1)$ denotes any quantity that tends to 0 as $s \to \infty$.

Note that M and M', and therefore also $U \setminus U'$ and $U' \setminus U$, lie within a bounded hyperbolic distance d of $\mathbf{B} \cap \{z = 0\}$. Fix an $h > d$. Let $\mathrm{Cyl}(s, h)$ denote the set of points in $\mathrm{Cyl}(s)$ that are at hyperbolic distance less than h from $\mathbf{B} \cap \{z = 0\}$. Let $U(s, h) = U \cap \mathrm{Cyl}(s, h)$. Now apply the divergence theorem to ν on $U(s, h)$:

$$\begin{aligned} 0 &= \int_{U(s,h)} \mathrm{Div}\,\nu\,dv \\ &= \int_{M \cap \mathrm{Cyl}(s,h)} \nu \cdot n\,dA + \int_{U \cap \partial \mathrm{Cyl}(s,h)} \nu \cdot n_{\mathrm{Cyl}(s,h)}\,dA. \end{aligned} \tag{17}$$

Similarly, applying the divergence theorem to ν on $U'(s, h) = U' \cap \mathrm{Cyl}(s, h)$ gives

$$0 = \int_{M' \cap \mathrm{Cyl}(s,h)} \nu \cdot n'\,dA + \int_{U' \cap \partial \mathrm{Cyl}(s,h)} \nu \cdot n_{\mathrm{Cyl}(s,h)}\,dA. \tag{18}$$

Combining (17) and (18) gives

$$\begin{aligned} &\int_{M \cap \mathrm{Cyl}(s,h)} \nu \cdot n\,dA + \int_{(U \setminus U') \cap \partial \mathrm{Cyl}(s,h)} \nu \cdot n_{\mathrm{Cyl}(s,h)}\,dA \\ &\qquad = \int_{M' \cap \mathrm{Cyl}(s,h)} \nu \cdot n'\,dA + \int_{(U' \setminus U) \cap \partial\,\mathrm{Cyl}(s,h)} \nu \cdot n_{\mathrm{Cyl}(s,h)}\,dA. \end{aligned}$$

By choice of h, none of these terms is changed if we replace $\mathrm{Cyl}(s,h)$ by $\mathrm{Cyl}(s)$:

$$\int_{M\cap \mathrm{Cyl}(s)} \nu\cdot n\,dA + \int_{(U\setminus U')\cap\partial\,\mathrm{Cyl}(s)} \nu\cdot n_{\mathrm{Cyl}(s)}\,dA$$
$$= \int_{M'\cap \mathrm{Cyl}(s)} \nu\cdot n'\,dA + \int_{(U'\setminus U)\cap\partial\,\mathrm{Cyl}(s)} \nu\cdot n_{\mathrm{Cyl}(s)}\,dA.$$

Since $\nu\equiv n$ on M, using (15), we have

$$\begin{aligned}
&\mathrm{area}(M\cap \mathrm{Cyl}(s))\\
&\quad= \mathrm{area}(M'\cap \mathrm{Cyl}(s)) + \int_{M'\cap \mathrm{Cyl}(s)} (\nu\cdot n' - 1)\,dA + o(1)\\
&\quad\le \mathrm{area}(M\cap \mathrm{Cyl}(s)) + \int_{M'\cap \mathrm{Cyl}(s)} (\nu\cdot n' - 1)\,dA + o(1)
\end{aligned}$$

by (16).

Subtracting $\mathrm{area}(M\cap\mathrm{Cyl}(s))$ from both sides and then letting $s\to\infty$ gives

$$0\le \int_{M'} (\nu\cdot n' - 1)\,dA$$

which implies that $\nu\equiv n'$ on M'. This implies that M' consists of rotationally invariant leaves of $\mathcal{R}$. q.e.d.

9. The Hyperbolic case III. Specifying the rotationally invariant leaves of a limit lamination: Section 6 in the hyperbolic case

For a relatively closed subset $T\subset(0,1)$, we defined in Section 6.1 a lamination $\mathcal{C}(T)$ of $\partial\mathbf{B}\setminus\{p^+,p^-\}$ and a lamination $\mathcal{M}(T)$ of $\mathbf{B}\setminus Z$.

Theorem 9.1. *Let* $\mathbf{B}$ *be the open unit ball with the Poincaré metric. Let* T *be a relatively closed subset of* $(0,1)$*. There exists a sequence of spanning minimal* θ*-graphs in* $\mathbf{B}\setminus Z$ *that converge to a limit lamination* $\mathcal{L}$ *whose rotationally invariant leaves are precisely* $\mathcal{M}(T)$*.*

This is the hyperbolic version of Theorem 6.3. That theorem is for Riemannian metrics on $\mathbf{B}$ that extend smoothly to $\overline{\mathbf{B}}$, something that is not true for the Poincaré metric. Nevertheless, we can use the results of the previous sections to prove this result.

Proof. We follow the proof of Theorem 6.3. Start with a sequence $\gamma_i\subset\partial B$ of θ-graphs with the bulleted properties that define $\mathcal{G}$. Choose them so that they that converge to a lamination $\mathcal{C}$ of $\partial B\setminus Z$ whose rotationally invariant leaves are precisely the circles in $c(T)$, where the convergence is smooth except possibly where $z=0$. By Theorem 7.1, we may assert the existence of a smooth, embedded, minimal θ-graph D_i with boundary $\gamma_i\cup(Z\cap\overline{\mathbf{B}})$. Since this boundary is μ-invariant it follows from Proposition 7.2 that D_i is also μ-invariant. Passing to a subsequence, we may assume that these θ-graphs converge smoothly to a lamination $\mathcal{L}$ of $\overline{B} \setminus Z$. This limit lamination is also μ-invariant. We must show that the limit leaves of $\mathcal{L}$ are precisely the rotationally invariant surfaces in $\mathcal{M}(T)$.

The limit leaves of $\mathcal{L}' \subset \mathcal{L}$ together with the rotations about Z of the non-limit leaves of $\mathcal{L}$ form an oriented minimal foliation $\mathcal{R}$ of $\mathbf{B} \setminus Z$ that is rotationally invariant about Z. Therefore, we may use Theorem 8.3. This theorem can be easily used to show that Corollaries 5.5 and 5.6 hold in hyperbolic space. The arguments in Steps 1 and 2 of the proof of Theorem 6.3 are now directly applicable to our situation, using Theorem 8.3 where Theorem 5.4 is invoked. q.e.d.

Remark 9.2. To be precise, the limit leaves are the annuli, if any, in $\mathcal{M}(T)$ together with the disks in $\mathcal{M}(T)$, if any, with their centers removed. By not removing the centers, we may consider $\mathcal{M}(T)$ as a lamination of $\mathbf{B}$.

As a application of Theorem 9.1, let a be small enough so that $\mathcal{M}(a)$ consists of one or more catenoids. (See (2) of Section 6.) Theorem 9.1 above tells us we may realize $\mathcal{M}(a)$ as the limit leaves of a limit lamination of $\mathbf{B} \setminus Z$. Doubling the nonlimit leaves of the limit lamination by reflection in Z produces a lamination of $\mathbf{B}$ with the same limit leaves $\mathcal{M}(a)$, one nonlimit leaf in the component of $\mathbf{B} \setminus \cup\mathcal{M}(a)$ that contains $\mathbf{B} \cap Z$, and two congruent leaves in every other component of $\mathbf{B} \setminus \cup\mathcal{M}(a)$. Consequently:

Theorem 9.3. *There exist complete, embedded, simply connected minimal surfaces in hyperbolic space that are not properly embedded. In particular for every area-minimizing catenoid C in hyperbolic space, there exist two complete, noncongruent, simply connected, embedded minimal surfaces (one on either side of C) that have C in their closure.*

Appendix A. The divergence theorem

Theorem A.1 (Generalized Divergence Theorem)**.** *Suppose that Ω is a domain with compact closure and with piecewise smooth boundary in a Riemannian $(m+1)$-manifold. Suppose that K is a compact subset of $\overline{\Omega}$ with Hausdorff m-dimensional measure 0, and that ν is a bounded C^1 vectorfield on $\overline{\Omega} \setminus K$ such that $\int |\operatorname{Div} \nu|\, dV < \infty$. Then*

$$\int_\Omega \operatorname{Div} \nu \, dV = \int_{\partial\Omega} \nu \cdot n \, dA,$$

where n is the unit normal to $\partial\Omega$ that points out of Ω.

Here dV and dA indicate integration with respect to $(m+1)$-dimensional volume and m-dimensional area (i.e., with respect to Hausdorff measure of those dimensions.)

Proof. Let ϵ_n be a sequence of positive numbers converging to 0. Note that for each n, we can cover K by open balls such that the sum of the areas of the boundaries of the balls is less than ϵ_n. Since K is compact, we can cover it by a finite collection of such balls. Let $W(n)$ be the union of those balls. Note that for each n, the balls may be chosen to have arbitrarily small radii. In particular, we can choose the balls at stage n so that $W(n) \subset W(n-1)$. Note that $\cap_n W(n) = K$. Applying the divergence theorem to $\Omega \setminus W(n)$ gives

$$\begin{aligned}\int_{\Omega\setminus W(n)} \operatorname{Div} \nu \, dV &= \int_{\partial(\Omega\setminus W(n))} \nu \cdot n \, dA \\ &= \int_{(\partial\Omega)\setminus W(n)} \nu \cdot n \, dA + \int_{\Omega\cap\partial W(n)} \nu \cdot n \, dA.\end{aligned}$$

This last integral is bounded in absolute value by the supremum of $|\nu|$ times the area of $\partial W(n)$; that area is bounded by ϵ_n by choice of $W(n)$. Thus

$$\int_{\Omega\setminus W(n)} \operatorname{Div} \nu \, dV = \int_{(\partial\Omega)\setminus W(n)} \nu \cdot n \, dA + O(\epsilon_n).$$

Now use the dominated convergence theorem to take the limit as $n \to \infty$. (Recall that the $W(n)$ are nested and that $\cap_n W(n) = K$.) q.e.d.

Appendix B. Minimal graphs

Let N be a smooth 2-manifold with boundary. Let g be a smooth Riemannian metric (not necessarily complete) on $N \times \mathbf{R}$ that is invariant under vertical translations. Suppose also that $N \times \mathbf{R}$ is strictly mean convex, i.e. the mean curvature vector of $\partial(N \times \mathbf{R})$ is a positive multiple of the inward-pointing unit normal. We will assume when necessary that N has been isometrically embedded in some Euclidean space $\mathbf{R}^k$, so that $N \times \mathbf{R}$ lies in $\mathbf{R}^{k+1}$. Thus translation and dilation make sense.

We are interested in graphs over N: Let

$$f : N \setminus \partial N \to \mathbf{R}$$

be a smooth function whose graph is a g-minimal surface M. *We will assume throughout this Appendix that M is such a graph, and that $\overline{M}$ is a smoothly embedded manifold-with-boundary, where the boundary is* $\Gamma := \overline{M} \cap \partial N$. We will also assume that the curvature of Γ and its derivative with respect to arclength are bounded above by some $\kappa < \infty$.

Letting N be a convex domain in $\mathbf{R}^2$ gives the simplest example of this setting. Here, the metric g on $N \times \mathbf{R}$ is the product metric, and it is the standard Euclidean metric. In this paper we are considering rotationally symmetric (around the z-axis Z) domains $W \subset \mathbf{R}^3$ endowed with Riemannian metrics that are also rotationally symmetric. The simply connected covering space of $W \setminus Z$ can be written in the form $N \times \mathbf{R}$, where

$$N = \{(x,z) : x > 0,\ (x,0,z) \in W\}.$$

Vertical translations in $N \times \mathbf{R}$ correspond to rotations in $W \setminus Z$. The metric g on $N \times \mathbf{R}$ lifted from the metric on $W \setminus Z$ is translation invariant but it is not the product metric.

Proposition B.1. *Suppose that K is any compact region in $(N \setminus \partial N) \times \mathbf{R}$ with piecewise-smooth, mean convex boundary. Then*

1. *The surface $M \cap K$ has less area than any other surface in K having the same boundary.*
2. *Furthermore,*

$$\text{area}(M \cap K) \leq \frac{1}{2}\,\text{area}(\partial K).$$

Theorem B.2. *If U is an open subset of N with compact closure and if $\mathcal{U} = U \times \mathbf{R}$, then*

$$B(M,p)\ \text{dist}(p, \mathcal{U}^c) < C,$$

for some constant $C = C(N, g, \mathcal{U}, \kappa) < \infty$.

Proof of Proposition B.1. To prove Statement 1, define

$$F : N \times \mathbf{R} \to \mathbf{R},$$
$$F(x,z) = z - f(x).$$

Note that the level sets of F are vertical translates of M, that these level sets foliate $N \times \mathbf{R}$, and that M is the level set $F = 0$. Now let S be the least-area surface (flat chain mod 2) in K having the same boundary as $M \cap K$. Then S is smooth except possibly at its boundary. Assume that S is not $M \cap K$. Then F is nonzero, say positive, at some point of S. Let q be the point in S at which F is a maximum. Since $F = 0$ on ∂S, q is an interior point. Thus S lies below the minimal surface $F = F(q)$ but touches it at q. By the maximum principle, the entire connected component S' of S that contains q must lie in the level set $F = F(q)$. Note that S' must have boundary points, since otherwise $S \setminus S'$ would have the same boundary as S but less area. However, $F = 0$ on ∂S, a contradiction. This completes the proof of Statement 1.

To prove Statement 2, note that $(\partial K) \cap \{F < 0\}$ and $(\partial K) \cap \{F \geq 0\}$ both have the same boundary as $M \cap K$, and their areas add up to $\text{area}(\partial K)$. Thus

$$\begin{aligned}\text{area}(M \cap K) &\leq \min\{\text{area}((\partial K) \cap \{F < 0\}), \text{area}((\partial K) \cap \{F \geq 0\})\} \\ &\leq \frac{1}{2}\,\text{area}(\partial K).\end{aligned}$$

q.e.d.

Proof of Theorem B.2. Suppose that the theorem is false. Then there is a sequence of examples M_n satisfying the hypotheses of the theorem such that

$$\sup_{p \in M_n \cap \overline{\mathcal{U}}} B(M_n, p)\ \mathrm{dist}(p, \mathcal{U}^c) \to \infty. \tag{19}$$

Since $M_n \cap \overline{\mathcal{U}}$ is compact (it is the graph of a smooth function over $\overline{U}$), the supremum in (19) is attained at some point $p_n \in M_n \cap \overline{\mathcal{U}}$. Thus:

$$B(M_n, p)\ \mathrm{dist}(p, \mathcal{U}^c) \le B(M_n, p_n)\ \mathrm{dist}(p_n, \mathcal{U}^c) \to \infty \tag{20}$$

for any $p \in M_n \cap \overline{\mathcal{U}}$. By vertically translating each M_n, we may assume that the height of p_n is 0. The assumption about bounds on the curvature of $\Gamma_n = \partial M_n$ imply that we can assume, by passing to a subsequence, that the Γ_n converge in $C^{2,\alpha}$ to an embedded curve Γ. (If ∂N is connected, then of course each Γ_n is connected. But Γ need not be connected because portions of Γ_n may go off to infinity.)

Now translate M_n, $\mathcal{U}$, and $N \times \mathbf{R}$ by $-p_n$ and dilate by $B(M_n, p_n)$ to get M_n', $\mathcal{U}_n'$, and $N' \times \mathbf{R}$. Note that

$$B(M_n', 0) = 1 \tag{21}$$

and, using (21), the scale invariance of the product $B(M_n, p)\ \mathrm{dist}(p, \mathcal{U}^c)$, and (20),

$$\begin{aligned} \mathrm{dist}(0, (\mathcal{U}_n')^c) &= B(M_n', 0)\ \mathrm{dist}(0, (\mathcal{U}_n')^c) \\ &= B(M_n, p_n)\ \mathrm{dist}(p_n, (\mathcal{U}_n)^c) \\ &\to \infty. \end{aligned}$$

In particular,

$$\mathrm{dist}(0, (\mathcal{U}_n')^c) \to \infty. \tag{22}$$

Choose $R < \mathrm{dist}(0, \partial \mathcal{U}_n')$, and let p be a point in M_n' satisfying $\mathrm{dist}(p, 0) < R$. (Here we are using the rescaled metric associated at the nth stage.) For such a choice of p we have from (20), scale invariance and (21) and the triangle inequality:

$$\begin{aligned} B(M_n', p) &\le B(M_n', 0) \frac{\mathrm{dist}(0, (\mathcal{U}_n')^c)}{\mathrm{dist}(p, (\mathcal{U}_n')^c)} \\ &= \frac{\mathrm{dist}(0, (\mathcal{U}_n')^c)}{\mathrm{dist}(p, (\mathcal{U}_n')^c)} \\ &\le \frac{\mathrm{dist}(0, (\mathcal{U}_n')^c)}{\mathrm{dist}(0, (\mathcal{U}_n')^c) - \mathrm{dist}(p, 0)}. \end{aligned}$$

Now choose any fixed $R > 0$. By (22), we have $R < \mathrm{dist}(0, \partial \mathcal{U}_n')$ for n large enough. Choose p so that $\mathrm{dist}(p, \partial \mathcal{U}_n') < R$. Then from the estimate above

$$B(M'_n, p) \leq \left(1 - \frac{\operatorname{dist}(p, 0)}{\operatorname{dist}(0, (\mathcal{U}'_n)^c)}\right)^{-1}$$
$$\leq \left(1 - \frac{R}{\operatorname{dist}(0, (\mathcal{U}'_n)^c)}\right)^{-1}.$$

This estimate is valid for any $R > 0$ and n sufficiently large. It follows that

$$\limsup_{n\to\infty} \left(\sup\{B(M'_n, p) \,:\, p \in M'_n,\ \operatorname{dist}(p, 0) < R\}\right) \leq 1. \tag{23}$$

Note that the dilation factors $B(M_n, p_n)$ are diverging. Hence the metrics $B(M_n, p_n)g$ are becoming the flat metric. The curvature estimate (23) implies that (after passing to a subsequence) the M'_n converge smoothly to an area-minimizing surface M' in a flat Euclidean space E. Whether E is all of $\mathbf{R}^3$ or not depends on what happens as $n \to \infty$ to $\partial N'_n \times \mathbf{R}$. If $\operatorname{dist}(0, \partial N_n \times \mathbf{R}) \to \infty$, then E is Euclidean three-space. If these distances are bounded, then E is a flat halfspace bounded by a plane corresponding to the limit (after passing to a further subsequence) of the boundaries $N'_n \times \mathbf{R}$. In the latter case, $\partial M'$ is a straight line lying in the plane ∂E. In either case, from (23) and (21), we can assert that

$$\sup B(M', \cdot) = B(M', 0) = 1. \tag{24}$$

Claim. *E is a halfspace, and $M' \subset E$ is a properly embedded, simply connected area-minimizing minimal surface with quadratic area growth, whose boundary $\partial M'$ is a line in the plane ∂E.*

Proof of Claim. Each M'_n is a graph. Hence M' is simply connected and properly embedded in E. Recall that M_n is stable in $N \times \mathbf{R}$. Hence M'_n is stable in $N'_n \times \mathbf{R}$. Stability gives us the estimate

$$B(M'_n, 0) \operatorname{dist}(0, \Gamma'_n \cup (\mathcal{U}'_n)^c) < c_0$$

for some constant c_0 independent of n. Therefore,

$$\operatorname{dist}(0, \Gamma'_n \cup (\mathcal{U}'_n)^c) < c_0$$

since $B(M'_n, 0) = 1$. Thus by (22),

$$\limsup_{n\to\infty} \operatorname{dist}(0, \Gamma'_n) < c.$$

It follows that (after passing to a subsequence) the Γ'_n converge smoothly to a straight line Γ' and that $\partial N'_n \times \mathbf{R}$ converges smoothly to a limit E that is isometric to a closed halfspace of $\mathbf{R}^3$. The boundary of E contains the line Γ'.

Observe that if $q \in \partial E$, then it follows from Statement 2 of Proposition B.1 that

$$\operatorname{area}(M' \cap \mathbf{B}(q, r)) \leq \frac{1}{2} \operatorname{area}(\partial(\mathbf{B}(q, r) \cap E)) = 3\pi r^2.$$

Thus M' has quadratic area growth. It follows from Statement 1 of that same proposition that M' is area minimizing. This completes the proof of the claim. q.e.d.

We now show that M' must be a halfplane, contradicting the fact that $B(M', 0) = 1$ (see (24)). This will complete the proof of Theorem B.2.

A properly embedded, area-minimizing minimal surface with quadratic area growth must be a halfplane or half of Enneper's surface. This was conjectured by one of us [23] and proved by Pérez [20]. (Here, area-minimizing is used in the classical sense. That is, the allowed comparison surfaces are obtained by compactly supported deformations that vanish on the boundary.) According to the claim above, M' satisfies all the hypotheses, so it must be either a halfplane or half of Enneper's surface. But M' lies in a halfspace, and half of Enneper's surface does not. So M' is a halfplane.

Here is another way to see that M' is a halfplane. Double M' by Schwartz reflection about its boundary line to produce a complete, simply connected, embedded, minimal surface. As established in the Claim above, M', has quadratic area growth in $\mathbf{R}^3$, so the same is true for its double. But finite topology together with quadratic area growth was shown by P. Li [15] (see Proposition 32 in [25]) to imply finite total curvature, and it is well known that the only complete, simply connected, embedded minimal surface of finite total curvature is the plane. q.e.d.

Appendix C. Hyperbolic catenoids

Consider the hyperbolic metric on the upper halfspace:

$$\frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let $r = \sqrt{x^2 + y^2}$ and $R = \sqrt{x^2 + y^2 + z^2}$. Let θ be the angle that the vector (x, y, z) makes with the horizontal:

$$\theta = \arcsin \frac{z}{R} \in [0, \pi/2].$$

The hemispheres $\{R = \text{constant}\}$ are totally geodesic surfaces of revolution about $Z = \{x = y = 0\} = \{\theta = \pi/2\}$. For $\alpha \in (0, \pi/2)$, the surfaces

$$\text{cone}(\alpha) := \{\theta = \alpha\} \tag{25}$$

are surfaces of revolution about Z orthogonal to the hemispheres.

The hyperbolic distance s from Z of a point $(x, y, z) \in \text{cone}(\alpha)$ to Z is given by the following, where $R^2 = x^2 + y^2 + z^2$:

$$s = \int_\alpha^{\pi/2} \frac{R\,d\theta}{z} = \int_\alpha^{\pi/2} \frac{d\theta}{\sin\theta} = |\ln\tan(\alpha/2)|. \tag{26}$$

From this, we see that cone(α) is the set of points at constant hyperbolic distance s from Z.

If we define Cyl(s) to be the points at hyperbolic distance equal to or less than s from Z, then

$$\begin{aligned} \mathrm{Cyl}(s) &= \{(x,y,z) : \pi/2 > \theta(x,y,z) \geq \alpha\}, \\ \partial\,\mathrm{Cyl}(s) &= \mathrm{cone}(\alpha), \end{aligned} \tag{27}$$

where α and s are related by (26).

In general, consider a surface Σ of revolution about Z. It can be expressed as

$$R = R(\tau), \quad \theta = \theta(\tau), \quad \tau \in I,$$

where $I \subset \mathbf{R}$ is some interval. Since the Euclidean distance to Z is $R\sin\theta$, the Euclidean area of an infinitesimal ribbon of Σ is given by

$$2\pi R\cos\theta\sqrt{dR^2 + R^2\,d\theta^2}.$$

Therefore the hyperbolic area of that ribbon is

$$\begin{aligned} \frac{2\pi R\cos\theta\sqrt{dR^2 + R^2\,d\theta^2}}{z^2} &= \frac{2\pi R^2\cos\theta}{z^2}\sqrt{(dR/R)^2 + d\theta^2} \\ &= \frac{2\pi\cos\theta}{\sin^2\theta}\sqrt{dt^2 + d\theta^2}, \end{aligned}$$

where $t = \log R$ (so $R = e^t$). Here we have used $z = R\sin\theta$.

Consequently, we see that a surface rotationally invariant about Z is a minimal surface if and only if the corresponding curve in

$$\{(\theta, t) \in (0, \pi/2] \times \mathbf{R}\}$$

is a geodesic with respect to the metric

$$\frac{2\pi\cos\theta}{\sin^2\theta}\sqrt{dt^2 + d\theta^2}. \tag{28}$$

Now suppose we have a geodesic given by

$$t = t(\theta), \quad \theta \in I,$$

where $I \subset (0, \pi/2]$ is an interval. Then the length is

$$\int_{\theta\in I}\frac{2\pi\cos\theta}{\sin^2\theta}\sqrt{dt^2 + d\theta^2} = \int_{\theta\in I}\frac{2\pi\cos\theta}{\sin^2\theta}\sqrt{t'(\theta)^2 + 1}\,d\theta.$$

Since the integrand does not depend on t, the Euler-Lagrange equation for this functional (i.e., the equation for a geodesic) is

$$\frac{d}{d\theta}\left(\frac{2\pi\cos\theta}{\sin^2\theta}\frac{t'(\theta)}{\sqrt{t'(\theta)^2 + 1}}\right) = 0$$

or

$$\frac{2\pi\cos\theta}{\sin^2\theta}\frac{t'(\theta)}{\sqrt{t'(\theta)^2 + 1}} = c \tag{29}$$

for some constant c.

From (29) we have the following result:

Theorem C.1. *For θ near 0,*

$$t'(\theta) = O(\theta^2) \tag{30}$$

and therefore

$$t(\theta) - t(0) = O(\theta^3). \tag{31}$$

Corollary C.2. *Consider two geodesics in $(0, \pi/2] \times \mathbf{R}$ converging to the same ideal boundary point. The vertical distance between them tends to 0 as $\theta \to 0$. That is, if $t_1(\cdot)$ and $t_2(\cdot)$ are two solutions of the Euler-Lagrange equation with $t_1(0) = t_2(0)$, and if $I(\theta)$ is the vertical segment joining $(\theta, t_1(\theta))$ and $(\theta, t_2(\theta))$, then the length of $I(\theta)$ (with respect to the metric (28)) is $O(\theta)$ as $\theta \to 0$.*

Proof. We can let $t_1(\cdot)$ be any solution $t(\cdot)$, and we may as well take $t_2(\theta)$ to be the horizontal geodesic $t_2(\theta) \equiv t(0)$. Now

$$\begin{aligned}
\operatorname{length}(I(\theta)) &= \frac{2\pi\cos\theta}{\sin^2\theta}\operatorname{length}_{\mathrm{eucl}} I(\theta) \\
&= \frac{2\pi\cos\theta}{\sin^2\theta}|t(\theta) - t(0)| \\
&= \frac{2\pi\cos\theta}{\sin^2\theta}O(\theta^3),
\end{aligned}$$

which is clearly $O(\theta)$. q.e.d.

Remark C.3. The length of $I(\theta)$ equals the area of the ribbon on cone(θ) between the rotational minimal surfaces that correspond to the two geodesics converging to the same ideal-boundary point. By (27), cone$(\theta) = \partial\,\mathrm{Cyl}(s)$, and by (26), $s \to \infty$ if and only if $\theta \to 0$. Therefore, Theorem 8.2 follows from Corollary C.2.

We now compute the curvature of the Riemannian metric (28).

Lemma C.4. *Let $\lambda = \lambda(\theta, t) = 2\pi\cos\theta/\sin^2\theta$. The Gauss curvature K of the metric $\lambda\sqrt{d\theta^2 + dt^2}$ on the strip $(\theta, t) \in (0, \pi/2) \times \mathbf{R}$ is given by*

$$K = K(\theta) = \frac{1}{4\pi^2}\tan^2\theta\left(\tan^2\theta - 2\right).$$

In particular, $K \geq 0$ if and only if $\theta \geq \alpha_0 := \arctan\sqrt{2}$.

Proof. We use the following formula for the Gauss curvature of a surface with a conformal metric $\lambda\sqrt{d\theta^2 + dt^2}$:

$$K = \frac{-\Delta \ln \lambda}{\lambda^2}. \tag{32}$$

We compute

$$(\ln \lambda)' = -\tan\theta + 2\cot\theta,$$
$$(\ln \lambda)'' = -\frac{1}{\cos^2\theta} + \frac{2}{\sin^2\theta}.$$

Thus

$$K = \frac{\sin^4\theta}{4\pi^2\cos^2\theta}\left(\frac{1}{\cos^2\theta} - \frac{2}{\sin^2\theta}\right) = \frac{1}{4\pi^2}\tan^2\theta\left(\tan^2\theta - 2\right).$$

q.e.d.

Proposition C.5. *Let C and C' be minimal annuli of rotation with a common axis Z in hyperbolic thee-space. Suppose that both of these annuli lie outside the cylinder* $\mathbf{Cyl}(\ln\tan(\alpha_0/2))$*, as defined in* (27)*. Then C and C' can intersect in at most one circle. In particular, no two such annuli have the same boundary.*

Here, $\alpha_0 = \arctan\sqrt{2}$ as in Lemma C.4 above. The proposition follows immediately from Lemma C.4 and the observation that on a surface of negative curvature, two distinct geodesics cannot cross more than once, a simple consequence of the Gauss-Bonnet formula. (By construction, geodesics in the strip $(0, \pi/2) \times \mathbf{R}$ correspond to minimal annuli of rotation in hyperbolic three-space.)

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