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MINIMAL SURFACES IN THE 3-SPHERE BY STACKING CLIFFORD TORI

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Abstract

Extending work of Kapouleas and Yang, for any integers $N \geq 2$, $k, \ell \geq 1$, and m sufficiently large, we apply gluing methods to construct in the round 3-sphere a closed embedded minimal surface that has genus $k\ell m^2(N-1)+1$ and is invariant under a $D_{km} \times D_{\ell m}$ subgroup of $O(4)$, where D_n is the dihedral group of order $2n$. Each such surface resembles the union of N nested topological tori, all small perturbations of a single Clifford torus $\mathbb{T}$, that have been connected by $k\ell m^2(N-1)$ small catenoidal tunnels, with $k\ell m^2$ tunnels joining each pair of neighboring tori. In the large-m limit for fixed N, k, and ℓ, the corresponding surfaces converge to $\mathbb{T}$ counted with multiplicity N.

1. Introduction

In [13] Kapouleas and Yang constructed a sequence of embedded minimal surfaces in the round 3-sphere $\mathbb{S}^3$ converging to a Clifford torus $\mathbb{T}$ counted with multiplicity 2; each surface consists of two small perturbations of $\mathbb{T}$ connected by many catenoidal annuli taking their centers at the sites of a square lattice on $\mathbb{T}$. Accordingly they called their surfaces *doublings* of the Clifford torus. Kapouleas announced these in [8] as the first examples of a general class of gluing constructions to double given minimal surfaces, subsequently discussed further in [9]. More recently in [10] he has doubled the equatorial 2-sphere in $\mathbb{S}^3$, and now additional such doublings with different configurations of catenoidal tunnels have been carried out by Kapouleas and McGrath [11]. Min-max methods have also been used to double minimal surfaces in $\mathbb{S}^3$. Pitts and Rubinstein proposed a variety of such constructions in [20]. One was completed by Ketover, Marques, and Neves in [16], where they too double the torus over square lattices, conjecturally producing the same surfaces as [13] when the lattice spacing becomes small, and in [15] Ketover has performed more min-max constructions, including doublings, previously described in [20]. Doublings appear in the free-boundary setting

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as well. Using variational rather than gluing methods, for each integer $n \geq 3$ Fraser and Schoen ([5]) have constructed orientable free-boundary minimal embeddings in the unit ball with genus 0 and n boundary components; for large n these surfaces look like doublings of the equatorial disc. Later, in [4], Folha, Pacard, and Zolotareva applied gluing techniques to double the equatorial disc, producing free-boundary examples with genus 0 (possibly the same as those in [5]) or 1 and a large number n of boundary components.

Returning to [13], the surfaces of Kapouleas and Yang are highly symmetric, admitting many *horizontal* symmetries, which preserve as sets each of the two sides of the doubled Clifford torus and permute the lattice sites, as well as *vertical* symmetries, each of which exchanges the two sides of the doubled torus but fixes as a set a catenoidal tunnel (and in fact every catenoidal tunnel centered on a certain great circle on $\mathbb{T}$). All these symmetries are enforced throughout the construction and exploited to simplify its execution. The present article undertakes less vertically symmetric doublings of the torus, with the symmetry broken in two ways. First, we allow the catenoidal tunnels to be arranged on rectangular rather than strictly square lattices. Any isometry of $\mathbb{S}^3$ exchanging the two sides of the doubled Clifford torus will fail to preserve such a lattice, unless it is square. Second, we interpret *doubling* in a generalized sense, realizing also triplings, quadruplings, and in fact embedded minimal surfaces resembling any prescribed finite number of slightly perturbed copies of $\mathbb{T}$ connected to one another by many small catenoidal tunnels. Whenever at least three copies are incorporated, even if these tunnels are centered on square lattices, the symmetry group will not act transitively on the collection of copies.

These new constructions add to the list of known closed minimal embeddings in $\mathbb{S}^3$, so far comprising those found in [18], [14], [13], [3], [10], [19], [16], [15], [11], [12], and [1]. The survey article [2] contains an outline of a few of the constructions just mentioned. The constructions at hand should be of interest not only as providing new examples in $\mathbb{S}^3$ but also as a basis for further doublings with asymmetric sides in a variety of settings. A program toward doubling constructions of increasing generality, including potential applications, is described in [9]. The present work naturally emulates, with a few departures, the approach of [13] and draws extensively from the general gluing technology developed by Kapouleas, much of which can be found summarized in [8] and was itself inspired by techniques applied by Schoen in [21]. Although the current article can be read without reference to [13] or any other gluing constructions, for the rest of this introduction we will make use, without detailed explanation, of terminology standardized by Kapouleas, so that the reader already acquainted with it may easily appreciate the principal differences between this construction and [13].

We now outline our procedure in rough terms. We first fix a Clifford torus $\mathbb{T}$, which by definition is the locus of points in $\mathbb{S}^3 \subset \mathbb{R}^4$ at distance $\frac{\pi}{4}$ from some great circle C_1. More generally, for each $r \in \left(0, \frac{\pi}{2}\right)$ the locus of points at distance r from C_1 is a torus of constant mean curvature, which is equivalently the locus of points at distance $\frac{\pi}{2} - r$ from the great circle C_2 defined as the intersection with $\mathbb{S}^3$ of the orthogonal complement in $\mathbb{R}^4$ of the plane containing C_1. We will refer to C_1 and C_2 as the *axes* of each such torus; directions tangential to any of these tori we will call *horizontal*, while the orthogonal direction we will call *vertical*. As basic data for the construction we take integers $k, \ell, m \geq 1$ and $N \geq 2$. Corresponding to a choice of such data an *initial surface* is built as follows.

We start with N constant-mean-curvature tori coaxial with $\mathbb{T}$, labeled $\mathbb{T}[1], \mathbb{T}[2], \ldots, \mathbb{T}[N]$ so that the index increases with distance from C_2. The precise placements of the tori (that is their signed distances from $\mathbb{T}$) cannot be freely prescribed but will be determined, as a function of the data, in the course of the construction. For now we mention only that as $m \to \infty$ every $\mathbb{T}[j]$ tends to $\mathbb{T}$. Next we will connect this collection of tori by first from each one excising discs centered on certain lattices (to be described momentarily) and then gluing in truncated approximate catenoids, which shrink to points on $\mathbb{T}$ as $m \to \infty$. Like the arrangement of the tori, the precise sizes of the catenoids and the heights (signed distances from $\mathbb{T}$) of their centers are variables whose values will be set by conditions—to be described later—necessary for the completion of the construction.

On the other hand, we impose enough horizontal symmetries (isometries of $\mathbb{S}^3$ preserving each side of $\mathbb{T}$) that the horizontal positioning of the catenoids and equivalently the locations of the discs deleted from the tori are directly determined by the data already listed. Specifically, we mark km equally spaced points on C_1 and ℓm equally spaced points on C_2, and we write $\mathcal{G}[k, \ell, m]$ for the subgroup of $O(4)$ preserving each of these sets of marked points. Thus $\mathcal{G}[k, \ell, m]$ is isomorphic to $D_{km} \times D_{\ell m}$, where D_n is the dihedral group of order $2n$. Note that $\mathcal{G}[k, \ell, m]$ is equally the subgroup of $O(4)$ preserving the union of the sets of marked points except when $k = \ell$, in which case $\mathcal{G}[k, \ell, m]$ is strictly contained in this last group, which admits also reflections through certain great circles on $\mathbb{T}$, exchanging C_1 and C_2.

We choose orientations on C_1 and C_2 (together selecting an orientation on $\mathbb{R}^4$), and for each $i \in \{0, 1\}$ we write $\mathsf{R}^{\theta}_{C_i}$ for the element of $O(4)$ fixing C_i pointwise and rotating the great circle in the orthogonally complementary plane through angle θ (according to its orientation). The group $\mathcal{G}[k, \ell, m]$ is then generated by (i) $\mathsf{R}^{2\pi/km}_{C_2}$, (ii) $\mathsf{R}^{2\pi/\ell m}_{C_1}$, (iii) reflection through any great sphere having equator C_2 and one pole a marked point on C_1, and (iv) reflection through any great sphere having equator

C_1 and one pole a marked point on C_2. (Of course the point antipodal to a given marked point on C_1 (or C_2) is itself marked if and only if km (or ℓm) is even.) Obviously $\mathcal{G}[k,\ell,m]$ preserves $\mathbb{T}$ and each $\mathbb{T}[j]$. We will design the initial surfaces (and the final minimal surfaces they approximate) to be likewise invariant under $\mathcal{G}[k,\ell,m]$.

Next we pick a marked point on C_1, a marked point on C_2, and the minimizing geodesic segment (quarter great circle) joining them. This segment intersects $\mathbb{T}$ at a single point, whose orbit under $\mathcal{G}[k,\ell,m]$ is a $km \times \ell m$ rectangular lattice on $\mathbb{T}$, which we call $L_{0,0}$. We will use $L_{0,0}$ to fix the horizontal positions of the catenoidal annuli connecting the N tori in our configuration. In fact there are precisely four $km \times \ell m$ rectangular lattices on $\mathbb{T}$ preserved by $\mathcal{G}[k,\ell,m]$. It would be possible to carry out the constructions in this paper using any of these lattices (to locate the catenoidal annuli) without introducing any additional technical difficulties, but to simplify the presentation we will make use of only $L_{0,0}$ and $L_{1,1} := \mathsf{R}_{C_2}^{\frac{\pi}{km}} \mathsf{R}_{C_1}^{\frac{\pi}{\ell m}} L_{0,0}$. (Of course there are also finer lattices preserved by the same group. Permitting such lattices in the construction of the initial surface would allow for different numbers of catenoidal tunnels connecting different pairs of tori while maintaining the high horizontal symmetry but would demand a more complicated approach.)

Now for each $j \in \{1, 2, \dots N-1\}$ and for each point in $L_{\frac{(-1)^j+1}{2}, \frac{(-1)^j+1}{2}}$ we locate the closest point on $\mathbb{T}[j]$ and the closest point on $\mathbb{T}[j+1]$, and we excise from the two tori two small discs having these nearest points as their respective centers. Then for each such pair of points, using local coordinates for $\mathbb{S}^3$ adapted to the tori, we smoothly glue the boundary circles of the deleted discs to the boundary circles of a catenoidal annulus centered on the geodesic segment connecting the two points. The radii of the deleted discs are chosen comparable to the lattice spacing but small enough so that all the discs are pairwise disjoint, and the annuli are shaped so that the resulting connected surface is invariant under $\mathcal{G}[k,\ell,m]$. As already noted, additional information is needed to specify the precise sizes and heights of these annuli, but right now we mention that, when suitably scaled, each tends with large m to a complete standard catenoid.

Thus we have produced a connected closed surface, the initial surface, which is preserved by $\mathcal{G}[k,\ell,m]$, is easily seen to have genus $N + (N-1)k\ell m^2 - (N-1) = k\ell m^2(N-1) + 1$ (since $N-1$ of the $(N-1)k\ell m^2$ catenoidal annuli are spent to connect the N tori, each of genus 1, while the remaining ones contribute genus), and, for large m, is approximately minimal in a certain sense. The construction will be completed by perturbing the surface to exact minimality. Two mechanisms of perturbation are applied in tandem. One sort of perturbation is realized by considering graphs of small functions over the initial surface. To select the right function is then to solve the elliptic quasilinear partial differential equation prescribing zero mean curvature for the corresponding graph. This equation can be studied by comparing the linearization of the operator governing the mean curvature of graphs to certain large-m limit operators on the limit catenoids and limit torus. In the simplest scenario imaginable one could solve the linearized equation on the toral and catenoidal components separately, combine these solutions through an iterative procedure, and finally invoke an inverse function theorem to solve the original nonlinear equation. However, the presence of nontrivial kernel to the limit operators gives rise to *approximate kernel* that obstructs the approach just described.

The space of admissible perturbing functions is constrained to respect the symmetries enjoyed by the initial surface, and so their imposition has the effect of reducing the dimension of the approximate kernel. Each torus turns out to carry one-dimensional approximate kernel of its own, but in [13] the two tori can be exchanged by reflections through certain great circles, and so together the tori contribute just one dimension to the approximate kernel in [13] versus N dimensions more generally. Furthermore, in [13] these reflections through circles render trivial the approximate kernel on the catenoidal tunnels. Following the approach of [13] in the absence of these symmetries, each tunnel would feature one-dimensional approximate kernel, but we bypass this kernel altogether by altering, as compared to [13], the initial data at the tunnel's waist for the rotationally invariant mode of the solution.

To overcome the obstruction posed by the approximate kernel [13] introduces *substitute kernel*, spanned by a single function supported on the tori away from the circles where they attach to the tunnels. By adding multiples of this function to the source term of the linearized equation, the so modified source can be made orthogonal to the approximate kernel, enabling the success of the above scheme, but at the cost of solving the original equation only modulo substitute kernel. For the same purpose the current construction introduces N-dimensional substitute kernel, spanned by functions each of which is supported on a single torus away from the tunnels. (Actually, in this construction we never explicitly identify the approximate kernel, nor do we invoke the h metric employed in [13] for its analysis, but our application of substitute kernel is morally identical.)

A further difficulty concerns the vast disparity in scale between the waist radii of the catenoidal tunnels on the one hand and the much greater spacing between the tori on the other. The norm of the initial surface's second fundamental form grows toward the waists of the catenoids to a value diverging with m from a value bounded uniformly in m on the tori, and the embeddedness of graphical perturbations is most precarious near the waists. For these reasons, as well as to ensure convergence of the iteratively defined global solution, it is necessary to arrange for solutions on the tori to decay toward the catenoidal waists.

All of the catenoids attaching to each of the two outermost tori—the only type of torus appearing in [13]—are equivalent modulo the symmetries, and adjustment of the source term by the substitute kernel suffices to achieve such decay on these catenoids. (Again, our actual approach deviates somewhat from this description, applicable to [13], but just superficially.) However, each of the intermediate tori, $N-2$ in number, attaches to catenoids of precisely two equivalence classes under $\mathcal{G}[k, \ell, m]$, and so the appropriate decay of solutions requires the introduction of another $N-2$ functions, linear combinations of which are added to the source term to arrange decay, a device originating in [7] but unneeded in [13]. In total we arrive at a $(2N-2)$-dimensional *extended substitute kernel*, the sum of the substitute kernel and the span of these additional functions, modulo which subspace we can, for large m, invert the linearized operator.

Thus an infinite-dimensional problem is reduced to a finite-dimensional one. The resolution of this latter problem requires the second type of perturbation and is best understood in terms of a correspondence, which Kapouleas ([7]) calls the *geometric principle*, between the initial geometry and the analytic obstructions that the extended substitute kernel represents. In a few words, elements of the extended substitute kernel can be generated, as components of the initial surface's mean curvature, by certain motions of its building blocks—here catenoids and tori—relative to one another. In accordance with this principle the other type of perturbation is realized by incorporating parameters, one for each dimension of extended substitute kernel, into the definition of the initial surface, whose variation repositions the component tori and catenoids. Thus for each choice of k, ℓ, m, and N we define not one initial surface but a $(2N-2)$-parameter family of initial surfaces. More specifically, two parameters may be associated with each of the $N-1$ classes of catenoids joining pairs of adjacent tori. One set of parameters, $\{\zeta_i\}_{i=1}^{N-1}$, controls the waist radii, while the other set, $\{\xi_i\}_{i=1}^{N-1}$, adjusts the heights of the centers. A degree of rigidity, in the form of *matching conditions*, is maintained to reposition the tori in response to the parameters, and the surface is smoothed using cutoff functions as needed. A single parameter ζ works for [13], since there $N=2$ and the symmetry between the sides of $\mathbb{T}$ forces $\xi=0$.

In the course of the construction it is necessary to solve for the proper parameter values along with the perturbing function. The parameter dependence of the "extended" components of the extended substitute kernel can be directly estimated with accuracy adequate for our purposes. It turns out that these components are primarily generated by *dislocations* resulting from antisymmetric variation in pairs of ξ parameters associated to catenoids adjoining a common torus. The parameter dependence of the substitute kernel itself is more conveniently monitored indirectly, as in [13], via *forces*. On each torus the elements of the approximate kernel, and so of the substitute kernel, may be identified with approximate translations of the torus relative to $\mathbb{T}$. In fact $\mathbb{S}^3$ admits an exact Killing field which, though it does not exactly generate this variation of the torus, does approximate it in the vicinity of a great circle orthogonally intersecting $\mathbb{T}$. The force in the direction of this Killing field through certain neighborhoods of a given torus then serves as an estimate of the projection of the mean curvature onto the approximate kernel and thereby as a proxy for the corresponding component of substitute kernel itself. The *balancing* equations and the analysis of the parameter dependence of the forces here are substantially more complicated than those of [13] but no different in principle.

Finally, estimates for the initial geometry, the linearized equation, the nonlinear terms, and the parameter dependence of the forces and dislocations are applied in conjunction with the Schauder fixed-point theorem to prove our main result, which we state informally now, a more refined version appearing as Theorem 6.50, which makes use of notation developed throughout in the paper.

Theorem 1.1 (Informal statement of the main theorem)**.** *Let $k, \ell \geq 1$, and $N \geq 2$ be given integers. For sufficiently large m there exist both a choice of parameters and a smooth, appropriately symmetric perturbing function such that the resulting surface (as described above) is minimal, invariant under $\mathcal{G}[k,\ell,m]$, and a small perturbation of the corresponding initial surface, so in particular embedded and of genus $k\ell m^2(N-1)+1$.*

Remark 1.2 (The full symmetry group)**.** As described earlier, the catenoidal tunnels joining a pair of adjacent tori in a given initial surface take their centers on geodesic arcs intersecting $\mathbb{T}$ at the sites of a $km \times \ell m$ rectangular lattice invariant under $\mathcal{G}[k,\ell,m]$. Because the minimal surfaces produced by the construction are small perturbations of the initial surfaces, the symmetry group of each resulting minimal surface—that is the subgroup of $O(4)$ preserving it as a set—must preserve each of these lattices. When $k \neq \ell$, we can therefore conclude that this symmetry group does not merely contain $\mathcal{G}[k,\ell,m]$ but coincides with it. On the other hand, when $k = \ell$, there are additional isometries of $\mathbb{S}^3$, not belonging to $\mathcal{G}[k,\ell,m]$ (namely vertical ones, exchanging the sides of $\mathbb{T}$), that preserve each lattice, which one could easily enforce in the construction to obtain minimal surfaces enjoying these extra symmetries as well. To avoid complicating the presentation, however, we do not carry out these details. Without making such modifications it is not immediately clear whether or not the minimal surfaces resulting from our construction in the $k = \ell$ case necessarily possess vertical symmetries; to prove they do it would suffice to establish uniqueness of the fixed point in the proof of Theorem 6.50.

Remark 1.3 (Choice of lattices)**.** We have also already mentioned that there are in fact four $km \times \ell m$ rectangular lattices on $\mathbb{T}$ invariant under $\mathcal{G}[k, \ell, m]$, but the constructions in the present article utilize only two (and of course just one in the special case that $N = 2$) in alternating fashion to distribute the catenoidal tunnels joining each pair of adjacent tori. It would be possible, without incurring any technical difficulties that we do not already confront, to avail ourselves of any of the four lattices when fixing the horizontal locations of the catenoidal annuli (subject only to the obvious constraint that two lattices corresponding to consecutive pairs of tori be distinct). By doing so, for $N > 2$, we could construct a variety of examples not congruent to one another but having the same genus and symmetry group. This same flexibility would also allow us to construct examples with $k = \ell$ and $N > 2$ which indubitably do not enjoy any vertical symmetries. (See Remark 1.2, just above.) To avoid complicating the definitions in this article any further we do not present our construction in this generality. In a more ambitious construction one could even attempt to allow a different (large) number of catenoidal annuli at each layer, but this modification would require a genuinely more elaborate approach.

Outline of the presentation. In Section 2 we define the initial surfaces. In Section 3 we analyze the dependence on the ζ and ξ parameters of the dislocations and vertical forces through various regions. In Section 4 we obtain estimates for the geometry of the initial surfaces. In Section 5 we study the linearized problem. In Section 6 we solve the nonlinear problem, proving the main theorem.

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2. Initial surfaces

In this section we must make a number of preliminary definitions before defining the initial surfaces themselves. We also try to offer some motivation for these definitions. The eager reader may wish to look ahead to (2.30) and the references immediately preceding it, consulting the intervening material only as needed.

We realize $\mathbb{S}^3$ as the unit sphere $\{(z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1\}$ in $\mathbb{C}^2$ and set

$$\mathbb{T} := \left\{(z_1, z_2) : |z_1| = |z_2| = \frac{1}{\sqrt{2}}\right\}, \tag{2.1}$$

the Clifford torus whose axes (as defined in Section 1) are simply the coordinate unit circles $C_1 := \{z_2 = 0\}$ and $C_2 := \{z_1 = 0\}$. We define the covering map

$$\begin{aligned} &\Phi : \mathbb{R} \times \mathbb{R} \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to \mathbb{S}^3 \backslash (C_1 \cup C_2) \text{ by} \\ &\Phi(\mathrm{x}, \mathrm{y}, \mathrm{z}) := \left(e^{i\sqrt{2}\mathrm{x}} \sin\left(\mathrm{z} + \frac{\pi}{4}\right), e^{i\sqrt{2}\mathrm{y}} \cos\left(\mathrm{z} + \frac{\pi}{4}\right)\right), \end{aligned} \tag{2.2}$$

which maps (i) horizontal planes to constant-mean-curvature tori having axes C_1 and C_2, with $\Phi(\{\mathrm{z} = 0\}) = \mathbb{T}$ in particular, (ii) vertical lines to quarter great circles orthogonal to C_1, C_2, and $\mathbb{T}$, (iii) vertical planes of constant x to great hemispheres with equator C_2, (iv) vertical planes of constant y to great hemispheres with equator C_1, and (v) vertical planes of constant $\mathrm{x} \pm \mathrm{y}$ to half Clifford tori through C_1 and C_2, orthogonally intersecting $\mathbb{T}$ along great circles. Writing g_S for the standard round metric on $\mathbb{S}^3$ and g_E for the standard flat metric on $\mathbb{R}^3$, we find

$$\Phi^* g_S = g_E + (\sin 2\mathrm{z}) \left(d\mathrm{x}^2 - d\mathrm{y}^2\right). \tag{2.3}$$

The initial surfaces will be assembled by applying Φ to a stack of horizontal planes connected by staggered catenoidal columns.

Half-catenoids bent to planes. We shall make frequent use of cutoff functions throughout the construction, so we fix now a smooth, nondecreasing $\Psi : \mathbb{R} \to [0, 1]$ with Ψ identically 0 on $(-\infty, -1]$, identically 1 on $[1, \infty)$, and such that $\Psi - \frac{1}{2}$ is odd. We then define, for any $a, b \in \mathbb{R}$, the function $\psi\,[a, b] : \mathbb{R} \to [0, 1]$ by

$$\psi\,[a, b] := \Psi \circ L_{a,b}, \tag{2.4}$$

where $L_{a,b} : \mathbb{R} \to \mathbb{R}$ is the linear function satisfying $L(a) = -3$ and $L(b) = 3$.

We write $B[(x, y), r]$ for the open Euclidean disc in $\mathbb{R}^2$ with radius r and center (x, y). Given $X, Y, \tau > 0$ with $\tau < \min\{X, Y\}$, we set

$$\mathbb{T}_{X,Y,\tau} := ([-X, X] \times [-Y, Y]) \backslash B[(0, 0), \tau], \tag{2.5}$$

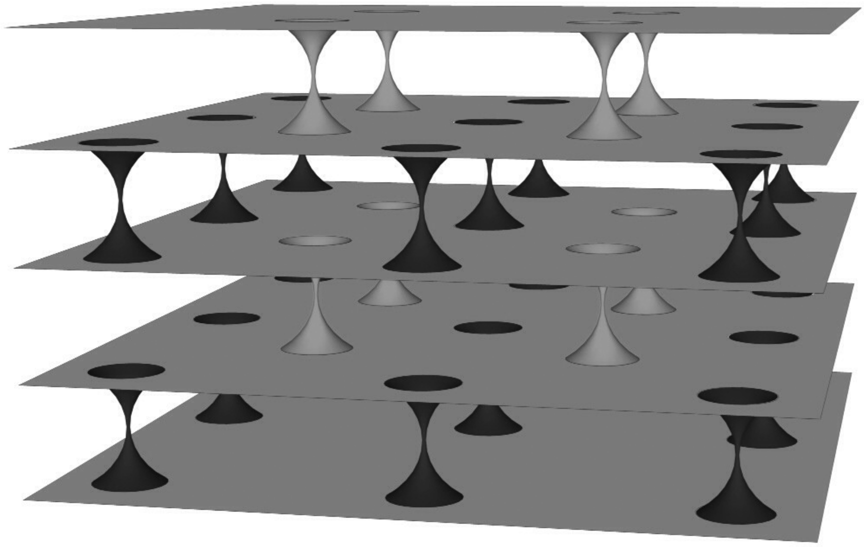

Figure 1. An $N = 5$ initial surface, cut by four hemispheres and viewed in the $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ coordinate system; for clarity the picture grossly exaggerates the ratio, which in fact tends to 0 with large m, of the vertical spacing between the tori to the horizontal spacing between the catenoids.

a solid rectangle with a disc removed from its center. Given also $\mathrm{z}_K, \mathrm{z}_T \in \mathbb{R}$ and $R > \tau$ with $2R < \min\{X, Y\}$, we define the function

$$\begin{aligned} &\phi = \phi[\mathrm{z}_K, \mathrm{z}_T, R, X, Y, \tau] : \mathbb{T}_{X,Y,\tau} \to \mathbb{R} \text{ by} \\ &\phi(\mathrm{x}, \mathrm{y}) := \mathrm{z}_K + (\mathrm{z}_T - \mathrm{z}_K)\psi\,[R, 2R]\left(\sqrt{\mathrm{x}^2 + \mathrm{y}^2}\right) \\ &\quad + \operatorname{sgn}(\mathrm{z}_T - \mathrm{z}_K)\left(\tau \operatorname{arcosh} \frac{\sqrt{\mathrm{x}^2 + \mathrm{y}^2}}{\tau}\right)\psi\,[2R, R]\left(\sqrt{\mathrm{x}^2 + \mathrm{y}^2}\right), \end{aligned} \tag{2.6}$$

where $\operatorname{arcosh} : [1, \infty) \to [0, \infty)$ is the inverse of the restriction to $[0, \infty)$ of the hyperbolic cosine function and the sign function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ takes the value 1 when its argument is nonnegative and the value -1 otherwise. Thus, inside the cylinder of radius R about the z-axis the graph of ϕ coincides with the portion between the planes $\mathrm{z} = \mathrm{z}_K$ and $\mathrm{z} = \mathrm{z}_T$ of the catenoid with vertical axis, center $(0, 0, \mathrm{z}_K)$, and waist radius τ, while outside the cylinder of radius $2R$ about the z-axis it coincides with the solid rectangle $[-X, X] \times [-Y, Y] \times \{\mathrm{z} = \mathrm{z}_T\}$. In between these two cylinders the cutoff function is used to bend the end of the half-catenoid to become exactly horizontal. With the additional data $(\mathrm{x}_0, \mathrm{y}_0) \in \mathbb{R}^2$ we define the embedding

$$\begin{aligned} &T_{ext}[(\mathrm{x}_0, \mathrm{y}_0), \mathrm{z}_K, \mathrm{z}_T, R, X, Y, \tau] : \mathbb{T}_{X,Y,\tau} \to \mathbb{R}^3 \text{ by} \\ &T_{ext}(\mathrm{x}, \mathrm{y}) := (\mathrm{x}_0 + \mathrm{x}, \mathrm{y}_0 + \mathrm{y}, \phi[\mathrm{z}_K, \mathrm{z}_T, R, X, Y, \tau](\mathrm{x}, \mathrm{y})), \end{aligned} \tag{2.7}$$

whose image is the graph of ϕ translated by $(\mathrm{x}_0, \mathrm{y}_0, 0)$.

Given also $\mathrm{z}'_K \in \mathbb{R}$ along with $\tau' > 0$, now assuming $\max\{\tau, \tau'\} < R < \frac{1}{4}\min\{X, Y\}$, we set

$$\begin{aligned} \mathbb{T}_{X,Y,\tau,\tau'} := ([-X, X] \times [-Y, Y]) \Big\backslash \Bigg(& B\left[\left(-\frac{X}{2}, -\frac{Y}{2}\right), \tau\right] \\ & \cup B\left[\left(\frac{X}{2}, \frac{Y}{2}\right), \tau'\right]\Bigg), \end{aligned} \tag{2.8}$$

and we define the embedding

$$\begin{aligned} &T_{int} = T_{int}[(\mathrm{x}_0, \mathrm{y}_0), \mathrm{z}_K, \mathrm{z}'_K, \mathrm{z}_T, R, X, Y, \tau, \tau'] : \mathbb{T}_{X,Y,\tau,\tau'} \to \mathbb{R}^3 \text{ by} \\ &T_{int}(\mathrm{x}, \mathrm{y}) := (\mathrm{x}_0 + \mathrm{x}, \mathrm{y}_0 + \mathrm{y}, \mathrm{z}(\mathrm{x}, \mathrm{y}),), \text{ where} \\ &\mathrm{z}(\mathrm{x}, \mathrm{y}) := \\ &\begin{cases} \phi\left[\mathrm{z}_K, \mathrm{z}_T, R, \frac{X}{2}, \frac{Y}{2}, \tau\right]\left(\mathrm{x} + \frac{X}{2}, \mathrm{y} + \frac{Y}{2}\right) \text{ on } [-X, 0] \times [-Y, 0] \\ \phi\left[\mathrm{z}'_K, \mathrm{z}_T, R, \frac{X}{2}, \frac{Y}{2}, \tau'\right]\left(\mathrm{x} - \frac{X}{2}, \mathrm{y} - \frac{Y}{2}\right) \text{ on } [0, X] \times [0, Y] \\ (\mathrm{x}_0 + \mathrm{x}, \mathrm{y}_0 + \mathrm{y}, \mathrm{z}_T) \text{ everywhere else.} \end{cases} \end{aligned} \tag{2.9}$$

Thus the image of T_{int} looks like a solid $2X \times 2Y$ rectangle in the $\mathrm{z} = \mathrm{z}_T$ plane with two discs centered at $(\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_T) \pm \left(\frac{X}{2}, \frac{Y}{2}, 0\right)$ replaced by catenoidal annuli terminating on waist circles at heights z'_K and z_K.

The initial surfaces will be built from various applications of T_{ext}, for extreme or outermost tori and adjoining half-catenoids, and of T_{int}, for the intermediate tori and pairs of adjoining half-catenoids. The horizontal positions of the catenoids (the values of $(\mathrm{x}_0, \mathrm{y}_0)$ in the parametrizations above) and the dimensions (X and Y) of the parametrizing solid rectangles (or equivalently the lattice edges) will be set directly by the data k, ℓ, and m. The radii of the annuli of transition (determined by R) will be chosen on the order of $\min\{X, Y\}$ (but smaller than it by a wide enough margin that the images of T_{ext} and T_{int} are horizontal near their boundaries as assumed above). There remain $N - 1$ selections of waist radii (τ and τ') and $N - 1$ more of waist heights (z_K and z'_K). *Balancing conditions* studied in the next section allow for the estimation of the values these $2N - 2$ unknowns must assume for the construction to succeed, but their precise specification is made by the ζ and ξ parameters described in Section 1. *Matching conditions* will then fix the height z_T of each torus, by requiring these heights to agree with the heights of the adjoining catenoids where they meet the transition annuli, in the case of the extreme tori, or to agree with the average of the heights of the upper and lower catenoids at the transition circles they adjoin, in the case of the intermediate tori.

The hierarchy of data. For any positive integers k, ℓ, and $N \geq 2$ the first phase of the construction produces a sequence, indexed by m, of $(2N-2)$-parameter families of initial surfaces. In order to obtain adequate estimates for the initial mean curvature, the linearized operator, and the nonlinear terms, we will routinely make the assumption that m is as large as needed in terms of k, ℓ, N, and all of the parameters. Of course we expect the ultimate parameter choices themselves to depend on m, so, in taking m large as just described, it is necessary to assume that the parameters are all bounded in absolute value by a constant $c > 0$ independent of m. Naturally we do not yet know what range is needed for the parameters, but eventually we will be able to pick c in terms of k, ℓ, and N so that for every sufficiently large m we will be able to find parameters bounded by c so that the corresponding initial surface can be perturbed to minimality.

To continue with the definition of the intial surfaces we assume we are given integers $k, \ell, m \geq 1$ and $N \geq 2$ as well as a constant $c > 0$ and parameters $\zeta, \xi \in [-c, c]^{N-1}$. For notational simplicity we assume

$$k \leq \ell \tag{2.10}$$

and we write $n = n[N]$ for the greatest integer no greater than $N/2$, so that

$$N = \begin{cases} 2n \text{ when } N \text{ is even} \\ 2n+1 \text{ when } N \text{ is odd.} \end{cases} \tag{2.11}$$

We acknowledge a certain redundancy in the minimal surfaces ultimately exhibited, one which is easily removed by taking k and ℓ relatively prime.

Catenoidal radii and vertical specifications. Modulo the symmetries that we will impose when defining our initial surfaces we have $N-1$ catenoidal waist radii $\tau_1, \ldots, \tau_{N-1}$ to prescribe, one for each pair of adjacent tori to be joined. Their selection is critical to the success of the construction, and the next section (Section 3) is devoted in part to making a viable choice. Specifically, in Lemma 3.18, when $N \geq 4$, we will determine a collection $\{b_j\}_{j=2}^{n}$ of $n-1$ positive real numbers (recalling (2.11) just above), which in turn we will use in conjunction with the ζ parameters to set the waist radii of our catenoidal tunnels. Each b_j will be a function of k, ℓ, N, and m but (for large m) will have an upper bound and a positive lower bound depending only on N; in particular b_2 will have an upper bound depending on N but always strictly less than 2. Of course in the simpler cases $N = 2$ and $N = 3$ we have $n - 1 = 0$, but it is nevertheless notationally convenient to set $b_2 := 0$ when $N = 2$ and $b_2 := 1$ when $N = 3$. We emphasize that each $b_j = b_j[N, k, \ell, m]$ depends at least on N and generally on the data k, ℓ, and m as well, but for brevity in our notation we will frequently suppress the expression of this dependence, as we do for many other quantities of interest. To summarize:

$$\begin{aligned}&b_2[N=2,k,\ell,m]:=0, \quad b_2[N=3,k,\ell,m]:=1, \quad \text{and}\\ &\{b_i[N\geq 4,k,\ell,m]\}_{i=2}^n \text{ is determined in Lemma 3.18,}\\ &\text{so there exists a constant } C[N,k,\ell]>0 \text{ such that}\\ &1<\max\{b_i\}_{i=2}^n<C[N,k,\ell] \quad \text{and} \quad b_2\leq 2-1/C[N,k,\ell]\\ &\text{whenever } m \text{ is large enough in terms of } N\geq 4.\end{aligned} \tag{2.12}$$

Having identified these numbers, we define the collection $\{\underline{\tau}_i[N,k,\ell,m]\}_{i=1}^{N-1}$ of waist radii when $\zeta=0$ by

$$\underline{\tau}_i := \begin{cases} \frac{1}{10\ell m} e^{-\frac{k\ell m^2}{4\pi}\left(1-\frac{1}{2}b_2[N,k,\ell,m]\right)} \text{ for } i=1 \\ b_i[N,k,\ell,m]\underline{\tau}_1[N,k,\ell,m] \text{ for } 2\leq i\leq n \\ \underline{\tau}_{N-i}[N,k,\ell,m] \text{ for } n+1\leq i\leq N-1 \end{cases} \tag{2.13}$$

(recalling (2.11)) and then for general ζ we define the radii $\tau_i=\tau_i[N,k,\ell,m,\zeta]$ by

$$\tau_i := \begin{cases} e^{\zeta_1}\underline{\tau}_1[N,k,\ell,m] \text{ for } i=1 \\ e^{\zeta_1+k^{-1}\ell^{-1}m^{-2}\zeta_i}\underline{\tau}_i[N,k,\ell,m] \text{ for } 1<i<N. \end{cases} \tag{2.14}$$

We next define the N heights $\mathrm{z}_i=\mathrm{z}_i[N,k,\ell,m,\zeta,\xi]$ of the tori by

$$\begin{aligned} \mathrm{z}_1 &:= \tau_1\xi_1 - 2^{N \bmod 2}\tau_n \ln\frac{1}{10\ell m\tau_n} - 2\sum_{j=1}^{n-1}\tau_j\ln\frac{1}{10\ell m\tau_j}, \\ \mathrm{z}_N &:= \tau_{N-1}\xi_{N-1} - 2^{N \bmod 2}\tau_n \ln\frac{1}{10\ell m\tau_n} + 2\sum_{j=n}^{N-1}\tau_j\ln\frac{1}{10\ell m\tau_j}, \\ \mathrm{z}_i &:= \frac{\tau_{i-1}\xi_{i-1}+\tau_i\xi_i}{2} - 2^{N \bmod 2}\tau_n\ln\frac{1}{10\ell m\tau_n} \\ &\quad + 2\sum_{j=1}^{i-1}\tau_j\ln\frac{1}{10\ell m\tau_j} - 2\sum_{j=1}^{n-1}\tau_j\ln\frac{1}{10\ell m\tau_j} \text{ for } 1<i<N, \end{aligned} \tag{2.15}$$

and the $N-1$ heights $\mathrm{z}_i^K=\mathrm{z}_i^K[N,k,\ell,m,\zeta,\xi]$ of the catenoids' centers by

$$\begin{aligned} \mathrm{z}_i^K := &\tau_i\xi_i + \tau_i\ln\frac{1}{10\ell m\tau_i} + 2\sum_{j=1}^{i-1}\tau_j\ln\frac{1}{10\ell m\tau_j} \\ &- 2\sum_{j=1}^{n-1}\tau_j\ln\frac{1}{10\ell m\tau_j} - 2^{N \bmod 2}\tau_n\ln\frac{1}{10\ell m\tau_n} \end{aligned} \tag{2.16}$$

for $1\leq i\leq N-1$.

Equivalently (suppressing from the notation in each of the below equations pieces of data which agree on the two sides and may otherwise be freely chosen) we have

$$\begin{aligned}
&\mathrm{z}_i^K[\xi] = \mathrm{z}_i^K[\xi=0] + \tau_i\xi_i \text{ for } 1 \leq i \leq N-1,\\
&\mathrm{z}_i[\xi] = \mathrm{z}_i[\xi=0] + \begin{cases} \tau_1\xi_1 \text{ for } i=1 \\ \tau_{N-1}\xi_{N-1} \text{ for } i=N \\ \frac{1}{2}\left(\tau_{i-1}\xi_{i-1}+\tau_i\xi_i\right) \text{ for } 1<i<N, \end{cases}\\
&\mathrm{z}_i^K[\xi=0] = \mathrm{z}_i[\xi=0] + \tau_i \ln\frac{1}{10\ell m\tau_i} \text{ for } 1 \leq i \leq N-1,\\
&\mathrm{z}_{i+1}[\xi=0] = \mathrm{z}_i[\xi=0] + 2\tau_i \ln\frac{1}{10\ell m\tau_i} \text{ for } 1 \leq i \leq N-1, \text{ and}\\
&\mathrm{z}_n^K[N=2n,\xi=0] = \mathrm{z}_{n+1}[N=2n+1,\xi=0] = 0,
\end{aligned} \tag{2.17}$$

recalling (2.11) in the last equation.

From definition (2.14) we have for $1 \leq i \leq N$

$$\ln\frac{1}{10\ell m\tau_i} = \frac{k\ell}{4\pi}\left(1-\frac{b_2}{2}\right)m^2 - \zeta_1 - (1-\delta_{i1})\left(\ln b_i + \frac{\zeta_i}{k\ell m^2}\right), \tag{2.18}$$

where $\delta_{i1} = \left\{\begin{smallmatrix} 1 \text{ if } i=1 \\ 0 \text{ otherwise,}\end{smallmatrix}\right.$ so by taking m sufficiently large in terms of ζ, ξ, and each b_i (which according to (2.12) satisfy bounds depending on just N, k, and ℓ), we can make every z_i and z_i^K as close to 0 as desired and we can also guarantee that

$$\mathrm{z}_i[\xi] < \mathrm{z}_i^K[\xi=0] < \mathrm{z}_{i+1}[\xi] \text{ for } 1 \leq i \leq N-1. \tag{2.19}$$

The definitions (2.15) and (2.16) can be understood as implementing the matching conditions mentioned earlier as well as the vertical offsets introduced by the ξ parameters. Here the logarithm is used—merely because it simplifies some expressions later—to capture the dominant part of the inverse hyperbolic cosine function for large values of its argument. Thus each logarithmic term, ignoring any powers of 2 appearing as prefactors, represents the height achieved by a corresponding catenoid above its waist plane a distance $\frac{1}{10\ell m}$ from its axis, where the catenoids are meant to transition to planes (tori under Φ). The factor of 10ℓ is chosen—10 somewhat arbitrarily and ℓ because we assume $k \leq \ell$—to ensure the transition is completed on the order—m^{-1}—of the lattice spacing but well away from neighboring catenoids. From the identity

$$\operatorname{arcosh} x = \ln x + \ln\left(1+\sqrt{1-x^{-2}}\right) \tag{2.20}$$

we have the estimate

$$\left|\tau_i \operatorname{arcosh}\frac{1}{10\ell m\tau_i} - \tau_i\ln\frac{1}{10\ell m\tau_i}\right| \leq \tau_i\ln 2, \tag{2.21}$$

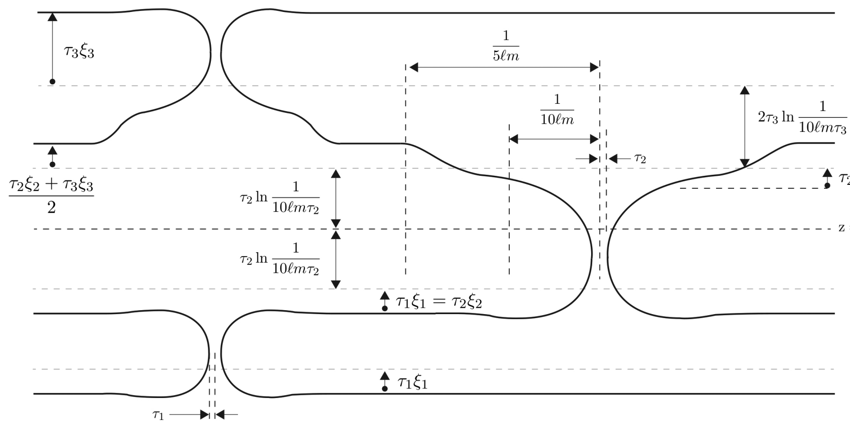

Figure 2. A schematic profile of an $N = 4$ initial surface. The surface depicted has $\tau_1\xi_1 = \tau_2\xi_2 \neq \tau_3\xi_3$, resulting in dislocation at the third torus from the bottom and nowhere else. The figure is not drawn to scale. In fact (see especially (2.14) and (2.18)) the vertical spacing $\tau_i \ln \frac{1}{10\ell m \tau_i}$ is of order $m^2\tau_1$ and $\lim_{m\to\infty} \frac{m^2\tau_1}{m^{-1}} = 0$. Note also that the schematic does not attempt to convey the height mismatch, of order τ_1, arising from the approximation of arcosh by ln (see (2.21)), because, unlike the dislocations controlled by ξ, it plays no role in the construction.

so our very substitution of ln for arcosh introduces additional height mismatch (beyond that contributed by the ξ parameters) where catenoidal annuli connect to tori, but one whose ratio to τ_1 is obviously bounded independently of the ζ and ξ parameters.

Symmetries and horizontal specifications. By the definition (2.1) of $\mathbb{T}$ as the locus of points in $\mathbb{S}^3$ equidistant from C_1 and C_2, the symmetry group $\mathcal{G}_{sym}[\mathbb{T}]$ of $\mathbb{T}$ in $\mathbb{S}^3$, that is the subgroup of $O(4)$ preserving $\mathbb{T}$ as a set, is precisely the symmetry group of $C_1 \cup C_2$. Accordingly $\mathcal{G}_{sym}[\mathbb{T}]$ is generated by the set of all reflections through great spheres containing either C_1 or C_2 together with the set of reflections through any great circle equidistant from C_1 and C_2 (so contained in a plane of the form $\{z_1 = e^{i\theta} z_2\}$ or $\{z_1 = e^{i\theta} \overline{z}_2\}$) and therefore lying on $\mathbb{T}$. (Here reflection through a great sphere (or circle) refers to the element of $O(4)$ that identically preserves the 3-space (or 2-plane) containing that sphere (or circle) and reflects the orthogonal complement through the origin.) Consequently $\mathcal{G}_{sym}[\mathbb{T}]$ also includes rotation in C_1 and C_2 by any angles as well as reflection through any great circle orthogonally intersecting $\mathbb{T}$ (and C_1 and C_2).

We equip $\mathbb{C}^2$, C_1, and C_2 with their standard orientations and, given an oriented circle C in $\mathbb{S}^3$, we write R_C^θ for rotation by θ about C, which by definition is the element of $O(4)$ fixing the plane containing C pointwise and rotating its orthogonal complement by angle θ in the direction consistent with the induced orientation. We write $\underline{\mathsf{X}}$ for reflection through the great sphere with equator C_2 and poles $(\pm 1, 0)$, $\underline{\mathsf{Y}}$ for reflection through the great sphere with equator C_1 and poles $(0, \pm 1)$, and $\underline{\mathsf{Z}}$ for reflection through the great circle in the plane $\{z_1 = z_2\}$ (so $\underline{\mathsf{Z}} = \mathsf{R}^\pi_{\{z_1=z_2\}\cap\mathbb{S}^3}$). Thus (writing $\overline{z}$ for the complex conjugate of $z \in \mathbb{C}$)

$$\begin{aligned}&\underline{\mathsf{X}}(z_1, z_2) = (\overline{z_1}, z_2), \quad \underline{\mathsf{Y}}(z_1, z_2) = (z_1, \overline{z_2}), \quad \underline{\mathsf{Z}}(z_1, z_2) = (z_2, z_1),\\ &\mathsf{R}^\theta_{C_2}(z_1, z_2) = (e^{i\theta} z_1, z_2), \quad \text{and} \quad \mathsf{R}^\theta_{C_1}(z_1, z_2) = (z_1, e^{i\theta} z_2)\end{aligned} \tag{2.22}$$

and

$$\mathcal{G}_{sym}[\mathbb{T}] = \left\langle \underline{\mathsf{X}}, \underline{\mathsf{Y}}, \underline{\mathsf{Z}}, \mathsf{R}^{\theta_1}_{C_1}, \mathsf{R}^{\theta_2}_{C_2} \; : \; \theta_1, \theta_2 \in \mathbb{R} \right\rangle, \tag{2.23}$$

where the right-hand side is the subgroup of $O(4)$ generated by the elements the angled brackets enclose.

Recalling (2.2), we also name some elements of $O(3)$ preserving the domain of Φ and the pullback metric $\Phi^* g_S$: we write $\widehat{\mathsf{T}}^h_{\text{x-axis}}$ and $\widehat{\mathsf{T}}^h_{\text{y-axis}}$ for translation by (real number) h in the (positive) x and y directions respectively, $\widehat{\underline{\mathsf{X}}}$ and $\widehat{\underline{\mathsf{Y}}}$ for reflection through the $\mathrm{x} = 0$ and $\mathrm{y} = 0$ planes respectively, and $\widehat{\underline{\mathsf{Z}}}$ for reflection through the line $\mathrm{x} = \mathrm{y}$ in the plane $\mathrm{z} = 0$. Thus

$$\begin{aligned}&\widehat{\mathsf{T}}^h_{\text{x-axis}}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x} + h, \mathrm{y}, \mathrm{z}), \quad \widehat{\mathsf{T}}^h_{\text{y-axis}}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x}, \mathrm{y} + h, \mathrm{z})\\ &\widehat{\underline{\mathsf{X}}}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (-\mathrm{x}, \mathrm{y}, \mathrm{z}), \quad \widehat{\underline{\mathsf{Y}}}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x}, -\mathrm{y}, \mathrm{z}), \quad \text{and}\\ &\widehat{\underline{\mathsf{Z}}}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{y}, \mathrm{x}, -\mathrm{z}).\end{aligned} \tag{2.24}$$

It is then easy to verify the relations

$$\begin{aligned}&\Phi \circ \widehat{\underline{\mathsf{X}}} = \underline{\mathsf{X}} \circ \Phi, \quad \Phi \circ \widehat{\underline{\mathsf{Y}}} = \underline{\mathsf{Y}} \circ \Phi, \quad \Phi \circ \widehat{\underline{\mathsf{Z}}} = \underline{\mathsf{Z}} \circ \Phi,\\ &\Phi \circ \widehat{\mathsf{T}}^h_{\text{x-axis}} = \mathsf{R}^{\sqrt{2}h}_{C_2} \circ \Phi, \quad \text{and} \quad \Phi \circ \widehat{\mathsf{T}}^h_{\text{y-axis}} = \mathsf{R}^{\sqrt{2}h}_{C_1} \circ \Phi \quad \forall h \in \mathbb{R},\end{aligned} \tag{2.25}$$

which play an essential role in identifying the symmetries of the initial surfaces, defined via Φ, and in exploiting these symmetries when solving the linearized problem.

Each initial surface will be built to be invariant under a certain subgroup of $\mathcal{G}_{sym}[T]$, with the symmetries of $\mathbb{T}$ broken in several ways. Recall that each initial surface is to be obtained by gluing together a collection of constant-mean-curvature tori coaxial with $\mathbb{T}$ using a collection of catenoidal annuli. Since among these tori only $\mathbb{T}$ itself is equidistant from C_1 and C_2, the symmetry group of every other such torus is just the subgroup $\left\langle \underline{\mathsf{X}}, \underline{\mathsf{Y}}, \mathsf{R}^{\theta_1}_{C_1}, \mathsf{R}^{\theta_2}_{C_2} \; : \; \theta_1, \theta_2 \in \mathbb{R} \right\rangle$ of horizontal symmetries of $\mathbb{T}$. If the toral components of an initial surface are not arranged symmetrically about $\mathbb{T}$, then of course the initial surface itself cannot possess any vertical symmetries. The arrangement of the catenoidal tunnels may also be inconsistent with vertical symmetry and inevitably breaks the continuous horizontal symmetries, leaving only a discrete subgroup. In this construction we impose the group

$$\mathcal{G} = \mathcal{G}[k,\ell,m] := \left\langle \mathsf{R}_{C_2}^{\frac{2\pi}{km}}, \mathsf{R}_{C_1}^{\frac{2\pi}{\ell m}}, \underline{\mathsf{X}}, \underline{\mathsf{Y}} \right\rangle < \mathcal{G}_{sym}[\mathbb{T}] < O(4), \tag{2.26}$$

which is the subgroup of $O(4)$ preserving the set of km^{th} roots of unity on C_1 as well as (separately) the set of ℓm^{th} roots of unity on C_2; as such, $\mathcal{G}[k,\ell,m]$ is therefore isomorphic to $D_{km} \times D_{\ell m}$, where D_q is the dihedral group of order $2q$. We will sometimes write $\mathcal{G}$ in place of $\mathcal{G}[k,\ell,m]$ when there is no danger of confusion.

The catenoidal tunnels connecting each pair of adjacent tori will be placed, via Φ, so as to take their centers on the great circles orthogonally intersecting $\mathbb{T}$ at the sites of a $\mathcal{G}[k,\ell,m]$-invariant $km \times \ell m$ rectangular lattice. There are precisely four such lattices, namely the $\mathcal{G}[k,\ell,m]$ orbits

$$\begin{aligned} L_{\sigma_{\mathrm{x}},\sigma_{\mathrm{y}}} &= L_{\sigma_{\mathrm{x}},\sigma_{\mathrm{y}}}[k,\ell,m] := \mathcal{G}[k,\ell,m]\left(e^{i\sigma_{\mathrm{x}}\pi/(km)}, e^{i\sigma_{\mathrm{y}}\pi/(\ell m)}\right), \\ &= \Phi\left(\widehat{L}_{\sigma_{\mathrm{x}},\sigma_{\mathrm{y}}} \times \{0\}\right), \text{ where} \\ \widehat{L}_{\sigma_{\mathrm{x}},\sigma_{\mathrm{y}}} &= \widehat{L}_{\sigma_{\mathrm{x}},\sigma_{\mathrm{y}}}[k,\ell,m] \subset \mathbb{R}^2 \\ &:= \left\{\left(\frac{\sigma_{\mathrm{x}}\pi}{\sqrt{2}km} + n_{\mathrm{x}}\frac{\sqrt{2}\pi}{km}, \frac{\sigma_{\mathrm{y}}\pi}{\sqrt{2}\ell m} + n_{\mathrm{y}}\frac{\sqrt{2}\pi}{\ell m}\right)\right\}_{n_{\mathrm{x}},n_{\mathrm{y}}\in\mathbb{Z}}, \end{aligned} \tag{2.27}$$

corresponding to the four choices of $\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}} \in \{0,1\}$. To avoid further complicating the definition of the initial surfaces in this article we make use of just the two lattices $L_{0,0}$ and $L_{1,1}$, but we emphasize that only obvious, minor modifications to our procedure would be required to take advantage of the other two lattices as well, which freedom would allow us to produce minimal surfaces not congruent to the ones we explicitly construct here. (It would also be natural to attempt to adapt the construction to permit more generally $\mathcal{G}[k,\ell,m]$-invariant $n_i km \times n_i \ell m$ lattices refining the above four, where the positive integer n_i could be allowed to vary from layer to layer.) See Remark 1.3.

Each initial surface will be invariant under the corresponding $\mathcal{G}[k,\ell,m]$, and we will later admit only deformations respecting this group, so in fact each minimal surface ultimately produced will also be invariant under the corresponding $\mathcal{G}[k,\ell,m]$. When $k \neq \ell$, $\mathcal{G}[k,\ell,m]$ is in fact the largest group preserving each lattice and the largest group preserving the set of centers of the catenoidal tunnels, and it will consequently be the full symmetry group of the resulting minimal surface. When $k = \ell$, however, there are choices of parameters ζ, ξ such that the corresponding initial surface is also preserved by reflection through certain great circles on $\mathbb{T}$. In this case it would be possible to modify the procedure that follows by cutting in half the number of free parameters, imposing constraints respecting these additional symmetries; the construction would then produce minimal surfaces also invariant under the extra symmetries. We will not pursue this modification here, instead enforcing just the smaller group $\mathcal{G}[k,\ell,m]$ even in the square case. From the analysis of this article alone it is unclear whether or not the resulting minimal surfaces nevertheless enjoy the additional symmetries. See Remark 1.2.

Assembly and basic properties. Suppose we are given the following data as above: integers $N \geq 2$, $\ell \geq k \geq 1$, and $m \geq 1$, as well as two vectors $\zeta, \xi \in \mathbb{R}^{N-1}$. We set $b_2 := 0$ if $N = 2$ and $b_2 := 1$ if $N = 3$, and for $N \geq 4$ we accept a collection $\{b_j[N,k,\ell,m]\}_{j=2}^n \subset (1,\infty)$ as described in Lemma 3.18 (namely the one determined in its proof). The set of waist radii $\{\tau_j\}_{j=1}^{N-1}$ is then defined by (2.14), the set of toral heights $\{\mathrm{z}_j\}_{j=1}^N$ by (2.15), and the set of catenoidal heights $\{\mathrm{z}_j^K\}_{j=1}^{N-1}$ by (2.16). Setting

$$X = X[k,\ell,m] := \frac{\pi}{\sqrt{2}km}, \quad Y = Y[k,\ell,m] := \frac{\pi}{\sqrt{2}\ell m}, \quad \text{and} \tag{2.28}$$
$$R = R[\ell,m] := \frac{1}{10\ell m},$$

and recalling (2.2), (2.7), and (2.9), for each integer $1 \leq i \leq N$ we define the parametrized surface-with-boundary Ω_i in $\mathbb{S}^3$ by

$$\begin{aligned}
&\Omega_i = \Phi\left(\widehat{\Omega}_i\right) \quad (\text{for } 1 \leq i \leq N), \quad \text{where}\\
&\widehat{\Omega}_1 := T_{ext}[(0,0), \mathrm{z}_1^K, \mathrm{z}_1, R, X, Y, \tau_1](\mathbb{T}_{X,Y,\tau_1}),\\
&\widehat{\Omega}_N := T_{ext}[(N-2)(X,Y), \mathrm{z}_{N-1}^K, \mathrm{z}_N, R, X, Y, \tau_{N-1}](\mathbb{T}_{X,Y,\tau_{N-1}}),\\
&\text{and for } 1 < i < N\\
&\widehat{\Omega}_i := T_{int}\left[(2i-3)\left(\frac{X}{2},\frac{Y}{2}\right), \mathrm{z}_{i-1}^K, \mathrm{z}_i^K, \mathrm{z}_i, R, X, Y, \tau_{i-1}, \tau_i\right]\\
&\qquad\qquad (\mathbb{T}_{X,Y,\tau_{i-1},\tau_i}).
\end{aligned} \tag{2.29}$$

Note that each Ω_i is a $\frac{\sqrt{2}\pi}{km} \times \frac{\sqrt{2}\pi}{\ell m}$ rectangular patch of the constant-mean-curvature torus at signed distance z_i (increasing toward C_1) from $\mathbb{T}$, within which patch one (for $i = 1$ or $i = N$) or two (for $1 < i < N$) discs have been replaced (via Φ and a cutoff function) by catenoidal annuli, so that the boundary of Ω_i is the union of a rectangle with one or two waist cicles, the nearest-point projection onto $\mathbb{T}$ of the center of each deleted disc is a site of either $L_{0,0}$ or $L_{1,1}$ (recalling (2.27)), and $\bigcup_{i=1}^N \Omega_i$ is a smooth connected surface whose boundary is the union of N rectangles.

Finally we define the initial surface

$$\Sigma = \Sigma[N,k,\ell,m,\zeta,\xi] := \mathcal{G}[k,\ell,m] \bigcup_{i=1}^{N} \Omega_i, \tag{2.30}$$

the orbit under $\mathcal{G}[k,\ell,m]$ of $\bigcup_{i=1}^{N} \Omega_i$.

Proposition 2.31 (Basic properties of the initial surfaces)**.** *Given a real number $c > 0$ and integers $N \geq 2$ and $\ell \geq k \geq 1$, there exists $m_0 = m_0[N,k,\ell,c] > 0$ such that for every integer $m \geq m_0$ and every choice of parameters $\zeta, \xi \in [-c,c]^{N-1}$ the initial surface $\Sigma[N,k,\ell,m,\zeta,\xi]$ (defined by* (2.30)*) is a smooth, closed surface, embedded in $\mathbb{S}^3$, of genus $k\ell m^2(N-1)+1$, and invariant as a set under the action of $\mathcal{G}[k,\ell,m]$ (defined in* (2.26)*).*

Proof. By (2.12), (2.14), and (2.18) for fixed N, k, ℓ, and c, we have

$$\lim_{m\to\infty} \left(\frac{|c|}{m} + \sum_{i=1}^{N-1} 10\ell m \tau_i + \sum_{i=1}^{N-1} \tau_i \ln \frac{1}{10\ell m \tau_i} \right) = 0, \tag{2.32}$$

ensuring embeddedness of $\bigcup_{i=1}^{N} \Omega_i$. All the claims are now clear from (2.25), (2.26), and the explicit construction of Σ. q.e.d.

Remark 2.33 (Smooth dependence on the parameters)**.** Since for any fixed N, k, ℓ, and m, the quantities (2.14), (2.15), and (2.16) all depend smoothly on the ζ, ξ parameters, it follows from (2.30) and the supporting definitions, particularly (2.7) and (2.9), that the initial surface $\Sigma[N,k,\ell,m,\zeta,\xi]$ depends smoothly on ζ and ξ in the sense that there exists a smooth map $I = I[N,k,\ell,m] : \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \times \Sigma[N,k,\ell,m,0,0] \to \mathbb{S}^3$ such that for any $\zeta, \xi \in \mathbb{R}^{N-1}$ the map $I[N,k,\ell,m](\zeta,\xi,\cdot)$ is an embedding (provided m is large enough in terms of $|\zeta|$ and $|\xi|$) with image $\Sigma[N,k,\ell,m,\zeta,\xi]$. In particular $I(\zeta,\xi,\cdot)$ is a diffeomorphism onto its image, and, in casual abuse of notation, we will routinely write $I(\zeta,\xi,\cdot)$ for this diffeomorphism, so that $I[N,k,\ell,m](\zeta,\xi,\cdot)^{-1} : \Sigma[N,k,\ell,m,\zeta,\xi] \to \Sigma[N,k,\ell,m,0,0]$ is the inverse diffeomorphism.

Of course it remains to specify $\{b_j\}_{j=2}^{n} \subset (1,\infty)$ when $N \geq 4$, a gap filled in the next section.

3. Forces and dislocations

Forces. As mentioned in Section 1, we will eventually discover (in Section 5) that on each toral region (not yet defined), as m tends to infinity, the Jacobi operator converges, after appropriate rescaling, to a limit operator on a corresponding limit region (likewise after rescaling), and this limit operator has one-dimensional kernel. The nontriviality of this kernel will compel us, when attempting to prescribe mean curvature at the linear level by graphical deformation, to modify the term to be prescribed by elements of the substitute kernel (as outlined in Section 1 and formally defined in Section 5). The details appear in Section 5, where the precise argument is couched in rather different terms, but, in order to help motivate the study undertaken in the current section, we offer in this paragraph the following rough sketch of the situation from a purely geometric point of view. In the next section (specifically (4.73)) we will define the i^{th} toral region $\mathcal{T}[i]$ to be Ω_i (2.29) less a certain portion of the half catenoid(s) attached. Of course the area of Ω_i (and so of $\mathcal{T}[i]$ too) shrinks to 0 as m goes to infinity, but if we scale the ambient spherical metric g_S up to $m^2 g_S$, then under the corresponding induced metric $\mathcal{T}[i]$ tends to a flat $\frac{\sqrt{2}\pi}{k} \times \frac{\sqrt{2}\pi}{\ell}$ solid rectangle (because the annuli deleted from Ω_i to define $\mathcal{T}[i]$ are sized so as to vanish in this limit). The corresponding limit Jacobi operator is just the flat Laplacian on this rectangle, its Jacobi operator as a submanifold of Euclidean $\mathbb{R}^3$ (viewed in the domain of the map Φ (2.2)). By enforcing the symmetry group $\mathcal{G}$ (2.26) throughout the construction we impose periodic boundary conditions on $\mathcal{T}[i]$, so that this limit Jacobi operator has kernel spanned by the constants. The Jacobi field 1 is induced by the Euclidean Killing field ∂_{z} (viewed in the domain of Φ) orthogonal to the rectangle.

Of course the vector field ∂_{z} is not Killing relative to $\Phi^* g_S$ (2.3), but it is approximately Killing relative to $m^2 \Phi^* g_S$, and, conveniently, on a neighborhood of Ω_i in $\mathbb{S}^3$, (viewed through Φ) ∂_{z} is itself approximated by the exact Killing field K on $(\mathbb{S}^3, g_S)$ that generates rotation, toward C_1 (defined just below (2.1)), along the great circle through the points $(1,0), (0,1) \in \mathbb{C}^2$. The geodesic segment joining these points is simply the closure of the image under Φ of the segment of the z-axis contained in the domain of Φ; thus $K \circ \Phi = \Phi_* \partial_{\mathrm{z}}$ along this segment. More generally (recalling (2.2)),

$$
\begin{aligned}
K \circ \Phi = & -\frac{1}{\sqrt{2}} \cot\left(\mathrm{z} + \frac{\pi}{4}\right) \sin \sqrt{2}\mathrm{x} \, \cos \sqrt{2}\mathrm{y} \; \Phi_* \partial_{\mathrm{x}} \\
& + \frac{1}{\sqrt{2}} \tan\left(\mathrm{z} + \frac{\pi}{4}\right) \cos \sqrt{2}\mathrm{x} \, \sin \sqrt{2}\mathrm{y} \; \Phi_* \partial_{\mathrm{y}} \\
& + \cos \sqrt{2}\mathrm{x} \, \cos \sqrt{2}\mathrm{y} \; \Phi_* \partial_{\mathrm{z}}.
\end{aligned} \tag{3.1}
$$

We will now calculate the K force (also called *flux* in the literature) through various regions of the initial surface and to study its dependence on the ζ, ξ parameters. These forces will indirectly measure the projection of the initial surface's mean curvature onto the substitute kernel, so in the final section we will apply the results of this section (specifically Lemma 3.33) to help manage the substitute kernel and complete the proof of the main theorem. More immediately we will impose *balancing conditions* ([17], [6], [8]) on the initial surface, such that the K force on various regions vanishes, at least within a margin on the order of the perturbations—by functions and parameters—that we will be making. This balancing will finally determine the waist radii, up to choice of ζ, thus completing the definition of the initial surfaces.

Let

$$\begin{aligned} \mathcal{F}_i = \mathcal{F}_i[N,k,\ell,m,\zeta,\xi] &:= \int_{\partial\Omega_i} (g_S \circ \iota)(K \circ \iota, \eta_i)\sqrt{|\iota^* g_S|_{\partial\Omega_i}|} \\ &= \int_{\Omega_i} (g_S \circ \iota)(K \circ \iota, \mathbf{H})\sqrt{|\iota^* g_S|}, \end{aligned} \tag{3.2}$$

the K *force* exerted by the region Ω_i (defined by (2.29)) on the rest of Σ, where $\iota : \Sigma \to \mathbb{S}^3$ is the inclusion map of Σ in $\mathbb{S}^3$, η_i is the outward conormal for Ω_i, $\mathbf{H} := \operatorname{tr}_{\iota^* g_S} D d\iota$ is the vector-valued mean curvature of Σ (D being the connection induced on $T^*\Sigma \otimes \iota^* T\mathbb{S}^3$ by g_S and ι), $\sqrt{|\iota^* g_S|}$ and $\sqrt{|\iota^* g_S|_{\partial\Omega_i}|}$ are respectively the area and length forms induced by ι and g_S, and the last equality follows trivially from the formula for the first variation of area and the fact that K is Killing. Since the initial surface should be designed to be approximately minimal, by virtue of the second equality we can now impose the approximate (in a sense made precise below) balancing condition

$$\mathcal{F}_i \approx 0 \text{ for each } 1 \leq i \leq N \tag{3.3}$$

in order to estimate the necessary waist radii, before varying the parameters or deforming the initial surface graphically. We will see that this heuristic approach leads to (2.13). We will also analyze how by adjusting the parameters we can fine-tune these radii as well as the heights of the catenoids and tori in order to control the forces.

The computation of the forces is simple. The boundary of each Ω_i consists of a rectangle on a constant-mean-curvature torus coaxial with $\mathbb{T}$ and one (for $i \in \{1, N\}$) or two (for $2 \leq i \leq N-1$) catenoidal waists. For each such component, by working in an $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ coordinate system defined via Φ (2.2), we will estimate the corresponding integral arising in (3.2). Suppose S is a catenoidal waist with center $(\mathrm{x}_i^K, \mathrm{y}_i^K, \mathrm{z}_i^K)$. The height z_i^K has been defined in (2.16), and the horizontal coordinates $\mathrm{x}_i^K, {\mathrm{y}_i}^K$, though not previously named, can be found in (2.29). The outward unit conormal η_S along S is, recalling (2.3), simply $\pm\partial_{\mathrm{z}}$ and, relative to g_E, S is a Euclidean circle of radius τ_i, so by (2.3) and (3.1)

$$\begin{aligned} \int_S g_S(K, \eta_S) = \int_0^{2\pi} &\cos\left[\sqrt{2}\left(\mathrm{x}_i^K + \tau_i \cos\theta\right)\right] \\ &\times \cos\left[\sqrt{2}\left(\mathrm{y}_i^K + \tau_i \sin\theta\right)\right] \\ &\times \tau_i \sqrt{1 - (\sin 2\mathrm{z}_i^K)(\cos 2\theta)}\, d\theta. \end{aligned} \tag{3.4}$$

Now suppose T is a rectangular component of $\partial\Omega_i$ for some admissible i, with outward conormal $\eta_{_T}$ and center $(\mathrm{x}_i, \mathrm{y}_i, \mathrm{z}_i)$. The height z_i has been defined in (2.15), and the horizontal coordinates $(\mathrm{x}_i, \mathrm{y}_i)$ can be found in (2.29). Thus T lies on the constant-mean-curvature torus $\{\mathrm{z} = \mathrm{z}_i\}$ and its complement in this torus has two connected components, the smaller of which (in terms of area) we call $\overline{T}$, a solid rectangle in $\{\mathrm{z} = \mathrm{z}_i\}$ having boundary $\partial\overline{T} = T$. Because of the way Ω_i is defined using cutoff functions, there is a neighborhood $\mathcal{U}$ of T in $\mathbb{S}^3$ such that $\mathcal{U} \cap \Omega_i = \mathcal{U} \cap \overline{T}$, and therefore $\eta_{_T}$ is equally the outward unit conormal for $\overline{T}$ along T. By invoking the first-variation-of-area formula (as in the second equality of (3.2)) we have $\int_T g_{_S}(K, \eta_{_T}) = \int_{\overline{T}} g_{_S}(K, \mathbf{H}_{_T})$, where $\mathbf{H}_{_T}$ is the mean curvature of $\overline{T}$. From (2.3) we find that the area form on $\{\mathrm{z} = \mathrm{z}_i\}$ is $\cos 2\mathrm{z}_i \, d\mathrm{x}\, d\mathrm{y}$, whence $\mathbf{H}_{_T} = 2\tan 2\mathrm{z}_i \partial_{\mathrm{z}}$, so using also (2.29) and (3.1) we get

$$\int_T g_{_S}(K, \eta_{_T}) = \int_{\mathrm{x}_i - \frac{\sqrt{2}\pi}{km}}^{\mathrm{x}_i + \frac{\sqrt{2}\pi}{km}} \int_{\mathrm{y}_i - \frac{\sqrt{2}\pi}{\ell m}}^{\mathrm{y}_i + \frac{\sqrt{2}\pi}{\ell m}} 2\cos(\sqrt{2}\mathrm{x})\cos(\sqrt{2}\mathrm{y})\tan(2\mathrm{z}_i) \times \cos(2\mathrm{z}_i)\, d\mathrm{x}\, d\mathrm{y}. \tag{3.5}$$

To estimate the integrals (3.4) and (3.5) we will make the approximations $\cos u \approx 1$, $\sin u \approx u$, and $\sqrt{1+u} \approx 1 + u/2$. On a heuristic level we could ignore the error in these approximations and proceed with the calculation to motivate (2.13) as advertised. On the other hand, the actual construction will demand more detailed estimates, so, to avoid repeating some calculations, we will instead keep track of the error as we go, predicating the estimates on (2.12) and (2.14), which themselves are suggested by the more cavalier approach. From (2.12), (2.14), (2.15), (2.16), and (2.29) we see that whenever m is sufficiently large in terms of N for (2.12) to hold, there is some constant $C[N, k, \ell] > 0$ (possibly larger than the one appearing in (2.12) but independent of ζ, ξ, and m) for which

$$\begin{aligned}
&\text{(i)} \quad \left|\mathrm{x}_i^{(K)}\right| + \left|\mathrm{y}_i^{(K)}\right| \le 10Nm^{-1} \\
&\qquad \text{for } \mathrm{x}_i,\ \mathrm{y}_i \text{ and } \mathrm{x}_i^K,\ \mathrm{y}_i^K \text{ in (3.4) and (3.5)}, \\
&\text{(ii)} \quad \lim_{m\to\infty} m^q \tau_1 = 0 \\
&\qquad \text{uniformly in } \zeta \in [-c, c]^{N-1} \text{ for any fixed } q,\ c,\ N,\ k,\ \ell, \\
&\text{(iii)} \quad \max\{\tau_i/\tau_1 + \tau_1/\tau_i\}_{i=2}^N \le C[N, k, \ell] e^{|\zeta_i|/m^2}, \text{ and} \\
&\text{(iv)} \quad \max\{|\mathrm{z}_i|\}_{i=1}^N + \max\{\left|\mathrm{z}_i^K\right|\}_{i=1}^{N-1} \\
&\qquad \le C[N, k, \ell] N (m^2 + |\zeta| + |\xi|) \tau_1 e^{|\zeta|/m^2} \le C[N, k, \ell] m^2 \tau_1,
\end{aligned} \tag{3.6}$$

where $|\zeta|$ and $|\xi|$ are the Euclidean norms of $\zeta, \xi \in \mathbb{R}^{N-1}$ and for the final inequality we assume m large in terms of $|\zeta|$ and $|\xi|$ (and allow a larger $C[N, k, \ell]$ on the right-hand side than previously needed).

Using (3.2), (3.4), (3.5), and (3.6), we conclude

$$
\begin{aligned}
&\left|2\pi\tau_1 + \frac{8\pi^2}{k\ell m^2}\mathrm{z}_1 - \mathcal{F}_1\right| \le Cm^{-2}\tau_1,\\
&\left|-2\pi\tau_{N-1} + \frac{8\pi^2}{k\ell m^2}\mathrm{z}_N - \mathcal{F}_N\right| \le Cm^{-2}\tau_1, \text{ and} \\
&\left|2\pi(\tau_i - \tau_{i-1}) + \frac{8\pi^2}{k\ell m^2}\mathrm{z}_i - \mathcal{F}_i\right| \le Cm^{-2}\tau_1 \text{ when } 2 \le i \le N-1
\end{aligned} \tag{3.7}
$$

for some constant $C = C[N,k,\ell] > 0$ (independent of m and ζ and ξ) whenever m is sufficiently large in terms of N, $|\zeta|$, and $|\xi|$.

It now follows from (3.7) and (2.15), recalling (2.11), that

$$
\begin{aligned}
&\left|2\pi\tau_1 - \frac{8\pi^2}{k\ell m^2}\left(2^{N \bmod 2}\tau_1 \ln\frac{1}{10\ell m\tau_1} - \tau_1\xi_1\right) - \mathcal{F}_n\right| \\
&\qquad \le Cm^{-2}\tau_1 \text{ if } n = 1,\\
&\left|2\pi(\tau_n - \tau_{n-1}) - \frac{8\pi^2}{k\ell m^2}\left(2^{N \bmod 2}\tau_n \ln\frac{1}{10\ell m\tau_n}\right.\right. \\
&\qquad \left.\left. - \frac{\tau_{n-1}\xi_{n-1} + \tau_n\xi_n}{2}\right) - \mathcal{F}_n\right| \le Cm^{-2}\tau_1 \text{ if } n \ge 2,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
&\left|-2\pi\tau_1 + \frac{8\pi^2}{k\ell m^2}\left(\tau_1 \ln\frac{1}{10\ell m\tau_1} + \tau_1\xi_1\right) - \mathcal{F}_{n+1}\right| \\
&\qquad \le Cm^{-2}\tau_1 \text{ if } N = 2,\\
&\left|2\pi(\tau_{n+1} - \tau_n) + \frac{8\pi^2}{k\ell m^2}\left([(N+1) \bmod 2]\tau_n \ln\frac{1}{10\ell m\tau_n}\right.\right. \\
&\qquad \left.\left. + \frac{\tau_n\xi_n + \tau_{n+1}\xi_{n+1}}{2}\right) - \mathcal{F}_{n+1}\right| \le Cm^{-2}\tau_1 \text{ if } N \ge 3,
\end{aligned} \tag{3.9}
$$

and, when $N \ge 3$,

$$
\begin{aligned}
&\left|2\pi(\tau_2 - 2\tau_1) + \frac{8\pi^2}{k\ell m^2}\left(2\tau_1 \ln\frac{1}{10\ell m\tau_1} + \frac{\xi_2\tau_2 - \xi_1\tau_1}{2}\right)\right. \\
&\qquad \left. - (\mathcal{F}_{i+1} - \mathcal{F}_i)\right| \le Cm^{-2}\tau_1 \text{ if } i = 1,\\
&\left|2\pi(\tau_{i+1} - 2\tau_i + \tau_{i-1}) + \frac{8\pi^2}{k\ell m^2}\left(2\tau_i \ln\frac{1}{10\ell m\tau_i}\right.\right. \\
&\qquad \left.\left. + \frac{\xi_{i+1}\tau_{i+1} - \xi_{i-1}\tau_{i-1}}{2}\right) - (\mathcal{F}_{i+1} - \mathcal{F}_i)\right| \\
&\qquad \le Cm^{-2}\tau_1 \text{ if } 2 \le i \le N-2, \text{ and} \\
&\Big|2\pi(-2\tau_{N-1} + \tau_{N-2})
\end{aligned} \tag{3.10}
$$

$$+\frac{8\pi^2}{k\ell m^2}\left(2\tau_{N-1}\ln\frac{1}{10\ell m\tau_{N-1}}+\frac{\xi_{N-1}\tau_{N-1}-\xi_{N-2}\tau_{N-2}}{2}\right)$$
$$-(\mathcal{F}_{i+1}-\mathcal{F}_i)\Bigg|\leq Cm^{-2}\tau_1 \text{ if } i=N-1$$

(where C can be taken to be twice the value of the C appearing in (3.7)).

Balancing. The force estimates (3.7)–(3.10) will play an indispensable role, via Lemma 3.33, in selecting viable parameter values in the final steps of the construction, but at the moment, with the goal of completing the specification of the waist radii in the initial surfaces, we set $\zeta=\xi=0$ and impose only the approximate balancing conditions (3.3) (since we will have to vary the parameters and graphically perturb the initial surface anyway to achieve minimality): temporarily ignoring the error bounded by the right-hand sides of (3.8)–(3.10), we demand

$$2\pi\underline{\tau}_1-2^{N \bmod 2}\frac{8\pi^2}{k\ell m^2}\cdot\underline{\tau}_1\ln\frac{1}{10\ell m\underline{\tau}_1}=0 \text{ if } n=1,$$

$$2\pi(\underline{\tau}_n-\underline{\tau}_{n-1})-2^{N \bmod 2}\frac{8\pi^2}{k\ell m^2}\cdot\underline{\tau}_n\ln\frac{1}{10\ell m\underline{\tau}_n}=0 \text{ if } n\geq 2,$$

and for $N\geq 3$

$$2\pi(\underline{\tau}_{n+1}-\underline{\tau}_n)+[(N+1) \bmod 2]\frac{8\pi^2}{k\ell m^2}\cdot\underline{\tau}_n\ln\frac{1}{10\ell m\underline{\tau}_n}=0,$$

(3.11)
$$2\pi(\underline{\tau}_2-2\underline{\tau}_1)+\frac{8\pi^2}{k\ell m^2}\cdot 2\underline{\tau}_1\ln\frac{1}{10\ell m\underline{\tau}_1}=0,$$

$$2\pi(\underline{\tau}_{i+1}-2\underline{\tau}_i+\underline{\tau}_{i-1})+\frac{8\pi^2}{k\ell m^2}\cdot 2\underline{\tau}_i\ln\frac{1}{10\ell m\underline{\tau}_i}=0$$

when $2\leq i\leq N-2$, and finally

$$2\pi(-2\underline{\tau}_{N-1}+\underline{\tau}_{N-2})+\frac{8\pi^2}{k\ell m^2}\cdot 2\underline{\tau}_{N-1}\ln\frac{1}{10\ell m\underline{\tau}_{N-1}}=0.$$

From the third equation of (3.11) we see that $\underline{\tau}_{n+1}=\underline{\tau}_n$ whenever $N\geq 3$ is odd, while from this same equation together with the second equation we see that $\underline{\tau}_{n+1}=\underline{\tau}_{n-1}$ whenever $N\geq 4$ is even (and of course $\underline{\tau}_{n+1}$ is undefined for $N=2$); thus $\underline{\tau}_{n+1}=\underline{\tau}_{N-(n+1)}$ whenever $N\geq 3$. If N is even, then $N=2n$, so obviously $\underline{\tau}_n=\underline{\tau}_{N-n}$, while if N is odd, then $N=2n+1$, so $\underline{\tau}_n=\underline{\tau}_{N-(n+1)}$ and $\underline{\tau}_{N-n}=\underline{\tau}_{n+1}$ but we have just established that $\underline{\tau}_{N-(n+1)}=\underline{\tau}_{n+1}$; thus we also have $\underline{\tau}_n=\underline{\tau}_{N-n}$ whenever $N\geq 2$. It now follows by induction on j, using the two equations obtained by taking $i=j-1$ and $i=N-(j-1)$ in the penultimate line of (3.11), having already dispensed in this paragraph with the cases $j=n$ and $j=n+1$, that $\underline{\tau}_j=\underline{\tau}_{N-j}$ for each $j\in\mathbb{Z}\cap[n,N-1]$. In fact it is clear that all the approximate balancing conditions in (3.11) will be satisfied if and only if we choose $\{\underline{\tau}_i\}_{i=1}^n$ satisfying (for $n = 1$) line 1 or (for $n \geq 2$) lines 2, 4, and 5 (with $2 \leq i \leq n-1$) of (3.11) and simultaneously set

$$\underline{\tau}_i := \underline{\tau}_{N-i} \text{ for } n+1 \leq i \leq N-1. \tag{3.12}$$

For $n = 1$ it therefore remains only to specify $\underline{\tau}_1$, which is uniquely determined by imposing line 1 of (3.11):

$$\underline{\tau}_1 := \begin{cases} \frac{1}{10\ell m} e^{-\frac{k\ell m^2}{4\pi}} & \text{if } N = 2 \\ \frac{1}{10\ell m} e^{-\frac{k\ell m^2}{8\pi}} & \text{if } N = 3. \end{cases} \tag{3.13}$$

For $n \geq 2$ (equivalently $N \geq 4$) we define

$$b_i := \underline{\tau}_i / \underline{\tau}_1 \text{ for } 1 \leq i \leq N-1, \tag{3.14}$$

so in particular $b_1 = 1$; dividing equations 4, 5, and 2 of (3.11) by $2\pi\underline{\tau}_1$, we need now only solve the n equations

$$\begin{aligned} b_2 - 2 &= \frac{8\pi}{k\ell m^2} \ln 10\ell m \underline{\tau}_1, \\ b_{i+1} - 2b_i + b_{i-1} &= \frac{8\pi}{k\ell m^2} b_i \ln 10\ell m b_i \underline{\tau}_1 \text{ for } 2 \leq i \leq n-1, \text{ and} \\ b_{n-1} - b_n &= \left(\frac{1}{2}\right)^{(N+1) \bmod 2} \frac{8\pi}{k\ell m^2} b_n \ln 10\ell m b_n \underline{\tau}_1 \end{aligned} \tag{3.15}$$

for the n unknowns $b_2, \dots, b_n$, and $\underline{\tau}_1$.

The first equation requires

$$\underline{\tau}_1 := \frac{1}{10\ell m} e^{-\frac{k\ell m^2}{4\pi}(1-b_2/2)}, \tag{3.16}$$

which we note even recovers (3.13) if we define $b_2 = 0$ for $N = 2$ and $b_2 = 1$ for $N = 3$. Assuming $N \geq 4$ (equivalently $n \geq 2$), we derive a system equivalent (presuming each $b_i \neq 0$) to the remaining $n-1$ equations of (3.15) by (i) for each $i \in \mathbb{Z} \cap [2, n-1]$ (a vacuous condition when $n = 2$) subtracting the middle equation of (3.15) from b_i times the top equation of (3.15) and (ii) subtracting $2^{(N+1) \bmod 2}$ times the bottom equation of (3.15) from b_n times the top equation of (3.15). In this way (recalling $b_1 = 1$) we obtain the system

$$\begin{aligned} &-b_{i-1} + b_2 b_i - b_{i+1} = -\frac{8\pi}{k\ell m^2} b_i \ln b_i \text{ for } 2 \leq i \leq n-1 \text{ and} \\ &-2^{(N+1) \bmod 2} b_{n-1} + (b_2 - N \bmod 2) b_n = -\frac{8\pi}{k\ell m^2} b_n \ln b_n. \end{aligned} \tag{3.17}$$

Lemma 3.18 (Determination of the waist ratios by the approximate balancing conditions)**.** *Let $N \geq 4$ be a given integer and recall (2.11). There exist $n-1$ real numbers $d_2[N] < d_3[N] < \cdots < d_n[N]$, with $d_2[N] \in (1, 2)$ and $d_2[N]$ strictly increasing in n (for a fixed parity of N), and furthermore there exists $m_0 = m_0[N] > 0$ such that for each* *integer* $m > m_0$ *and for all integers* $\ell \geq k \geq 1$ *there are* $n-1$ *real numbers* $b_2[N,k,\ell,m], b_3[N,k,\ell,m], \ldots, b_n[N,k,\ell,m]$ *solving* (3.17) *and satisfying* $\lim_{m\to\infty} b_i[N,k,\ell,m] = d_i[N]$ *for any fixed* k *and* ℓ.

Proof. Bear in mind that balancing has been accomplished by (3.13) for $N=2$ and $N=3$. Momentarily ignoring the logarithmic terms, for $N=4$ the system (3.17) reduces to $d_2^2 = 2$, so $d_2[4] = \sqrt{2} \in (1,2)$, while for $N=5$ we get $d_2^2 - d_2 - 1 = 0$, yielding $d_2[5] = \frac{1+\sqrt{5}}{2} \in (1,2)$. Now the functions $b_2 \mapsto b_2^2$ and $b_2 \mapsto b_2^2 - b_2 - 1$ have nonzero derivatives at these respective values, so the lemma is established for $N=4$ and $N=5$ by applying the inverse function theorem and taking m large. Thus we may assume $n \geq 3$ and pursue an elaboration of the same strategy.

For real β we define the $(n-1)\times(n-1)$ matrices

$$A_{2n}(\beta) := \begin{pmatrix} \beta & -1 & 0 & 0 & \cdots & 0 \\ -1 & \beta & -1 & 0 & \cdots & 0 \\ 0 & -1 & \beta & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \beta & -1 \\ 0 & 0 & \cdots & 0 & -2 & \beta \end{pmatrix} \tag{3.19}$$

and

$$A_{2n+1}(\beta) := \begin{pmatrix} \beta & -1 & 0 & 0 & \cdots & 0 \\ -1 & \beta & -1 & 0 & \cdots & 0 \\ 0 & -1 & \beta & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \beta & -1 \\ 0 & 0 & \cdots & 0 & -1 & \beta-1 \end{pmatrix}, \tag{3.20}$$

so that the system obtained by temporarily replacing all the logarithmic terms in (3.17) by 0 is equivalent (recalling $b_1 = 1$) to the equation

$$A_N(\beta) \begin{pmatrix} d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ with } \beta = d_2 \text{ and } d_i > 0 \text{ for } 2 \leq i \leq n. \tag{3.21}$$

Using Cramer's rule and expansion by minors, we find

$$\begin{aligned} & d_i[N] = \frac{P_{n-i+1}[N \bmod 2](\beta)}{P_n[N \bmod 2](\beta)} \quad \text{for } 2 \leq i \leq n, \text{ where} \\ & P_i[0](\lambda) = \det A_{2i}(\lambda) \text{ and } P_i[1](\lambda) = \det A_{2i+1}(\lambda) \text{ for } i \geq 3, \\ & P_2[0](\lambda) = \lambda \quad \text{and} \quad P_2[1] = \lambda - 1, \text{ and} \\ & P_1[0](\lambda) = 2 \quad \text{and} \quad P_1[1](\lambda) = 1. \end{aligned} \tag{3.22}$$

Further expansion by minors reveals the recursive relations (independent of the parity of N)

$$P_{i+1}(\lambda) = \lambda P_i(\lambda) - P_{i-1}(\lambda) \text{ for } i \geq 2. \tag{3.23}$$

On the other hand, by applying the constraint $d_2 = \beta$, the expression for d_2 given by (3.22) can be rewritten as

$$\beta P_n(\beta) = P_{n-1}(\beta), \tag{3.24}$$

whence (3.23) delivers

$$P_{n+1}(\beta) = 0. \tag{3.25}$$

We now claim that for each $i \geq 3$ (and either parity of N)

(i) $P_i = P_i[N \bmod 2]$ has a root strictly greater than 1; if $\gamma_i = \gamma_i[N \bmod 2]$ is its largest such root, then
(ii) $P_{i-1}(x) > 0$ whenever $x \geq \gamma_i$,
(iii) $P_{i+1}(\gamma_i) < 0$, and
(iv) γ_i is strictly increasing in i.

These claims can be established by induction on i. The case $i = 3$ is easily verified: $P_2[0](x) = x$, $P_3[0](x) = x^2 - 2$, and $P_4[0](x) = x^3 - 3x$, so $\gamma_3[0] = \sqrt{2}$, $P_2[0](x \geq \gamma_3[0]) > 0$, and $P_4[0](\gamma_3[0]) = 2\sqrt{2} - 3\sqrt{2} < 0$, while $P_2[1](x) = x - 1$, $P_3[1](x) = x^2 - x - 1$, and $P_4[1](x) = x^3 - x^2 - 2x + 1$, so $\gamma_3[1] = \frac{1+\sqrt{5}}{2}$, $P_2[1](x \geq \gamma_3[1]) \geq \frac{\sqrt{5}-1}{2} > 0$, and $P_4[1](\gamma_3[1]) = \frac{1-\sqrt{5}}{2} < 0$. Now suppose that claims (i)–(iii) hold for $i = j$. By claim (i) γ_j exists and $\gamma_j > 1$. According to claim (iii) $P_{j+1}(\gamma_j) < 0$, but P_{j+1} is clearly monic, so $P_{j+1}(x) > 0$ for large x, which implies that P_{j+1} has a root greater than γ_j, so γ_{j+1} exists and $\gamma_{j+1} > \gamma_j > 1$ (verifying claims (i) and (iv)). Therefore $P_j(x \geq \gamma_{j+1}) > 0$ by the maximality of γ_j (verifying claim (ii)). Finally, using (3.23), $P_{j+2}(\gamma_{j+1}) = \gamma_{j+1} P_{j+1}(\gamma_{j+1}) - P_j(\gamma_{j+1})$, which is negative, since the first term vanishes and the second has just been established positive (verifying claim (iii) and so completing the proof of claims (i)–(iv)).

Thus $\beta = \gamma_{n+1}$ solves (3.25) and is strictly increasing in n (for each fixed parity of N). We have already checked that $\beta > 1$; now we claim that $\beta < 2$. In fact we assert that for each $i \geq 2$ (regardless of the parity of N)

(v) $P_i(x) - P_{i-1}(x) \geq 0$ and $P_i(x) > 0$ whenever $x \geq 2$,

which is proven by induction on i. For $i = 2$ and $x \geq 2$ we have $P_2(x) - P_1(x) = x - 2 \geq 0$ (whatever the parity of N) and clearly both $P_2[0](x) = x > 0$ and $P_2[1](x) = x - 1 > 0$. Assuming then that claim (v) holds for $i = j$, we get from (3.23), assuming still $x \geq 2$,

$$\begin{aligned} P_{j+1}(x) - P_j(x) &= xP_j(x) - P_{j-1}(x) - P_j(x) \\ &= (x-1)P_j(x) - P_{j-1}(x) \\ &\geq P_j(x) - P_{j-1}(x) \geq 0 \end{aligned} \tag{3.26}$$

and therefore $P_{j+1}(x) > 0$ as well. We conclude that for every $i \geq 2$ we have $P_i(x) > 0$ whenever $x \geq 2$, so all roots of P_i lie to the left of 2, establishing the bound on $d_2 = \beta = \gamma_{n+1}$.

To show that for each fixed N $d_j[N]$ is a strictly increasing function of j we claim that for any $n \geq 3$ and either parity of N

(vi) $P_{j+1}(\gamma_{n+1}) - P_j(\gamma_{n+1}) < 0$ whenever $1 \leq j \leq n$,

which we prove by induction on j. Since $P_2(x) - P_1(x) = x - 2$ (whatever the parity of N), the case $j = 1$ follows immediately from the fact, proved in the preceding paragraph, that $\gamma_{n+1} < 2$. Assuming now that claim (vi) holds for a given $j \in \mathbb{Z} \cap [1, n-1]$ and applying (3.23) with $i = j + 1$ along with the same inequality $\gamma_{n+1} < 2$, we get

$$\begin{aligned} P_{j+2}(\gamma_{n+1}) - P_{j+1}(\gamma_{n+1}) &= (\gamma_{n+1} - 1)P_{j+1}(\gamma_{n+1}) - P_j(\gamma_{n+1}) \\ &< P_{j+1}(\gamma_{n+1}) - P_j(\gamma_{n+1}) < 0, \end{aligned} \tag{3.27}$$

confirming (vi). It then follows from (3.22) that $1 < d_2 < d_3 < d_4 < \cdots < d_n$.

It remains to reintroduce the logarithmic terms. Recalling the definitions (3.19) and (3.20), for each integer $N \geq 2$ we define the function $F_N : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F_N \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} := A_N(x_2) \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \tag{3.28}$$

and calculate its derivative at $(d_i)_{i=2}^n$:

$$dF_N|_{(d_i)_{i=2}^n} = A_N(d_2) + \begin{pmatrix} d_2 & 0 & 0 & \cdots & 0 \\ d_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.29}$$

whose determinant is $\det A_N(d_2) + \det B$, where B is the matrix

$$\begin{pmatrix} d_2 & -1 & 0 & 0 & \cdots & 0 \\ d_3 & d_2 & -1 & 0 & \cdots & 0 \\ d_4 & -1 & d_2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ d_{n-1} & 0 & \cdots & -1 & d_2 & -1 \\ d_n & 0 & \cdots & 0 & -2^{(N+1) \bmod 2} & d_2 - N \bmod 2 \end{pmatrix}. \tag{3.30}$$

Using just expansion by minors along with the inequalities, proven above, $d_i > 1 > 0$ for $2 \leq j \leq n$ and $P_j(d_2) > 0$ for $1 \leq j \leq n$, we then obtain

$$\det dF_N|_{(d_i)_{i=2}^n} = P_n(d_2) + \sum_{i=2}^{n-1} d_i P_{n-i+1}(d_2) + d_n > 0. \tag{3.31}$$

We conclude by invoking the inverse function theorem and taking m large in terms of $d_2, \ldots, d_n$. q.e.d.

Henceforth, given integers $N \geq 2$, $\ell \geq k \geq 1$, and $m \geq 1$, along with $N-1$ real numbers $\zeta_1, \ldots, \zeta_{N-1}$, we define (i) $b_2[N=2,k,\ell,m] := 0$, $b_2[N=3,k,\ell,m] := 1$, and $\{b_i[N \geq 4,k,\ell,m]\}_{i=2}^n$ as in the proof of Lemma 3.18, (ii) $\underline{\tau}_1[N,k,\ell,m]$ as defined by (3.16), (iii) $\underline{\tau}_i[N,k,\ell,m] := b_i[N,k,\ell,m]\underline{\tau}_1[N,k,\ell,m]$ for $2 \leq i \leq n$, (iv) $\underline{\tau}_i[N,k,\ell,m]$ and $b_i[N,k,\ell,m]$ for $n+1 \leq i \leq N-1$ (and $b_1[N,k,\ell,m] = 1$) in accordance with (3.12) and (3.14), and (v) $\tau_i[N,k,\ell,m,\zeta]$ for $1 \leq i \leq N$ as defined by (2.14). These quantities complete the specification of the initial surfaces defined in (2.30).

Parameter dependence. The ζ and ξ parameters influence the forces, which will be analyzed later to manage the substitute kernel (described in Section 1 and formally introduced in Section 5), as well as the *dislocations*

$$\begin{aligned} \mathcal{D}_i &= \mathcal{D}_i[N,k,\ell,m,\zeta,\xi] \\ &:= \begin{cases} \frac{1}{2}\tau_i\xi_i - \frac{1}{2}\tau_{i-1}\xi_{i-1} \text{ for } 2 \leq i \leq N-1 \\ 0 \text{ for } i = 1 \text{ and } i = N, \end{cases} \end{aligned} \tag{3.32}$$

which will be used to manage the extended part of the extended substitute kernel (again see Sections 1 and 5) and each of which measures the antisymmetric part of the vertical displacement of a pair of adjacent inequivalent (under the action of $\mathcal{G}$ (2.26)) catenoidal regions relative to the toral region they share. Morally, the next lemma ensures that, by adjusting the parameters, we can freely prescribe any set of suitably bounded forces and dislocations. Indeed this surjectivity assertion could be stated precisely and proved as a corollary of the lemma by applying the Brouwer fixed-point theorem. Because, however, we will also need to allow graphical deformations of the initial surfaces, we bypass this step and instead will more directly apply the lemma to control the extended substitute kernel in the proof of Theorem 6.50.

Lemma 3.33 (Parametric dependence of the forces and dislocations)**.** *Given $c > 0$ and integers $N \geq 2$ and $\ell \geq k \geq 1$, there exist real numbers $C = C[N,k,\ell], m_0 = m_0[N,k,\ell,c] > 0$ and an invertible linear map $\Theta = \Theta[N,k,\ell,m] : \mathbb{R}^{2N-2} \to \mathbb{R}^{2N-2}$ such that whenever $m > m_0$*

$$\|\Theta\| + \|\Theta^{-1}\| \leq C[N,k,\ell], \tag{i}$$

where $\|\cdot\|$ *denotes the operator norm induced by the Euclidean norm on* $\mathbb{R}^{2N-2}$*, and whenever* $\zeta, \xi \in [-c, c]^{2N-2}$

$$\text{(ii)} \qquad \left| \tau_1^{-1} \begin{pmatrix} m^2\mathcal{F}_1[N,k,\ell,m,\zeta,\xi] \\ \vdots \\ m^2\mathcal{F}_N[N,k,\ell,m,\zeta,\xi] \\ \mathcal{D}_2[N,k,\ell,m,\zeta,\xi] \\ \vdots \\ \mathcal{D}_{N-1}[N,k,\ell,m,\zeta,\xi] \end{pmatrix} - \Theta \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{N-1} \\ \xi_1 \\ \vdots \\ \xi_{N-1} \end{pmatrix} \right| \leq C[N,k,\ell],$$

where $|\cdot|$ *denotes the Euclidean norm on* $\mathbb{R}^{2N-2}$*.*

We emphasize that the estimates made in Lemma 3.33 are independent of m and the size of the parameters: $C[N,k,\ell]$ does not depend on m or c (but $m_0[N,k,\ell,c]$ does depend on c). This independence will be crucial in the proof of the main theorem when defining the nonlinear map whose fixed point will give us our final minimal surface. In particular it will be needed to establish that we can choose the parameter factor of the domain of this map to be compact, an ingredient in the justification of the applicability of the Schauder fixed-point theorem in the proof of Theorem 6.50. (Roughly, the independence from c of these estimates ensures that attempts to control the extended substitute kernel (via the forces and dislocations) by varying the parameters will not drive any of these parameters off to infinity.)

Proof. To begin, it is easy to see that the linear map $T = T[N,k,\ell] : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ defined by

$$(3.34) \quad T : \begin{pmatrix} m^2\mathcal{F}_1 \\ m^2\mathcal{F}_2 \\ \vdots \\ m^2\mathcal{F}_{N-1} \\ m^2\mathcal{F}_N \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{N-1} \end{pmatrix} \mapsto \begin{pmatrix} \frac{k\ell m^2}{2\pi}(\mathcal{F}_1 - \mathcal{F}_2) + 4\pi(\mathcal{D}_1 + \mathcal{D}_2) \\ \frac{k\ell m^2}{2\pi}(\mathcal{F}_2 - \mathcal{F}_3) + 4\pi(\mathcal{D}_2 + \mathcal{D}_3) \\ \vdots \\ \frac{k\ell m^2}{2\pi}(\mathcal{F}_{N-1} - \mathcal{F}_N) + 4\pi(\mathcal{D}_{N-1} + \mathcal{D}_N) \\ \frac{k\ell m^2}{8\pi^2}\mathcal{F}_1 \\ 2\mathcal{D}_2 \\ \vdots \\ 2\mathcal{D}_{N-1} \end{pmatrix}$$

is invertible (since, the lowest $N-2$ components on the right determine all dislocations, which with the top N components then determine all the forces too) with inverse bounded independently of m and c. To prove the lemma it will therefore suffice to identify an invertible linear map $\widetilde{\Theta} = \widetilde{\Theta}[N,k,\ell,m] : \mathbb{R}^{2N-2} \to \mathbb{R}^{2N-2}$ such that

$$\begin{aligned} &\left\|\widetilde{\Theta}\right\| + \left\|\widetilde{\Theta}^{-1}\right\| \leq C[N,k,\ell] \quad \text{and} \\ &\left|\tau_1^{-1} T \begin{pmatrix} m^2\mathcal{F} \\ \mathcal{D} \end{pmatrix} - \widetilde{\Theta}\begin{pmatrix} \zeta \\ \xi \end{pmatrix}\right| \leq C[N,k,\ell] \end{aligned} \tag{3.35}$$

for some constant $C[N,k,\ell] > 0$ whenever $\zeta, \xi \in [-c,c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, and c.

In fact we will show that we can take

$$\widetilde{\Theta}\begin{pmatrix} \zeta \\ \xi \end{pmatrix} := \begin{pmatrix} Z & 0 \\ B & \Xi \end{pmatrix}\begin{pmatrix} \zeta \\ \xi \end{pmatrix}, \tag{3.36}$$

where 0 is the $(N-1)\times(N-1)$ zero matrix, $B = B[N,k,\ell,m]$ is an $(N-1)\times(N-1)$ matrix bounded independently of c and m, and $Z = Z[N,k,\ell,m]$ and $\Xi = \Xi[N,k,\ell,m]$ are the $(N-1)\times(N-1)$ matrices

$$Z := \begin{pmatrix} 8\pi b & Q \end{pmatrix}, \quad \text{with} \quad b = \begin{pmatrix} b_1 & b_2 & \cdots & b_{N-1} \end{pmatrix}^\top \quad \text{and}$$

$$Q := \begin{pmatrix} -b_2 & 0 & \cdots & 0 & 0 \\ b_1 + b_3 & -b_3 & 0 & \cdots & 0 \\ -b_2 & b_2 + b_4 & -b_4 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -b_{N-3} & b_{N-3} + b_{N-1} & -b_{N-1} \\ 0 & \cdots & 0 & -b_{N-2} & b_{N-2} + b_N \end{pmatrix}, \tag{3.37}$$

and

$$\Xi := \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 & 0 \\ -b_1 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & -b_2 & b_3 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -b_{N-3} & b_{N-2} & 0 \\ 0 & 0 & \cdots & 0 & -b_{N-2} & b_{N-1} \end{pmatrix}, \tag{3.38}$$

recalling (2.12) and understanding $b_N := 0$. (We acknowledge that $b_1 = b_{N-1} = 1$ but refrain from making these substitutions above so as to avoid obscuring the structure of the matrices.) We will now verify that $\widetilde{\Theta}$ so defined satisfies (3.35), identifying the matrix B along the way.

First we check that Z and Ξ are invertible. Invertibility of Ξ is obvious, since it is lower-triangular with all its diagonal entries strictly positive. Next, we inductively alter the middle $N-3$ columns of Z, starting with column $N-2$ and working our way to the left until column 2, by replacing each by its sum with the column immediately to its right; we also divide the first column by 8π. The resulting matrix (for the computation of which we recall that $b_N = 0$)

$$\widetilde{Z} := \begin{pmatrix} 1 & -b_2 & 0 & \cdots & 0 & 0 \\ b_2 & b_1 & -b_3 & 0 & \cdots & 0 \\ b_3 & 0 & b_2 & -b_4 & \ddots & \vdots \\ b_4 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & b_{N-3} & -b_{N-1} \\ b_{N-1} & 0 & \cdots & 0 & 0 & b_{N-2} \end{pmatrix} \tag{3.39}$$

is invertible if and only if Z is. The only nonzero entries of $\widetilde{Z}$ lie in (i) the first column, whose entries are all strictly positive, (ii) the diagonal, whose entries are also all strictly positive, and (iii) the superdiagonal, whose entries are all strictly negative. It is easy to see that any square matrix of this form has strictly positive determinant. For example we can compute the determinant by cofactor expansion along the bottom row. Starting with the entry in the bottom row and first column (b_{N-1} in $\widetilde{Z}$), we see that the submatrix obtained by deleting the bottom row and first column is lower-triangular with all diagonal entries strictly negative. Counting minus signs, including possibly one contributed by the position of the entry in question, we find that the corresponding cofactor is strictly positive and, the entry itself being strictly positive as well, therefore the corresponding term in the expansion is also strictly positive. The remaining term, corresponding to the entry in the bottom row and last column (b_{N-2} in $\widetilde{Z}$) is clearly the product of a strictly positive number, the entry itself, with the determinant of a smaller matrix of the same form under consideration. Since our claim is obvious in the 1×1 case, the general case now follows by induction. Thus we see that Z is invertible as well. Consequently, for any choice of B (to be identified shortly) in (3.36), the map $\widetilde{\Theta}$ is indeed invertible, and the first inequality in (3.35) is now ensured by (2.12) (and the bound $\|B\| \leq C[N,k,\ell]$ established below).

Now we estimate $\frac{k\ell m^2}{2\pi\tau_1}(\mathcal{F}_i - \mathcal{F}_{i+1}) + \frac{4\pi}{\tau_1}(\mathcal{D}_i + \mathcal{D}_{i+1})$ for $1 \leq i \leq N-1$. When $N = 2$, from (3.8) and (3.9) we find

$$\left| \mathcal{F}_1 - \mathcal{F}_2 - \left(4\pi\tau_1 - \frac{16\pi^2}{k\ell m^2}\tau_1 \ln \frac{1}{10\ell m \tau_1} \right) \right| \leq Cm^{-2}\tau_1, \tag{3.40}$$

whence with (2.14)

$$\begin{aligned} &\Bigg| \mathcal{F}_1 - \mathcal{F}_2 \\ &\quad -2e^{\zeta_1} \left(2\pi\underline{\tau}_1 - \frac{8\pi^2}{k\ell m^2}\underline{\tau}_1 \left(-\zeta_1 + \ln \frac{1}{10\ell m \underline{\tau}_1} \right) \right) \Bigg| \leq Cm^{-2}\tau_1, \end{aligned} \tag{3.41}$$

so by the balancing condition (3.11)

$$\left|\mathcal{F}_1-\mathcal{F}_2-\frac{16\pi^2\tau_1}{k\ell m^2}\zeta_1\right|\le Cm^{-2}\tau_1, \tag{3.42}$$

while $\mathcal{D}_1=\mathcal{D}_2=0$ by (3.32), which proves

$$\left|\frac{k\ell m^2}{2\pi\tau_1}(\mathcal{F}_1-\mathcal{F}_2)+\frac{4\pi}{\tau_1}(\mathcal{D}_1+\mathcal{D}_2)-8\pi\zeta_1\right|\le C \text{ when } N=1, \tag{3.43}$$

taking the constant C possibly larger (depending on k and ℓ) than the one immediately above but still independent of c and m.

When $N\ge 3$ and $1\le i\le N-1$, from (3.10) and (3.32) we find

$$\begin{aligned}\Bigg|&\frac{k\ell m^2}{2\pi\tau_1}\left(\mathcal{F}_i-\mathcal{F}_{i+1}\right)+\frac{4\pi}{\tau_1}\left(\mathcal{D}_i+\mathcal{D}_{i+1}\right)\\ &-\frac{k\ell m^2}{2\pi\tau_1}\Bigg(2\pi\left(-\tau_{i-1}+2\tau_i-\tau_{i+1}\right)\\ &\qquad-\frac{16\pi^2}{k\ell m^2}\tau_i\ln\frac{1}{10\ell m\tau_i}\Bigg)\Bigg|\le C,\end{aligned} \tag{3.44}$$

understanding $\tau_0=\tau_N:=0$. Using (2.13), (2.14), and Taylor expansion, for $2\le j\le N-1$ we have

$$\begin{aligned}\left|\tau_j-e^{\zeta_1}\underline{\tau}_j-k^{-1}\ell^{-1}m^{-2}b_j\zeta_j\tau_1\right|&\le m^{-4}c^2e^{2c/m^2}b_j\tau_1\\ &\le Cm^{-2}\tau_1,\end{aligned} \tag{3.45}$$

where to ensure the last inequality we take m sufficiently large in terms of c. Thus, when $3\le i\le N-1$,

$$\begin{aligned}\Big|&(-\tau_{i-1}+2\tau_i-\tau_{i+1})-e^{\zeta_1}\left(-\underline{\tau}_{i-1}+2\underline{\tau}_i-\underline{\tau}_{i+1}\right)\\ &-\frac{\tau_1}{k\ell m^2}\left(-b_{i-1}\zeta_{i-1}+2b_i\zeta_i-b_{i+1}\zeta_{i+1}\right)\Big|\le Cm^{-2}\tau_1\end{aligned} \tag{3.46}$$

and, further using (2.13) and (2.14), when $2\le i\le N-1$

$$\begin{aligned}\Bigg|&\tau_i\ln\frac{1}{10\ell m\tau_i}\\ &-\left(e^{\zeta_1}\underline{\tau}_i\ln\frac{1}{10\ell m\underline{\tau}_i}-\tau_1b_i\zeta_1+\frac{\tau_1b_i\zeta_i}{4\pi}-\frac{\tau_1b_2b_i\zeta_i}{8\pi}\right)\Bigg|\le C\tau_1\end{aligned} \tag{3.47}$$

provided m is sufficiently large in terms of c. Note that by virtue of (2.13) and (3.17)

$$|b_2b_i-(b_{i-1}+b_{i+1})|\le Cm^{-2} \text{ for } 1\le i\le N-1 \tag{3.48}$$

(understanding $b_0 = b_N = 0$), so the last term on the left-hand side of (3.47) may be replaced by $-\frac{\tau_1(b_{i-1}+b_{i+1})\zeta_i}{8\pi}$. By applying estimates (3.46)–(3.48) to (3.44) and imposing the balancing condition (3.11) we obtain

$$\begin{aligned}(3.49)\qquad &\left|\frac{k\ell m^2}{2\pi\tau_1}(\mathcal{F}_i - \mathcal{F}_{i+1}) + \frac{4\pi}{\tau_1}(\mathcal{D}_i + \mathcal{D}_{i+1})\right.\\ &\qquad \left. - \left(8\pi b_i\zeta_1 - b_{i-1}\zeta_{i-1} + (b_{i-1}+b_{i+1})\zeta_i - b_{i+1}\zeta_{i+1}\right)\right| \leq C\end{aligned}$$

when $N \geq 3$ and $3 \leq i \leq N-1$ (understanding $b_N := 0$).

In the remaining cases that $N \geq 3$ but $i \in \{1,2\}$ a similar computation (using simply $\tau_1 = e^{\zeta_1}\underline{\tau}_1$ in place of (3.45) when $j=1$) reveals

$$\begin{aligned}(3.50)\qquad &\left|\frac{k\ell m^2}{2\pi\tau_1}(\mathcal{F}_1 - \mathcal{F}_2) + \frac{4\pi}{\tau_1}(\mathcal{D}_1 + \mathcal{D}_2) - (8\pi b_1\zeta_1 - b_2\zeta_2)\right| \leq C \text{ and}\\ &\left|\frac{k\ell m^2}{2\pi\tau_1}(\mathcal{F}_2 - \mathcal{F}_3) + \frac{4\pi}{\tau_1}(\mathcal{D}_2 + \mathcal{D}_3)\right.\\ &\qquad \left. - \left(8\pi b_2\zeta_1 + (b_1+b_3)\zeta_2 - b_3\zeta_3\right)\right| \leq C.\end{aligned}$$

Together, (3.43), (3.49), and (3.50) show that

$$(3.51)\qquad \left|\frac{k\ell m^2}{2\pi\tau_1}(\mathcal{F}_i - \mathcal{F}_{i+1}) + \frac{4\pi}{\tau_1}(\mathcal{D}_i + \mathcal{D}_{i+1}) - (Z\zeta)_i\right| \leq C$$

for some constant $C = C[N,k,\ell] > 0$ whenever $N \geq 2$, $1 \leq i \leq N-1$, $\zeta, \xi \in [-c,c]^{N-1}$, and m is sufficiently large in terms of c.

Next, it is obvious from (3.32), using (3.45) as necessary and continuing to take m large in terms of c, that

$$(3.52)\qquad \left|\frac{2}{\tau_1}\mathcal{D}_i - (b_i\xi_i - b_{i-1}\xi_{i-1})\right| \leq C \text{ for } 2 \leq i \leq N-1.$$

Furthermore, from (2.15), (3.7), and (3.11), using (3.47) and (3.48) again and still taking m large in terms of c, we have

$$\begin{aligned}(3.53)\qquad &\left|\frac{k\ell m^2}{8\pi^2\tau_1}\mathcal{F}_1 - \left(b_1\xi_1 + 2^{N \bmod 2}\zeta_1\right)\right| \leq C \text{ if } 2 \leq N \leq 3 \text{ and}\\ &\left|\frac{k\ell m^2}{8\pi^2\tau_1}\mathcal{F}_1 - \left[b_1\xi_1 + \left(2^{N \bmod 2}b_n + 2\sum_{j=1}^{n-1} b_j\right)\zeta_1\right.\right.\\ &\qquad + \frac{1}{4\pi}\sum_{j=2}^{n-1}(b_{j-1} - 2b_j + b_{j+1})\zeta_j\\ &\qquad \left.\left. + \frac{2^{N \bmod 2}}{8\pi}(b_{n-1} - 2b_n + b_{n+1})\zeta_n\right]\right| \leq C \text{ if } N \geq 3.\end{aligned}$$

Together, (3.51), (3.52), and (3.53) establish the second inequality of (3.35) with $\widetilde{\Theta}$ defined by (3.36), the entries of the first row of the matrix B given by the coefficients of the components of ζ appearing in (3.53), and the remaining entries of B vanishing. This completes the proof. q.e.d.

4. Estimates of the initial geometry

Norms and spaces of sections. To state the estimates for the geometry of the initial surfaces and to carry out the rest of the construction we must first identify certain norms and corresponding spaces of sections. For the most part our notation is standard and speaks for itself. Given a smooth vector bundle E over a smooth manifold M (possibly with boundary), a nonnegative integer j, and a real number $\alpha \in (0,1)$, we write $C^j_{loc}(E)$ and $C^{j,\alpha}_{loc}(E)$ for the space of sections of E having component functions of class C^j_{loc} or $C^{j,\alpha}_{loc}$ respectively relative to any (so every) smooth local chart and smooth trivialization; we set $C^\infty(E) := \bigcap_{j=0}^{\infty} C^j_{loc}(E)$. When E is the trivial bundle $M \times \mathbb{R}$, we write simply M in place of $M \times \mathbb{R}$ in our notation for the spaces just defined and also for the spaces below, as is standard for spaces of real-valued functions.

All of the vector bundles of interest to us are derived from tangent bundles by a combination of duality, tensor product, pullback, and projection; a Riemannian metric on M or on another manifold in which it is immersed will determine canonical metrics and connections on all these bundles. When there is no danger of confusion, we write simply $|\cdot|$ for the corresponding pointwise norm and D for the connection. Given a section u of a bundle E over M thus equipped, we define the standard global norms

$$\|u\|_j = \|u\|_{C^j(E,g)} = \left\|u : C^j(E,g)\right\| := \sum_{i=0}^{j} \sup_{p\in M} \left|D^i u(p)\right| \tag{4.1}$$

as well as the Hölder seminorm

$$[u]_\alpha := \sup_{\gamma:[0,1]\to M} \frac{\left|u(\gamma(1)) - P_0^1[\gamma]u(\gamma(0))\right|}{|\gamma|^\alpha}, \tag{4.2}$$

where the supremum is taken over all piecewise C^1 paths, $|\gamma|$ denotes the length of such a path, and $P_0^1[\gamma] : E_{\gamma(0)} \to E_{\gamma(1)}$ is the parallel transport map along γ from the fiber over $\gamma(0)$ to the fiber over $\gamma(1)$.

Then we can define also the Hölder norms

$$\|u\|_{j,\alpha} = \left\|u : C^{j,\alpha}(E,g)\right\| := \|u\|_j + \left[D^j u\right]_\alpha. \tag{4.3}$$

Note that for functions on convex open subsets of Euclidean space these Hölder norms agree with the conventional ones. Generally, the spaces

$C^{j,\alpha}(E,g)$ and $C^j(E,g)$ consisting of sections with finite corresponding norm enjoy many of the properties familiar from the Euclidean case; in particular $C^{j,\beta}(E,g)$ embeds compactly in $C^{j,\alpha}(E,g)$ whenever M is compact and $0<\alpha<\beta<1$. In the construction we will routinely wish to compare norms of the above type induced by different metrics on a single manifold. The definitions make it easy to see that $\left\|u : C^{j,\alpha}(E,h)\right\| \leq C \left\|u : C^{j,\alpha}(E,g)\right\|$, where C is controlled by the g-norms of h, its inverse, and finitely many g-derivatives of h (the maximum order needed depending in a transparent way on j, α, and on the bundle E); of course if M is not compact, then C may blow up, depending on g and h.

If M is a two-sided hypersurface immersed in a Riemannian manifold N and $\mathcal{G}_M$ is a group of isometries of N preserving M as a set, then $\mathcal{G}_M$ acts on a section u of the normal bundle of M by $(\mathfrak{g}.u)(p) := \mathfrak{g}_*[u\left(\mathfrak{g}^{-1}(p)\right)]$ for each element $\mathfrak{g} : N \to N$ of $\mathcal{G}_M$. Because this bundle is just the trivial $\mathbb{R}$ bundle over M, its sections can be identified with functions (for us representing mean curvature or normal perturbations) on which the corresponding action of $\mathcal{G}_M$ is given by

$$(\mathfrak{g}.f)(p) := \begin{cases} f\left(\mathfrak{g}^{-1}(p)\right) & \text{if } \mathfrak{g} \text{ preserves each side of } M \\ -f\left(\mathfrak{g}^{-1}(p)\right) & \text{if } \mathfrak{g} \text{ reverses the sides of } M. \end{cases} \tag{4.4}$$

All the elements of the symmetry group $\mathcal{G} = \mathcal{G}[k,\ell,m]$ (defined in (2.26)) of the construction can be seen to fix each side of the initial surface $\Sigma = \Sigma[N,k,\ell,m,\zeta,\xi]$ (defined in (2.30)), so its action is always given by the first line of (4.4). (As mentioned earlier (Remark 1.2), when $k=\ell$, we have the option of enforcing a larger symmetry group in the construction, one admitting reflections through great circles on $\mathbb{T}$. When N is odd, such reflections reverse the sides of the initial surface Σ.)

Notation 4.5. In general, if M is a two-sided hypersurface immersed in a Riemannian manifold N and $\mathcal{G}_M$ is a group of isometries of N preserving M as a set, we will append the subscript $\mathcal{G}_M$ to a space of functions to designate the subspace consisting of functions which are equivariant under the $\mathcal{G}_M$ action (4.4).

Finally we will often wish to work with weighted versions of the above norms. For this construction the following definition suffices:

$$\left\|u : C^{j,\alpha}(E,g,f)\right\| := \sup_{p\in M} \frac{\left\|u : C^{j,\alpha}\left(E|_{B[p,1,g]},g\right)\right\|}{f(p)}, \tag{4.6}$$

where $f : M \to (0,\infty)$ is a given weight function and $B[p,1,g] \subseteq M$ is the g metric ball of radius 1 centered at $p \in M$. We will also make use of weighted C^j norms, with the obvious definition.

The χ metric. It is the primary task of this section to estimate the intrinsic and extrinsic geometry of the initial surfaces. We continue to write $\iota = \iota[N,k,\ell,m,\zeta,\xi] : \Sigma[N,k,\ell,m,\zeta,\xi] \to \mathbb{S}^3$ for the inclusion map of Σ in $\mathbb{S}^3$, and we define

$$g = g[N,k,\ell,m,\zeta,\xi] := \iota^* g_S, \tag{4.7}$$

the metric induced on Σ by g_S and ι. To fix the extrinsic quantities we pick on each initial surface Σ the global unit normal $\nu = \nu[N,k,\ell,m,\zeta,\xi]$ which is directed toward C_1 at the points of Σ closest to C_1. We then define

$$\begin{aligned} &A = A[N,k,\ell,m,\zeta,\xi] := (g_S \circ \iota)(D d\iota, \nu) \quad \text{and} \\ &H = H[N,k,\ell,m,\zeta,\xi] := \operatorname{tr}_{\iota^* g} A, \end{aligned} \tag{4.8}$$

respectively the scalar-valued second fundamental form and mean curvature of Σ relative to ν and g_S, D being the connection induced on $T^*\Sigma \otimes \iota^* T\mathbb{S}^3$ by g_S and ι. In particular $H = (g_S \circ \iota)(\mathbf{H}, \nu)$, recalling the vector-valued mean curvature $\mathbf{H}$ defined below (3.2).

Every initial surface admits by virtue of its construction a natural decomposition into overlapping regions, each of which resembles either a portion of a torus coaxial with $\mathbb{T}$ or (via Φ (2.2)) a truncated catenoid. Modulo the horizontal symmetries, there are N such toral regions, one for each torus incorporated in the construction, and there are $N-1$ catenoidal regions, one for each pair of adjacent tori. Definitions are made in the subsections below. The estimates will then be obtained by treating the catenoidal regions as perturbations of Euclidean catenoids and the toral regions as graphs over the Clifford torus.

Because all these regions shrink with increasing m and because even on a fixed initial surface the characteristic scale m^{-1} of the toral regions dwarfs the characteristic scale τ_1 near the waists, it will be advantageous to uniformize the problem (and flatten Σ) by working with a metric χ on each initial surface conformal to the natural one $g = \iota^* g_S$. We will set

$$\chi = \chi[N,k,\ell,m,\zeta,\xi] := \rho^2 g, \tag{4.9}$$

where the conformal factor $\rho = \rho[N,k,\ell,m,\zeta,\xi] : \Sigma \to \mathbb{R}$ is defined so that its reciprocal ρ^{-1} is a $\mathcal{G}$-equivariant function (i) measuring on each catenoidal region the $\Phi^{-1*} g_E$ distance to the axis and (ii) transitioning smoothly to the constant m^{-1} by the edge of the toral regions.

To be precise, recalling (2.4) and setting

$$d(\mathrm{x},\mathrm{y},\mathrm{z}) := \sqrt{\mathrm{x}^2 + \mathrm{y}^2} \text{ on } \mathbb{R}^3, \tag{4.10}$$

we first define $\widetilde{\rho}[\mathrm{z}_1', \mathrm{z}_2'] : \mathbb{R}^3 \to \mathbb{R}$, for given $\mathrm{z}_1' < \mathrm{z}_2' \in \mathbb{R}$, by

$$\widetilde{\rho}[\mathrm{z}_1', \mathrm{z}_2'] := \begin{cases} \left(\frac{1}{d} - m\right) \cdot \psi\left[\frac{1}{5\ell m}, \frac{1}{10\ell m}\right] \circ d \text{ on } \{\mathrm{z} \in [\mathrm{z}_1', \mathrm{z}_2']\} \\ 0 \text{ elsewhere} \end{cases}, \tag{4.11}$$

we define $\widetilde{\rho}_q[\mathrm{z}_1', \mathrm{z}_2'] : \mathbb{R}^3 \to \mathbb{R}$, for given point $q = (\mathrm{x}_0, \mathrm{y}_0) \in \mathbb{R}^2$, by

$$\widetilde{\rho}_q[\mathrm{z}_1', \mathrm{z}_2'](\mathrm{x}, \mathrm{y}, \mathrm{z}) := \widetilde{\rho}[\mathrm{z}_1', \mathrm{z}_2'] \left(\mathrm{x} - \mathrm{x}_0, \mathrm{y} - \mathrm{y}_0, \mathrm{z}\right), \tag{4.12}$$

and we define $\widehat{\rho} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\widehat{\rho} := m + \sum_{i=0}^{N-2} \sum_{q \in \widehat{L}_{i \bmod 2, i \bmod 2}} \widetilde{\rho}_q \left[\mathrm{z}_i^K, \mathrm{z}_{i+2}^K\right], \tag{4.13}$$

recalling (2.16) and (2.27) and taking $\mathrm{z}_0^K := -\frac{\pi}{5}$ and $\mathrm{z}_N^K := \frac{\pi}{5}$. Then, recalling (2.2), $\rho \in C^\infty_{\mathcal{G}}(\Sigma)$ is uniquely defined by requiring

$$\rho \circ \Phi = \widehat{\rho} \quad \text{on } \Phi^{-1}(\Sigma). \tag{4.14}$$

Evidently from (2.12), (2.14), and (2.30)

$$\frac{m}{C[N, k, \ell]} \leq \rho \leq \frac{C[N, k, \ell]}{\tau_1} \tag{4.15}$$

for some constant $C[N, k, \ell] > 0$ whenever $\zeta, \xi \in [-c, c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, and c. Equipped with the χ metric, each catenoidal region tends with large m to the flat cylinder of radius 1, while each toral region tends, away from the catenoids adjoining it, to a flat $\frac{\sqrt{2}\pi}{k} \times \frac{\sqrt{2}\pi}{\ell}$ rectangle. Details are provided in the next two subsections.

Before proceeding, we briefly mention a couple differences of our approach from [13]. First, our catenoidal and toral regions (defined below) correspond to their *extended standard regions*, but since we never view their *standard regions* or *transition regions* in isolation, we omit the modifier *extended*. Second, our use of the χ metric follows theirs to study the mean curvature equation on the initial surfaces globally, but whereas Kapouleas and Yang introduce another metric (the h metric) conformal to g in order to analyze the approximate kernel, we will apply the χ metric to this problem as well, in the next section.

Catenoidal regions. We define the standard cylinder

$$\mathbb{K} := \mathbb{R} \times \mathbb{S}^1, \tag{4.16}$$

where $\mathbb{S}^1 := \{z \in \mathbb{C} \ : \ |z| = 1\}$. We write t for the standard coordinate on the $\mathbb{R}$ factor and θ for the standard coordinate on the universal cover $\mathbb{R}$ of the $\mathbb{S}^1$ factor given by $\theta \mapsto e^{i\theta}$. We will routinely and implicitly define functions on $\mathbb{K}$ by defining functions on this universal cover that are invariant under the deck transformations. We equip $\mathbb{K}$ with its usual flat metric

$$\widehat{\chi}_K := dt^2 + d\theta^2 \tag{4.17}$$

and we define the embedding

$$\widehat{\kappa} : \mathbb{K} \to \mathbb{R}^3 \qquad \text{by} \qquad \widehat{\kappa}(t, \theta) = (\cosh t \, \cos\theta, \cosh t \, \sin\theta, t), \tag{4.18}$$

whose image is the catenoid with waist radius 1, axis of symmetry the z-axis, and horizontal plane of the symmetry the z = 0 plane. It is easy to see that $\widehat{\kappa}$ is conformal, with

$$\widehat{\kappa}^* g_{{}_E} = \cosh^2 t \, \widehat{\chi}_{{}_K}, \tag{4.19}$$

$g_{{}_E}$ being the Euclidean metric on $\mathbb{R}^3$, and that its unit normal pointing outward at the waist is

$$\widehat{\nu} = \operatorname{sech} t \, \cos\theta \, \partial_{\mathrm{x}} + \operatorname{sech} t \, \sin\theta \, \partial_{\mathrm{y}} - \tanh t \, \partial_{\mathrm{z}}. \tag{4.20}$$

Of course the catenoid is famously minimal, and it is elementary to check that, more specifically, it has second fundamental form (relative to $\widehat{\nu}$)

$$\widehat{A} = dt^2 - d\theta^2. \tag{4.21}$$

Given $a > 0$, we define the truncated cylinder

$$\mathbb{K}_a := [-a, a] \times \mathbb{S}^1, \tag{4.22}$$

and, given also $(\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0) \in \mathbb{R}^3$ and $\tau > 0$, we define the embedding

$$\begin{aligned} &\widehat{\kappa}[(\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0), \tau, a] : \mathbb{K}_a \to \mathbb{R}^3 \text{ by} \\ &(t, \theta) \mapsto (\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0) + \tau(\cosh t \, \cos\theta, \cosh t \, \sin\theta, t), \end{aligned} \tag{4.23}$$

whose image is a truncated, translated, and scaled catenoid with vertical axis of symmetry. For $1 \leq i \leq N-1$ we set

$$\begin{aligned} a_i = a_i[N, k, \ell, m, \zeta, \xi] &:= \operatorname{arcosh} \frac{1}{10\ell m \tau_i} \\ &= \frac{k\ell}{4\pi} \left(1 - \frac{b_2}{2}\right) m^2 - \zeta_1 - (1 - \delta_{i1}) \left(\ln b_i + \frac{\zeta_i}{k\ell m^2}\right) \\ &\quad + \ln\left(1 + \sqrt{1 - 100\ell^2 m^2 \tau_i^2}\right) \end{aligned} \tag{4.24}$$

(writing δ_{ij} for the Kronecker delta and recalling (2.12), (2.14), (2.18), and (2.20)) and, recalling (2.2), we define the map

$$\begin{aligned} &\kappa_i = \kappa_i[N, k, \ell, m, \zeta, \xi] : \mathbb{K}_{a_i} \to \mathbb{S}^3 \text{ by} \\ &\kappa_i := \Phi \circ \widehat{\kappa} \left[\left(\frac{(i-1)\pi}{\sqrt{2}km}, \frac{(i-1)\pi}{\sqrt{2}\ell m}, \mathrm{z}_i^K \right), \tau_i, a_i \right]. \end{aligned} \tag{4.25}$$

Then (referring to (2.30) and particularly (2.29)) the image of κ_i is entirely contained in the initial surface Σ and defines the catenoidal region

$$\mathcal{K}[i] = \mathcal{K}[i; N, k, \ell, m, \zeta, \xi] := \kappa_i \left(\mathbb{K}_{a_i}\right), \tag{4.26}$$

so that κ_i is a diffeomorphism onto its image; in innocuous abuse of notation we will routinely write $\kappa_i^{-1} : \mathcal{K}[i] \to \mathbb{K}_{a_i}$ for the inverse of this diffeomorphism.

Note that, recalling (4.14),

$$\kappa_i{}^*\rho = \tau_i^{-1}\operatorname{sech} t \tag{4.27}$$

and, by applying (2.12) and (2.14) to (4.24),

$$C[N,k,\ell]^{-1}m^2 \leq a_i \leq C[N,k,\ell]m^2 \tag{4.28}$$

for some constant $C[N,k,\ell] > 0$ whenever $\zeta,\xi \in [-c,c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, and c. From (2.14), (2.16), and (4.26) it is clear that

$$\mathcal{K}[i] \cap \mathcal{K}[i'] \neq \emptyset \text{ if and only if } i = i'. \tag{4.29}$$

Since on small scales the covering map Φ is an approximate isometry, we expect that in a suitably rescaled sense each catenoidal region will converge in the large-m limit to an exact catenoid in Euclidean space. The next proposition quantifies this convergence.

Proposition 4.30 (Estimates of the geometry of the catenoidal regions)**.** *Given a real number $c > 0$ and integers $N \geq 2$, $\ell \geq k \geq 1$, and $j \geq 0$, there exist real numbers $m_0 = m_0[N,k,\ell,c] > 0$ and $C = C[N,k,\ell,j] > 0$ such that whenever $\zeta,\xi \in [-c,c]^{N-1}$ and $m > m_0$, for $1 \leq i \leq N-1$*

(i) $\left\|\kappa_i{}^*\chi - \widehat{\chi}_K : C^j\left(T^*\mathbb{K}_{a_i}^{\otimes 2}, \widehat{\chi}_K\right)\right\| \leq Cm^2\tau_1$,
(ii) $\left\|\rho : C^j(\mathcal{K}[i],\chi,\rho)\right\| + \left\|\rho^{-1} : C^j(\mathcal{K}[i],\chi,\rho^{-1})\right\| \leq C$,
(iii) $\left\|\mathrm{z} : C^j(\mathcal{K}[i],\chi)\right\| \leq Cm^2\tau_1$,
(iv) $\left\|A - (-1)^{N-i}\kappa_i^{-1*}\tau_i\left(dt^2 - d\theta^2\right) : C^j\left(T^*\mathcal{K}[i]^{\otimes 2},\chi,\tau_1\left|\mathrm{z}\right| + \rho^{-2}\right)\right\| \leq C$,
(v) $\left\|\rho^{-2}\left|A\right|_g^2 - 2\tau^2\rho^2 : C^j(\mathcal{K}[i],\chi,\tau_1 + \left|\mathrm{z}\right| + \rho^{-2})\right\| \leq C$, *and*
(vi) $\left\|\rho^{-2}H : C^j\left(\mathcal{K}[i],\chi,\tau_1\left|\mathrm{z}\right| + \rho^{-2}\left|\mathrm{z}\right| + \tau_1^2\right)\right\| \leq C$,

where $\mathrm{z} : \mathbb{S}^3 \to \mathbb{R}$ is defined via (2.2) and we also recall (2.14), (4.6), (4.7), (4.8), (4.9), (4.14), (4.17), (4.22), (4.25), and (4.26).

Proof. From (2.3), (4.23), (4.25), and (4.27) we calculate

$$\begin{aligned}\kappa_i{}^*\chi - \widehat{\chi}_K = \left(\sin 2\left(\mathrm{z}_i^K + \tau_i t\right)\right)&\left(\tanh^2 t\,\cos 2\theta\,dt^2\right. \\ &\left.-2\tanh t\,\sin 2\theta\,dt\,d\theta - \cos 2\theta\,d\theta^2\right).\end{aligned} \tag{4.31}$$

It follows from (2.14), (3.6) (the bottom line), and (4.28) that $\left|\mathrm{z}_i^K + \tau_i t\right| \leq C[N,k,\ell]m^2\tau_1$ everywhere on $\mathbb{K}_{a_i}$ for some $C[N,k,\ell] > 0$ whenever m is sufficiently large in terms of N, k, ℓ, and c. Since $\widehat{\chi}_K = dt^2 + d\theta^2$ is flat, it is trivial to differentiate (4.31), so item (i) now follows immediately, as do items (ii) and (iii) in turn, using also (4.27) and

$$\kappa_i{}^*\mathrm{z} = \mathrm{z}_i^K + \tau_i t. \tag{4.32}$$

Next, using (4.25), for any vector fields V and W on $\mathbb{K}_{a_i}$ we have

$$
\begin{aligned}
{\kappa_i}^* A(V,W) &= A[\widehat{\kappa}_i, \Phi^* g_S](V,W) \\
&= \nu[\widehat{\kappa}_i, \Phi^* g_S]_c \left(D[\Phi^* g_S]_{d\widehat{\kappa}_i V} d\widehat{\kappa}_i W \right)^c \\
&= \frac{\nu[\widehat{\kappa}_i, g_E]_c}{|\nu[\widehat{\kappa}_i, g_E]|_{\Phi^* g_S \circ \widehat{\kappa}_i}} \left(D[g_E]_{d\widehat{\kappa}_i V} d\widehat{\kappa}_i W \right. \\
&\quad \left. + \left(D[\Phi^* g_S] - D[g_E] \right) (d\widehat{\kappa}_i V, d\widehat{\kappa}_i W) \right)^c \\
&= |\nu[\widehat{\kappa}_i, g_E]|^{-1}_{\Phi^* g_S \circ \widehat{\kappa}_i} \left(A[\widehat{\kappa}_i, g_E](V,W) + B(V,W) \right),
\end{aligned} \tag{4.33}
$$

where $\widehat{\kappa}_i := \widehat{\kappa}\left[\left(\frac{(i-1)\pi}{\sqrt{2}km}, \frac{(i-1)\pi}{\sqrt{2}\ell m}, \mathrm{z}_i^K\right), \tau_i, a_i\right]$ (recalling (4.23)), $\nu[\widehat{\kappa}_i, \Phi^* g_S]$ is the unit normal for $\widehat{\kappa}_i$ relative to g_S directed so that $d\Phi\nu[\widehat{\kappa}_i, \Phi^* g_S] = \nu \circ \kappa_i$ and $\nu[\widehat{\kappa}_i, \Phi^* g_S]_c$ is its $\Phi^* g_S$ metric dual, $\nu[\widehat{\kappa}_i, g_E]$ is the unit normal for $\widehat{\kappa}_i$ relative to g_E directed so that its g_E metric dual $\nu[\widehat{\kappa}_i, g_E]_c$ is a positive multiple of $\nu[\widehat{\kappa}_i, \Phi^* g_S]_c$, $A[\widehat{\kappa}_i, \cdot]$ is the second fundamental of $\widehat{\kappa}_i$ with respect to the ambient metric $\cdot$ and the unit normal $\nu[\widehat{\kappa}_i, \cdot]$, $D[\cdot]$ is the Levi-Civita connection induced by the metric $\cdot$, and B is the symmetric tensor

$$
\begin{aligned}
B_{\alpha\beta} &:= \nu[\widehat{\kappa}_i, g_E]_c ({\Gamma^c}_{ab} \circ \widehat{\kappa}_i) {(d\widehat{\kappa}_i)^a}_\alpha {(d\widehat{\kappa}_i)^b}_\beta \text{ with} \\
{\Gamma^c}_{ab} &:= \frac{1}{2} (\Phi^* g_S)^{cd} \left(D[g_E]_b (\Phi^* g_S)_{ad} \right. \\
&\quad \left. + D[g_E]_a (\Phi^* g_S)_{bd} - D[g_E]_d (\Phi^* g_S)_{ab} \right).
\end{aligned} \tag{4.34}
$$

Recalling that we have chosen the unit normal ν for Σ pointing toward C_1 at the points closest to C_1 (so ν has positive inner product with ∂_{z} at the top of Σ, in an $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ coordinate system defined via Φ), we see from (4.23) that

$$
\nu[\widehat{\kappa}_i, g_E] = (-1)^{N-i} \widehat{\nu}, \tag{4.35}
$$

recalling (4.20), so in particular, using (2.3),

$$
\begin{aligned}
&|\nu[\widehat{\kappa}_i, g_E]|_{\Phi^* g_S \circ \widehat{\kappa}_i} (t, \theta) \\
&\quad = \frac{\sqrt{1 - \operatorname{sech}^2 t \, \cos 2\theta \, \sin 2{\kappa_i}^* \mathrm{z} - \tanh^2 t \, \sin^2 2{\kappa_i}^* \mathrm{z}}}{\cos 2{\kappa_i}^* \mathrm{z}}.
\end{aligned} \tag{4.36}
$$

It is also easy to see from (2.3) that the only Christoffel symbols (in $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ coordinates) for $\Phi^* g_S$ not vanishing identically are

$$
\begin{aligned}
&{\Gamma^{\mathrm{z}}}_{\mathrm{xx}} = -{\Gamma^{\mathrm{z}}}_{\mathrm{yy}} = -\cos 2\mathrm{z}, \qquad {\Gamma^{\mathrm{x}}}_{\mathrm{xz}} = {\Gamma^{\mathrm{x}}}_{\mathrm{zx}} = \frac{\cos 2\mathrm{z}}{1 + \sin 2\mathrm{z}}, \quad \text{and} \\
&{\Gamma^{\mathrm{y}}}_{\mathrm{yz}} = {\Gamma^{\mathrm{y}}}_{\mathrm{zy}} = \frac{-\cos 2\mathrm{z}}{1 - \sin 2\mathrm{z}}.
\end{aligned} \tag{4.37}
$$

Returning to (4.34) (and again using (4.20) and (4.23)) we now find

$$(4.38)\qquad \begin{aligned} (-1)^{N-i}B_{tt} &= {\kappa_i}^*\rho^{-2}\tanh^3 t\,\cos 2\theta\,\cos 2{\kappa_i}^*\mathrm{z} \\ &\quad + 2\tau_i^2\tanh t\,\cos 2\theta\,\sec\,2{\kappa_i}^*\mathrm{z} \\ &\quad - 2\tau_i^2\tanh t\,\tan 2{\kappa_i}^*\mathrm{z}, \\ (-1)^{N-i}B_{t\theta} &= -{\kappa_i}^*\rho^{-2}\tanh^2 t\,\sin 2\theta\,\cos 2{\kappa_i}^*\mathrm{z} \\ &\quad - \tau_i^2\sin 2\theta\,\sec 2{\kappa_i}^*\mathrm{z},\ \text{and} \\ (-1)^{N-i}B_{\theta\theta} &= -{\kappa_i}^*\rho^{-2}\tanh t\,\cos 2\theta\,\cos 2{\kappa_i}^*\mathrm{z}. \end{aligned}$$

Thus, applying (4.36) and (4.38) in (4.33), we have computed A on $\mathcal{K}[i]$, proving (iv). From (4.31) we also compute

$$(4.39)\qquad \begin{aligned} (\widehat{\kappa}_i^*\Phi^* g_S)^{-1} = &\frac{1}{1-\operatorname{sech}^2 t\,\cos 2\theta\,\sin 2{\kappa_i}^*\mathrm{z} - \tanh^2 t\,\sin^2 2{\kappa_i}^*\mathrm{z}} \\ &\times {\kappa_i}^*\rho^2\left(\partial_t^2+\partial_\theta^2+(\sin 2{\kappa_i}^*\mathrm{z})\left(-\cos 2\theta\,\partial_t^2\right.\right. \\ &\left.\left. +2\tanh t\,\sin 2\theta\,\partial_t\,\partial_\theta + \tanh^2 t\,\cos 2\theta\,\partial_\theta^2\right)\right), \end{aligned}$$

which in conjunction with (iv) immediately yields the estimate (v). The estimate (vi) requires slightly more care, but from (4.21), (4.33), (4.36), and (4.39) we compute the exact mean curvature

$$(4.40)\qquad \begin{aligned} {\kappa_i}^*H = &\frac{(-1)^{N-i}\left({\kappa_i}^*\rho^2\right)}{\left(1-\operatorname{sech}^2 t\,\cos 2\theta\,\sin 2{\kappa_i}^*\mathrm{z}-\tanh^2 t\,\sin^2 2{\kappa_i}^*\mathrm{z}\right)^{3/2}} \\ &\times\left[-\frac{1}{2}\tau_i(\sin 4{\kappa_i}^*\mathrm{z})(1+\tanh^2 t)\cos 2\theta\right. \\ &\quad -{\kappa_i}^*\rho^{-2}(\sin 4{\kappa_i}^*\mathrm{z})(\cos 2{\kappa_i}^*\mathrm{z})\tanh^3 t \\ &\quad +\tau_i^2\tanh t\,\cos 2\theta - 4\tau_i^2(\sin 2{\kappa_i}^*\mathrm{z})\tanh t \\ &\quad \left.+3\tau_i^2(\sin^2 2{\kappa_i}^*\mathrm{z})\tanh t\,\cos 2\theta\right], \end{aligned}$$

delivering (vi) and completing the proof. q.e.d.

Graphs over immersions. The estimates away from the catenoidal regions will be obtained by treating the initial surface there as a graph over the torus, as an application of the following lemma, which will be used again in the final section to estimate the contributions to the mean curvature of the perturbed surface which are nonlinear in the perturbing function and to estimate the perturbation to the corresponding forces (3.2). We first clarify some notation used in the statement of the lemma. Suppose (M,g) is a Riemannian manifold. If $u\in C^2_{loc}(M)$, we write $D[g]^2{}_{ab}u$ and $\Delta_g u := g^{ab}D[g]^2{}_{ab}u$ for the Hessian and Laplacian of u under g. Given $p\in M$ and $r>0$, we write $\overline{B}[p,r,(M,g)]$ for the closed metric ball in (M,g) with center p and radius r. We adopt the sign and indexing conventions that the Riemann curvature tensor R_{abcd} of (M,g) is defined by

$$R_{abcd}V^aW^bX^cY^d = g\left(D_VD_WX - D_WD_VX - D_{[V,W]}X, Y\right), \tag{4.41}$$

for any vector fields V,W,X,Y on M; then $R_{ab} := R_{ac}{}^c{}_b = R_{cab}{}^c$ is the Ricci curvature of (M,g). Suppose also that $\phi : S \to M$ is a C^2_{loc} codimension-one immersion of a manifold S into M and that ν is a global unit normal for ϕ. We write $A[\phi,\nu] := (g\circ\phi)(\nu, D[g]d\phi)$ and $H[\phi,\nu] := \operatorname{tr}_{\phi^*g} A[\phi,\nu]$ for the corresponding second fundamental form and mean curvature (here $D[g]$ being the canonical connection on the bundle $\phi^*TM\otimes T^*S$ defined by the Levi-Civita connections induced by g and ϕ^*g). Finally we point out that we reserve the right to denote evaluation of a section X at a point p by either of the standard options $X|_p = X(p)$.

Lemma 4.42 (Graphs over immersions)**.** *Let $\phi : S \to M$ be a smooth two-sided (codimension-one) immersion of a smooth manifold S into a smooth complete Riemannian manifold M with smooth metric g. Let $\nu \in \phi^*(TM)$ be a global unit normal for ϕ and write $A := A[\phi,\nu]$ and $H := H[\phi,\nu]$ for the corresponding second fundamental form and mean curvature. For each $t\in\mathbb{R}$ and $u \in C^2_{loc}(S)$ we define the maps $\phi_t, \phi[u] : S \to M$ by*

$$\begin{aligned} &\phi_t(p) := \exp_{\phi(p)} t\nu(p) \quad \textit{and} \\ &\phi[u](p) := \exp_{\phi(p)} u(p)\nu(p) = \phi_{u(p)}(p), \end{aligned} \tag{4.43}$$

where exp $: TM \to M$ *is the exponential map of (M,g). Suppose that for a given $u\in C^2_{loc}(S)$ and $p\in S$*

$$|u(p)|\,|A(p)|_{\phi^*g} + |u(p)|^2 \left\| |R|_g : C^0(\overline{B}[p,|u(p)|,(M,g)])\right\| < \frac{1}{3}. \tag{4.44}$$

Then $\phi[u]$ is a C^2_{loc} immersion on a neighborhood of p, as is ϕ_t for every t between 0 *and $u(p)$. On this neighborhood ϕ_t and $\phi[u]$ admit respective C^1_{loc} unit normals ν_t and $\nu[\phi[u]]$ satisfying $\nu_t(p) = \frac{d}{dt}\exp_{\phi(p)} t\nu(p)$ and $g\left(\nu[\phi[u]]|_p, \nu_t|_p\right) > 0$. If (working near p) we set $g^t := \phi_t^*g$, $g_t := (\phi_t^*g)^{-1}$, and $A^t := A[\phi_t,\nu_t]$, then*

$$\begin{aligned} &\partial_t g^t{}_{\alpha\beta} = -2A^t{}_{\alpha\beta} \quad \textit{and} \\ &\partial_t A^t{}_{\alpha\beta} = \nu_t{}^a\nu_t{}^d\phi_t{}^b{}_{,\alpha}\phi_t{}^c{}_{,\beta}R_{abcd}\circ\phi_t - A^t{}_{\alpha\gamma}A^t{}_{\beta\delta}g_t{}^{\gamma\delta}. \end{aligned} \tag{i}$$

If we also set $A[u] := A[\phi[u],\nu[\phi[u]]]$ and $H[u] := H[\phi[u],\nu[\phi[u]]]$ and we define on S the symmetric 2*-tensors $g^u := g^{u(\cdot)}$, $g_u := g_{u(\cdot)}$, and $A^u := A^{u(\cdot)}$, as well as the function $H^u := g_u{}^{\alpha\beta}A^u{}_{\alpha\beta}$ and the section $\phi_u{}^c{}_{,\alpha} := \phi_t{}^c{}_{,\alpha}\big|_{t=u(\cdot)}$ of $\phi[u]^*TM$, then*

$$\phi[u]^*g = g^u + du\otimes du, \tag{ii}$$

(iii) $$A[u]_{\alpha\beta} = \frac{{A^u}_{\alpha\beta} + {D[g^u]^2}_{\alpha\beta}u - 2{g_u}^{\gamma\delta}{A^u}_{\gamma(\alpha}u_{,\beta)}u_{,\delta}}{\sqrt{1+|du|^2_{g^u}}},$$ *and*

(iv)

$$\begin{aligned}
H[u] &= H + \left(\Delta_{\phi^*g} + |A|^2_g + \nu^a\nu^b R_{ab}\circ\phi\right)u \\
&+ u_{,\gamma}{g_u}^{\gamma\delta}{g_u}^{\alpha\beta}\int_0^1 \left(2u{A^{tu}}_{\alpha\delta;\beta} + 2u_{,\beta}{A^{tu}}_{\alpha\delta} - u{A^{tu}}_{\alpha\beta;\delta} - u_{,\delta}{A^{tu}}_{\alpha\beta}\right)dt \\
&- \frac{u_{,\gamma}u_{,\delta}{g_u}^{\alpha\gamma}{g_u}^{\beta\delta}}{\sqrt{1+|du|^2_{g^u}}}\left(\frac{H^u+\Delta_{g^u}u}{1+\sqrt{1+|du|^2_{g^u}}}{g^u}_{\alpha\beta} + \frac{{D[g^u]^2}_{\alpha\beta}u + 3{A^u}_{\alpha\beta}}{1+|du|^2_{g^u}}\right) \\
&+ u^2\int_0^1 (t-1)\Big[2{g_{tu}}^{\alpha\beta}{g_{tu}}^{\gamma\delta}{A^{tu}}_{\alpha\beta}{\nu_{tu}}^a{\nu_{tu}}^d{\phi_{tu}}^b{}_{,\gamma}{\phi_{tu}}^c{}_{,\delta}R_{abcd}\circ\phi[u] \\
&+ 2{A^{tu}}_{\alpha\beta}{A^{tu}}_{\gamma\delta}{A^{tu}}_{\epsilon\zeta}{g_{tu}}^{\beta\gamma}{g_{tu}}^{\delta\epsilon}{g_{tu}}^{\alpha\zeta} + {\nu_{tu}}^a{\nu_{tu}}^b{\nu_{tu}}^c R_{ab|c}\circ\phi[u]\Big]\,dt,
\end{aligned}$$

where the vertical bar and semicolon before an index indicate differentiation under the Levi-Civita connection induced by g and ϕ^*g respectively.

Remark 4.45. Note that in the special case of Lemma 4.42 that $(M,g) = (\mathbb{S}^3, \lambda^2 g_{_S})$ is the round 3-sphere of radius $\lambda > 0$ we have

$$\begin{aligned}
&{\nu_t}^a{\nu_t}^d{\phi_t}^b{}_{,\alpha}{\phi_t}^c{}_{,\beta}R_{abcd}\circ\phi_t = \lambda^{-2}{g^t}_{\alpha\beta}, \\
&\nu^a\nu^b R_{ab}\circ\phi = 2\lambda^{-2}, \qquad \text{and} \qquad R_{ab|c} = 0.
\end{aligned} \tag{4.46}$$

Proof. We begin with a few basic generalities concerning connections on pullbacks of vector bundles. Suppose $\varphi : P \to M$ is a smooth map (not necessarily an immersion) between smooth manifolds and the target M is equipped with a smooth Riemannian metric g. We will write $D[TM]$ for the Levi-Civita connection on TM induced by g. We omit the elementary verification of the following observations. There is a unique connection $D[\varphi^*TM]$ on the pullback bundle φ^*TM satisfying the chain rule

$$D[\varphi^*TM]_V(X\circ\varphi) = (D[TM]_{\varphi_*V}X)\circ\varphi \tag{4.47}$$

for all $V \in C^0_{loc}(TP)$ and $X \in C^1_{loc}(TM)$. Moreover, $D[\varphi^*TM]$ is torsion-free in the sense that

$$D[\varphi^*TM]_V\varphi_*W - D[\varphi^*TM]_W\varphi_*V = \varphi_*[V,W] \tag{4.48}$$

for all $V, W \in C^1_{loc}(TP)$; $D[\varphi^*TM]$ is compatible with g in the sense that

$$\begin{aligned}
V(g\circ\varphi)(X,Y) =&(g\circ\varphi)(D[\varphi^*TM]_VX, Y) \\
&+ (g\circ\phi)(X, D[\varphi^*TM]_VY)
\end{aligned} \tag{4.49}$$

for all $V \in C^0_{loc}(TP)$ and $X, Y \in C^1_{loc}(\varphi^*TM)$; and $D[\varphi^*TM]$ inherits the curvature of M: for all $V, W \in C^1_{loc}(TP)$ and $X \in C^2_{loc}(\varphi^*TM)$

$$D_V D_W X - D_W D_V X - D_{[V,W]} X = (R \circ \varphi)(\varphi_* V, \varphi_* W) X, \tag{4.50}$$

in which every instance of D is $D = D[\varphi^*TM]$.

Now let (M, g), S, ϕ, and ν be as in the statement of the lemma. We define the map

$$\Phi : S \times \mathbb{R} \to M \quad \text{by} \quad \Phi(p, t) := \exp_{\phi(p)} t\nu(p), \tag{4.51}$$

so that $\phi_t = \Phi(\cdot, t)$. Suppose $V \in C^\infty(TS)$ and write $\mathbb{V}$ for the unique vector field on $\Sigma \times \mathbb{R}$ such that $(\mathbb{V} f)(p, t) = (V f(\cdot, t))(p)$ for all $f \in C^\infty(S \times \mathbb{R})$, $p \in S$, and $t \in \mathbb{R}$. Then $\Phi_* \mathbb{V}|_{(p,t)} = d\phi_t V|_p$ and $\Phi_* \partial_t|_{(p,t)} = \frac{d}{dt} \exp_{\phi(p)} t\nu(p)$. Given $s, t \in \mathbb{R}$, we write $P^t_s : \Phi^*TM|_{(\cdot,s)} \to \Phi^*TM|_{(\cdot,t)}$ for the map of parallel translation (relative to $D[\Phi^*TM]$) along the $\mathbb{R}$ cross-sections of $S \times \mathbb{R}$. Using (4.48) and (4.50) as well as the fact that $D[\Phi^*TM]_{\partial_t} \Phi_* \partial_t = 0$, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} P^0_t (d\phi_t V) &= \left.\frac{d}{dt}\right|_{t=0} P^0_t (\Phi_* \mathbb{V}) = D[\Phi^*TM]_{\partial_t} \Phi_* \mathbb{V}|_{(\cdot,0)} \\
&= D[\Phi^*TM]_{\mathbb{V}} \Phi_* \partial_t|_{(\cdot,0)} = D[\phi^*TM]_V \nu \text{ and} \\
\frac{d^2}{dt^2} P^0_t (d\phi_t V) &= \frac{d^2}{dt^2} P^0_t (\Phi_* \mathbb{V}) = \frac{d}{dt} P^0_t (D[\Phi^*TM]_{\partial_t} \Phi_* \mathbb{V}) \\
&= \frac{d}{dt} P^0_t (D[\Phi^*TM] D_{\mathbb{V}} \Phi_* \partial_t) \\
&= P^0_t (D[\Phi^*TM]_{\partial_t} D_{\mathbb{V}} \Phi_* \partial_t) \\
&= P^0_t ((R \circ \phi_t)(\Phi_* \partial_t, d\phi_t V) \Phi_* \partial_t).
\end{aligned} \tag{4.52}$$

Thus

$$\begin{aligned}
P^0_t d\phi_t V =& d\phi V + t D[\phi^*TM]_V \nu \\
&+ \int_0^t (t - s) P^0_s ((R \circ \phi_s)(\Phi_* \partial_t, d\phi_s V) \Phi_* \partial_t) \, ds,
\end{aligned} \tag{4.53}$$

so, noting that $\Phi_* \partial_t$ is unit, for all $p \in S$ and $t \in \mathbb{R}$ (replacing $[0, t]$ below by $[t, 0]$ if $t < 0$)

$$\sup_{s \in [0,t]} |d\phi_s V_p|_g \leq \frac{1 + |t| \, |A(p)|_{\phi^* g}}{1 - t^2 \left\| |R|_g : C^0(\overline{B}[p, |t|, (M, g)]) \right\|} |d\phi V_p|_g \tag{4.54}$$

and also consequently

$$\frac{|d\phi_t V_p|_g}{|d\phi V_p|_g} \geq 1 - |A(p)|_{\phi^* g} |t| - K \frac{1 + |t| \, |A(p)|_{\phi^* g}}{1 - K t^2} t^2, \tag{4.55}$$

where

$$K := \left\| |R|_g : C^0(\overline{B}[p, |t|, (M, g)]) \right\|, \tag{4.56}$$

confirming that ϕ_t is an immersion near p whenever

$$|t|\,|A(p)|_{\phi^* g} + t^2 \left\| |R|_g : C^0(\overline{B}[p,|t|,(M,g)]) \right\| < 1/3 \tag{4.57}$$

(which condition is obviously not sharp).

Using (4.48) and (4.49) as well as the fact that $D[\Phi^* TM]_{\partial_t}\Phi_*\partial_t = 0$, we also compute

$$\begin{aligned} (g\circ\Phi)(D[\Phi^*TM]_{\mathbb{V}}\Phi_*\partial_t,\partial_t) &= \frac{1}{2}\mathbb{V}(g\circ\Phi)(\Phi_*\partial_t,\Phi_*\partial_t) = 0 \text{ and} \\ \frac{d}{dt}(g\circ\Phi)(\Phi_*\mathbb{V},\Phi_*\partial_t) &= (g\circ\Phi)(D[\Phi^*TM]_{\partial_t}\Phi_*\mathbb{V},\Phi_*\partial_t) \\ &= \mathbb{V}(g\circ\Phi)(\Phi_*\partial_t,\Phi_*\partial_t) = 0. \end{aligned} \tag{4.58}$$

Since $\Phi_*\partial_t|_{(\cdot,0)} = \nu$, in fact $\Phi_*\partial_t$ is everywhere and always orthogonal to $\Phi_*\mathbb{V}$. Thus wherever $|t|$ is small enough that ϕ_t is locally an immersion, $\Phi_*\partial_t$ is a smooth local unit normal for ϕ_t, designated ν_t in the statement of the lemma. Then, letting W be another vector field on S with $\mathbb{W}$ the canonically corresponding vector field on $S\times\mathbb{R}$, (4.48)–(4.50),

$$\begin{aligned} \frac{d}{dt}\phi_t^* g(V,W) &= (g\circ\Phi)(D[\Phi^*TM]_{\mathbb{V}}\Phi_*\partial_t,\Phi_*\mathbb{W}) \\ &\quad + (g\circ\Phi)(\Phi_*\mathbb{V}, D[\Phi^*TM]_{\mathbb{W}}\Phi_*\partial_t) \text{ and} \\ \frac{d}{dt}A[\phi_t,\nu_t](V,W) &= (g\circ\Phi)\left((R\circ\Phi)(\Phi_*\partial_t,\Phi_*\mathbb{V})\mathbb{W}\right. \\ &\quad \left. + D[\Phi^*TM]_{\mathbb{V}}D[\Phi_*TM]_{\mathbb{W}}\Phi_*\partial_t,\Phi_*\partial_t\right), \end{aligned} \tag{4.59}$$

proving item (i) of the lemma.

Now suppose $u \in C^2_{loc}(S)$ and take $\phi[u]$ as defined in the statement of the lemma. Let $\pi : S\times\mathbb{R}\to S$ be the canonical projection onto S and let ν_u be the section of $\phi[u]^*TM$ defined by

$$\nu_u := \nu_{u(\cdot)} \tag{4.60}$$

(so $\nu_u(p) = \nu_{u(p)}(p)$, ν_t having been defined in the statement of the lemma). Since $\phi[u](p) = \Phi(p,u(p))$,

$$d\phi[u]V = d\Phi\left(\mathbb{V} + (\mathbb{V}\pi^*u)\partial_t\right) = \left.(d\phi_t)\right|_{t=u(\cdot)} V + (Vu)\nu_u, \tag{4.61}$$

which implies item (ii) of the lemma. In particular, because $du\otimes du$ is nonnegative, $\phi[u]$ is an immersion on a neighborhood of p whenever $\phi_{u(p)}$ is, so in particular, in view of (4.55), provided (4.44) holds. It also follows that the corresponding metric on the cotangent space satisfies

$$(\phi[u]^*g)^{-1} = g_u - \frac{1}{1+|du|^2_{g^u}}\nabla_{g^u}u\otimes\nabla_{g^u}u, \tag{4.62}$$

where g_u and g^u are as defined in the statement of the lemma and ∇_{g^u} is the gradient operator on S induced by the metric g^u. (Equation (4.62) is a trivial consequence of item (ii) of the lemma at points where du

vanishes; at any other point p it is easily derived by working relative to a g^u orthogonal basis one of whose elements is $\nabla_{g^u} u|_p$.)

Clearly the 1-form $dt - d\pi^* u$ on $S \times \mathbb{R}$ annihilates all tangent vectors to the graph of u in $S \times \mathbb{R}$, so, relative to the metric $\Phi^* g = \phi_t^* g + dt^2$, the upward unit normal to this graph is

$$\frac{\partial_t - \nabla_{\Phi^* g} \pi^* u}{\sqrt{1 + |d\pi^* u|^2_{\Phi^* g}}}. \tag{4.63}$$

Noting that $d\Phi \left(\nabla_{\Phi^* g} \pi^* u\right)|_{(p,t)} = d\phi_t \left(\nabla_{\phi_t^* g} u\right)|_p$, we see that the unit normal $\nu[\phi[u]]$ for $\phi[u]$ identified in the statement of the lemma satisfies

$$\nu[\phi[u]] = \frac{\nu_u - (d\phi_t)|_{t=u(\cdot)} \nabla_{g^u} u}{\sqrt{1 + |du|^2_{g^u}}}. \tag{4.64}$$

The corresponding second fundamental form is

$$A[u] = (g \circ \phi[u])(\nu[\phi[u]], D[\phi[u]^* TM]_V d\phi[u] W), \tag{4.65}$$

but

$$\begin{aligned} D[\phi[u]^* TM]_V d\phi[u] W|_p &= D[\Phi^* TM]_{\mathbb{V} + (\mathbb{V}\pi^* u)\partial_t} (\Phi_* \mathbb{W} \\ &\quad + (\mathbb{W}\pi^* u)\Phi_* \partial_t)\,|_{(p,u(p))} \\ &= D[\phi^*_{u(p)} TM]_V \left(d\phi_t|_p \right)\Big|_{t=u(p)} W|_p \\ &\quad + (VWu)\nu[\phi_{u(p)}]|_p \\ &\quad + (Wu) D[\phi^*_{u(p)} TM]_V \nu[\phi_{u(p)}]|_p \\ &\quad + (Vu) D[\phi^*_{u(p)} TM]_W \nu[\phi_{u(p)}]|_p, \end{aligned} \tag{4.66}$$

whose inner product with (4.64) yields item (iii) of the lemma. By contracting item (ii) of the lemma with item (iii) we obtain

$$\begin{aligned} H[u] &= \frac{H^u + \Delta_{g^u} u}{\sqrt{1 + |du|^2_{g^u}}} - u_{,\gamma} u_{,\delta} \frac{D[g^u]^2{}_{\alpha\beta} u + 3A^u{}_{\alpha\beta}}{\left(1 + |du|^2_{g^u}\right)^{3/2}} {g_u}^{\alpha\gamma} {g_u}^{\beta\delta} \\ &= H^u + \Delta_{g^u} u - \frac{u_{,\gamma} u_{,\delta} {g_u}^{\alpha\gamma} {g_u}^{\beta\delta}}{\sqrt{1 + |du|^2_{g^u}}} \left(\frac{H^u + \Delta_{g^u} u}{1 + \sqrt{1 + |du|^2_{g^u}}} g^u{}_{\alpha\beta} \right. \\ &\qquad \left. + \frac{D[g^u]^2{}_{\alpha\beta} u + 3A^u{}_{\alpha\beta}}{1 + |du|^2_{g^u}} \right), \end{aligned} \tag{4.67}$$

but, using item (i) of the lemma,

$$\begin{aligned}\Delta_{g^u} u &= \Delta_{\phi^* g} u + u_{,\gamma} {g_u}^{\gamma\delta} {g_u}^{\alpha\beta} \left(\frac{1}{2} D[\phi^* g]_\delta {g^u}_{\alpha\beta} - D[\phi^* g]_\beta {g^u}_{\alpha\delta} \right) \\ &= \Delta_{\phi^* g} u + u_{,\gamma} {g_u}^{\gamma\delta} {g_u}^{\alpha\beta} \int_0^1 \left(2u {A^{tu}}_{\alpha\delta;\beta} \right. \\ &\qquad \left. + 2u_{,\beta} {A^{tu}}_{\alpha\delta} - u {A^{tu}}_{\alpha\beta;\delta} - u_{,\delta} {A^{tu}}_{\alpha\beta} \right) \, dt \end{aligned} \tag{4.68}$$

and

$$\begin{aligned} H^u =& H + u \left(|A|_g^2 + (R_{ab} \circ \phi) \nu^a \nu^b \right) + u^2 \int_0^1 (t-1) \\ & \times \left[2 {g_{tu}}^{\alpha\beta} {g_{tu}}^{\gamma\delta} {A^{tu}}_{\alpha\beta} (R_{abcd} \circ \phi[u]) {\nu_{tu}}^a {\nu_{tu}}^d {\phi_{tu}}^b{}_{,\gamma} {\phi_{tu}}^c{}_{,\delta} \right. \\ & \quad + 2 {A^{tu}}_{\alpha\beta} {A^{tu}}_{\gamma\delta} {A^{tu}}_{\epsilon\zeta} {g_{tu}}^{\beta\gamma} {g_{tu}}^{\delta\epsilon} {g_{tu}}^{\alpha\zeta} \\ & \quad \left. + \left(R_{ab|c} \circ \phi[u] \right) {\nu_{tu}}^a {\nu_{tu}}^b {\nu_{tu}}^c \right] \, dt, \end{aligned} \tag{4.69}$$

establishing item (iv) and completing the proof. q.e.d.

Toral regions. Recalling (2.5), (2.8), (2.14), and (2.28) we define, for $1 \le i \le N$, the closed domains $\mathbb{T}_i \subset \mathbb{R}^2$ by

$$\begin{aligned} \mathbb{T}_i &= \mathbb{T}_i[N, k, \ell, m, \zeta, \xi] \\ &:= \begin{cases} m\mathbb{T}_{X,Y,\sqrt{\tau_1}} \text{ if } i = 1 \\ m\mathbb{T}_{X,Y,\sqrt{\tau_{N-1}}} \text{ if } i = N \\ m\mathbb{T}_{X,Y,\sqrt{\tau_{i-1}},\sqrt{\tau_i}} \text{ if } 2 \le i \le N-1, \end{cases} \end{aligned} \tag{4.70}$$

so that $\mathbb{T}_i$ is a $\sqrt{2}\pi/k \times \sqrt{2}\pi/\ell$ rectangle with one or two discs removed, each having radius of order $m\sqrt{\tau_1}$. By virtue of the second line of (3.6) we see that $\mathbb{T}_i$ tends with large m to

$$\begin{aligned} \widehat{\mathbb{T}}_i = \widehat{\mathbb{T}}_i[N, k, \ell] &:= \left[-\frac{\pi}{\sqrt{2}k}, \frac{\pi}{\sqrt{2}k} \right] \times \left[-\frac{\pi}{\sqrt{2}\ell}, \frac{\pi}{\sqrt{2}\ell} \right] \\ & \quad \Big\backslash \begin{cases} \{(0,0)\} \text{ if } i \in \{1, N\} \\ \left\{ \pm \left(\frac{\pi}{2\sqrt{2}k}, \frac{\pi}{2\sqrt{2}\ell} \right) \right\} \text{ if } 1 < i < N. \end{cases} \end{aligned} \tag{4.71}$$

Recalling (2.2), (2.7), (2.9), (2.15), and (2.16), we also define the maps $T_i = T_i[N, k, \ell, m, \zeta, \xi] : \mathbb{T}_i \to \mathbb{S}^3$ by

$$\begin{aligned} & T_i(\mathrm{x}, \mathrm{y}) := \Phi \left(\widehat{T}_i \left(\frac{\mathrm{x}}{m}, \frac{\mathrm{y}}{m} \right) \right), \quad \text{where} \\ & \widehat{T}_N := T_{ext} \left[(N-2)(X,Y), \mathrm{z}_{N-1}^K, \mathrm{z}_N, R, X, Y, \tau_{N-1} \right], \\ & \widehat{T}_1 := T_{ext} \left[(0,0), \mathrm{z}_1^K, \mathrm{z}_1, R, X, Y, \tau_1 \right], \quad \text{and for } 1 < i < N \\ & \widehat{T}_i := T_{int} \left[(2i-3) \left(\frac{X}{2}, \frac{Y}{2} \right), \mathrm{z}_{i-1}^K, \mathrm{z}_i^K, \mathrm{z}_i, R, X, Y, \tau_{i-1}, \tau_i \right]. \end{aligned} \tag{4.72}$$

Then, referring to (2.30) and particularly (2.29), the image of each T_i is entirely contained in the initial surface $\Sigma[N,k,\ell,m,\zeta,\xi]$ and defines the corresponding toral region

$$\mathcal{T}[i] = \mathcal{T}[i;N,k,\ell,m,\zeta,\xi] := T_i\left(\mathbb{T}_i\right), \tag{4.73}$$

so that T_i a diffeomorphism onto its image. Abusing notation slightly we denote the inverse of this diffeomorphism by T_i^{-1}. From (2.14), (2.15), (2.16), (2.26), (2.29), (2.30), (4.26), and (4.73) it is clear that

$$\begin{aligned}
&\mathcal{T}[i] \cap \mathcal{T}[i'] \neq \emptyset \text{ if and only if } i = i', \\
&\mathcal{T}[i] \cap \mathcal{K}[i'] \neq \emptyset \text{ if and only if } i - i' \in \{0,1\}, \\
&\bigcup_{i=1}^{N} \Omega_i = \bigcup_{i=1}^{N-1} \mathcal{K}[i] \cup \bigcup_{i=1}^{N} \mathcal{T}[i], \quad \text{and} \quad \Sigma = \bigcup_{i=1}^{N-1} \mathcal{G}\mathcal{K}[i] \cup \bigcup_{i=1}^{N} \mathcal{G}\mathcal{T}[i].
\end{aligned} \tag{4.74}$$

Each limit region $\widehat{\mathbb{T}}_i \supset \mathbb{T}_i$ naturally carries the flat metric $g_{E} = d\mathrm{x}^2 + d\mathrm{y}^2$, but we also equip it with the conformal metric

$$\begin{aligned}
&\widehat{\chi}_i = \widehat{\chi}_i[N,k,\ell] := \widehat{\rho}_i^2 g_{E}, \text{ having conformal factor} \\
&\widehat{\rho}_i = \widehat{\rho}_i[N,k,\ell] : \widehat{\mathbb{T}}_i \to (0,\infty) \text{ defined by} \\
&\widehat{\rho}_i(\mathrm{x},\mathrm{y}) := \psi\left[\frac{1}{10\ell}, \frac{1}{5\ell}\right](d_i(\mathrm{x},\mathrm{y})) \\
&\qquad\qquad + \frac{1}{d_i(\mathrm{x},\mathrm{y})} \cdot \psi\left[\frac{1}{5\ell}, \frac{1}{10\ell}\right](d_i(\mathrm{x},\mathrm{y})), \text{ where} \\
&d_i(\mathrm{x},\mathrm{y}) \text{ is the Euclidean distance in } \mathbb{R}^2 \text{ from the set} \\
&\begin{cases} \{(0,0)\} \text{ if } i \in \{1,N\} \\ \left\{\pm\left(\frac{mX}{2}, \frac{mY}{2}\right)\right\} \text{ if } 1 < i < N, \end{cases}
\end{aligned} \tag{4.75}$$

recalling (2.4). Under the $\widehat{\chi}_i$ metric $\widehat{\mathbb{T}}_i$ looks like a flat $\sqrt{2}\pi/k \times \sqrt{2}\pi/\ell$ rectangle with one or two discs of radius $1/5\ell$ replaced by one or two infinite half-cylinders of radius 1, each attached smoothly along an annulus. We emphasize that $\widehat{\rho}_i$ is independent of m as well as the parameters ζ, ξ, and, in view of (4.14), we observe that on each domain $\mathbb{T}_i$

$$T_i^*\rho = m\widehat{\rho}_i. \tag{4.76}$$

In the next section we will define the *extended substitute kernel* needed to complete the construction, as outlined in Section 1. Then, in the final section, the role of the dislocations will become clear: the dislocation $\mathcal{D}_i$ (recalling (3.32)) on the toral region $\mathcal{T}[i]$ will be varied to cancel the "extended" portion of the extended substitute kernel supported there. For this reason it is necessary to isolate the dominant contribution of each dislocation to the mean curvature, and to that end for $2 \leq i \leq N-1$

we define $v_i \in C^\infty(\widehat{\mathbb{T}}_i)$ by

$$\begin{aligned} v_i(\mathrm{x},\mathrm{y}) :=&\psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right]\left(\sqrt{\left(\mathrm{x}-\frac{mX}{2}\right)^2+\left(\mathrm{y}-\frac{mY}{2}\right)^2}\right) \\ &-\psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right]\left(\sqrt{\left(\mathrm{x}+\frac{mX}{2}\right)^2+\left(\mathrm{y}+\frac{mY}{2}\right)^2}\right) \end{aligned} \tag{4.77}$$

and we define $\underline{w}_i \in C^\infty_{\mathcal{G}}(\Sigma)$ to be the unique $\mathcal{G}$-invariant (recalling (2.26) and (4.4)) function satisfying

$$\begin{aligned} &\underline{w}_i|_{\Sigma\setminus\mathcal{G}\mathcal{T}[i]} := 0 \qquad \text{and} \\ &T_i^*\underline{w}_i := \begin{cases} (-1)^{N-i}\left(\Delta_{\widehat{\chi}_i}v_i\right)\big|_{\mathbb{T}_i} \text{ if } 1<i<N \\ 0 \text{ if } i\in\{1,N\}, \end{cases} \end{aligned} \tag{4.78}$$

the alternating sign included to account for the alternating direction of the unit normal on the toral regions and the exceptional cases $i=1$ and $i=N$ included merely for convenience of notation. (We could have alternatively built the alternating sign into the definition of the dislocations.) The function $v_i|_{\mathbb{T}_i}\circ T_i^{-1}$ should be regarded as the section of the normal bundle graphically generating dislocations on $\mathcal{T}[i]$, and in the following proposition we will see that the function $\underline{w}_i$ then captures the principal effect of dislocation on the mean curvature. Later the collection $\{\underline{w}_i\}_{i=2}^{N-1}$ will reappear as the defining basis for the extended part of the extended substitute kernel. Right now we estimate the geometry of the toral regions.

Proposition 4.79 (Estimates of the geometry of the toral regions)**.** *Given a real number $c>0$ and integers $N\geq 2$, $\ell\geq k\geq 1$, and $j\geq 0$, there exist real numbers $m_0=m_0[N,k,\ell,c]>0$ and $C=C[N,k,\ell,j]>0$ such that whenever $\zeta,\xi\in[-c,c]^{N-1}$ and $m>m_0$, for $1\leq i\leq N$*

(i) $\left\|\chi - T_i^{-1*}\widehat{\chi}_i : C^j\left(T^*\mathcal{T}[i]^{\otimes 2},\chi\right)\right\| \leq Cm^2\tau_1$;
(ii) $\left\|\rho : C^j(\mathcal{T}[i],\chi,\rho)\right\| + \left\|\rho^{-1} : C^j(\mathcal{T}[i],\chi,\rho)\right\| \leq C$,
(iii) $\left\|A-(-1)^{N-i}m^{-2}T_i^{-1*}(d\mathrm{y}^2-d\mathrm{x}^2) : C^j\left(\mathcal{T}[i]\setminus\left(\mathcal{K}[i-1]\cup\mathcal{K}[i]\right),\chi\right)\right\| \leq Cm\tau_1$,
(iv) $\left\|\rho^{-2}\left|A\right|_g^2 : C^j\left(\mathcal{T}[i],\chi\right)\right\| \leq Cm^{-2}$, *and*
(v) $\left\|\rho^{-2}H-\mathcal{D}_i\underline{w}_i : C^j\left(\mathcal{T}[i],\chi,m^2\rho^{-2}\tau_1+m^2\tau_1^2\right)\right\| \leq C$,

recalling (2.14), (3.32), (4.6), (4.7), (4.8), (4.9), (4.14), (4.72), (4.73), (4.75), *and* (4.78).

Proof. We first observe, using (4.17), (4.25), (4.27), (4.72), (4.75), and (4.76), that if $\mathcal{T}[i] \cap \mathcal{K}[i'] \neq \emptyset$, then

$$\widehat{\chi}_K|_{\kappa_{i'}^{-1}(\mathcal{T}[i])} - \kappa_{i'}^* T_i^{-1*} \widehat{\chi}_i\Big|_{\mathcal{T}[i]\cap\mathcal{K}[i']} = \operatorname{sech}^2 t\, dt^2 \\ = \tau_{i'}^2 \left(\rho \circ \kappa_{i'}\right)^2 \, dt^2, \tag{4.80}$$

but $\rho|_{\kappa_{i'}^{-1}(\mathcal{T}[i])} \leq \tau_{i'}^{-1/2}$ by (4.73), so by applying items (i) and (ii) of Proposition 4.30 we have proven item (i) of the present proposition on the overlap of the toral and catenoidal regions. On this same intersection the remaining items (with (iii) obviously excluded) also follow from the corresponding ones in Proposition 4.30.

We will finish the proof by verifying the estimates on $\mathcal{T}[i] \cap \{\rho \leq 10\ell m\} = \mathcal{T}[i] \backslash (\mathcal{K}[i-1] \cup \mathcal{K}[i])$ (understanding $\mathcal{K}[0] = \mathcal{K}[N] = \emptyset$). For this we set

$$D_i := T_i^{-1}\left(\{\rho \leq 10\ell m\}\right) \subset \mathbb{T}_i \tag{4.81}$$

and apply Lemma 4.42, viewing $T_i|_{D_i}$ as a perturbation $\phi_i[u_i]$ of the embedding $\phi_i := \varpi \circ T_i : D_i \to (\mathbb{S}^3, m^2 g_S)$ of D_i into the Clifford torus $\mathbb{T}$ with $m^2 g_S$ unit normal $\nu[\phi_i]$ directed toward C_1; here $\varpi : \mathbb{S}^3 \backslash (C_1 \cup C_2) \to \mathbb{T}$ is nearest-point projection in $(\mathbb{S}^3, m^2 g_S)$ onto $\mathbb{T}$ and the function u_i generating the perturbation is identified below. Thus, recalling (2.2) and (2.3),

$$\begin{aligned} &\varpi\left(\Phi(\mathrm{x},\mathrm{y},\mathrm{z})\right) = \Phi(\mathrm{x},\mathrm{y},0), \\ &\phi_i(\mathrm{x},\mathrm{y}) = \Phi\left(\frac{\mathrm{x}}{m} + \mathrm{x}_i, \frac{\mathrm{y}}{m} + \mathrm{y}_i, 0\right), \qquad \phi_i^* m^2 g_S = g_E, \\ &\nu[\phi_i]|_{(\mathrm{x},\mathrm{y})} = m^{-1}\Phi_* \partial_{\mathrm{z}}\big|_{\left(\frac{\mathrm{x}}{m}+\mathrm{x}_i, \frac{\mathrm{y}}{m}+\mathrm{y}_i, 0\right)} \qquad \text{and} \\ &\nu[\phi_i[u_i]]|_{(\mathrm{x},\mathrm{y})} = (-1)^{N-i} m^{-1} \nu|_{T_i(\mathrm{x},\mathrm{y})}, \end{aligned} \tag{4.82}$$

where $\mathrm{x}_i, \mathrm{y}_i$ give the appropriate lattice site appearing in (4.72), $\nu[\phi_i[u_i]]$ is the $m^2 g_S$ unit normal for $\phi_i[u_i]$ specified in Lemma 4.42, and ν is the g_S unit normal we chose for Σ just above (4.8). Writing $A[u_i]$ and $H[u_i]$ for the second fundamental form and mean curvature of $\phi_i[u_i]$ relative to $m^2 g_S$ and $\nu[\phi_i[u_i]]$, as in Lemma 4.42, and recalling (4.8) and the definition of $\iota : \Sigma \to \mathbb{S}^3$ as the inclusion map of the initial surface in $\mathbb{S}^3$, it follows that

$$\begin{aligned} &\iota^* g_S|_{T_i(D_i)} = m^{-2} \phi_i[u_i]^* m^2 g_S, \\ &A|_{T_i(D_i)} = (-1)^{N-i} m^{-1} A[u_i], \text{ and} \\ &H|_{T_i(D_i)} = (-1)^{N-i} m H[u_i]. \end{aligned} \tag{4.83}$$

Setting

$$\mathrm{r}_0(\mathrm{x},\mathrm{y}) := \sqrt{\mathrm{x}^2 + \mathrm{y}^2} \text{ and } \mathrm{r}_{\pm 1}(\mathrm{x},\mathrm{y}) := \mathrm{r}_0\left(\mathrm{x} \mp \frac{mX}{2}, \mathrm{y} \mp \frac{mY}{2}\right) \tag{4.84}$$

and referring to (4.72) and the supporting definitions (including in particular (2.15) and (2.16)) and recalling (3.32), we have as the function generating the perturbation (that is playing the role of u in the statement of Lemma 4.42) $u_i : D_i \to \mathbb{R}$ given by

$$\begin{aligned}\frac{u_1}{m}(\mathrm{x},\mathrm{y}) =& \mathrm{z}_1 + \psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right](\mathrm{r}_0(\mathrm{x},\mathrm{y})) \\ &\times \tau_1\left(\ln\frac{1}{10\ell m\tau_1} - \operatorname{arcosh}\frac{\mathrm{r}_0(\mathrm{x},\mathrm{y})}{m\tau_1}\right),\end{aligned} \tag{4.85}$$

$$\begin{aligned}\frac{u_N}{m}(\mathrm{x},\mathrm{y}) =& \mathrm{z}_N + \psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right](\mathrm{r}_0(\mathrm{x},\mathrm{y})) \\ &\times \tau_{N-1}\left(\operatorname{arcosh}\frac{\mathrm{r}_0(\mathrm{x},\mathrm{y})}{m\tau_{N-1}} - \ln\frac{1}{10\ell m\tau_{N-1}}\right),\end{aligned} \tag{4.86}$$

and for $1 < i < N$

$$\begin{aligned}\frac{u_i}{m}(\mathrm{x},\mathrm{y}) =& \mathrm{z}_i + \psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right](\mathrm{r}_{-1}(\mathrm{x},\mathrm{y})) \\ &\times\left(-\mathcal{D}_i + \tau_{i-1}\operatorname{arcosh}\frac{\mathrm{r}_{-1}(\mathrm{x},\mathrm{y})}{m\tau_{i-1}} - \tau_{i-1}\ln\frac{1}{10\ell m\tau_{i-1}}\right) \\ &+ \psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right](\mathrm{r}_1(\mathrm{x},\mathrm{y})) \\ &\times\left(\mathcal{D}_i + \tau_i\ln\frac{1}{10\ell m\tau_i} - \tau_i\operatorname{arcosh}\frac{\mathrm{r}_1(\mathrm{x},\mathrm{y})}{m\tau_i}\right).\end{aligned} \tag{4.87}$$

Using (2.21), the last item of (3.6), the inequality $|\mathcal{D}_i| \leq C[N,k,\ell]m\tau_1$ (by (2.12) and (2.14) whenever $m \geq c$), along with the estimates

$$\begin{aligned}&\left\|\psi\left[\frac{1}{5\ell},\frac{1}{10\ell}\right]\circ\mathrm{r}_0 : C^j(T^*D_1{}^{\otimes j}, g_{\scriptscriptstyle E})\right\| \leq C[\ell,j], \\ &\left\|D[g_{\scriptscriptstyle E}]^j\operatorname{arcosh}\frac{\mathrm{r}_0}{m\tau_i} : C^0(T^*D_1{}^{\otimes j}, g_{\scriptscriptstyle E})\right\| \leq C[\ell,j] \text{ for } j>0, \\ &\text{and } \sum_{i=1}^{N-1}\left\|\tau_i\operatorname{arcosh}\frac{\mathrm{r}_0}{m\tau_i} : C^0(D_1)\right\| \leq C[N,k,\ell]m^2\tau_1\end{aligned} \tag{4.88}$$

(again using (2.12) and (2.14) for the third line), we obtain

$$\begin{aligned}&\left\|u_i : C^0(D_i)\right\| \leq C[N,k,\ell]m^3\tau_1 \text{ and} \\ &\left\|D[g_{\scriptscriptstyle E}]^j u_i : C^0(T^*D_i^{\otimes j}, g_{\scriptscriptstyle E})\right\| \leq C[N,k,\ell,j]m^2\tau_1 \text{ for } j \geq 1.\end{aligned} \tag{4.89}$$

Using (2.3) and recalling the notation of Lemma 4.42 (remembering in particular that we are taking g in its statement to be $m^2 g_{\scriptscriptstyle S}$) we also have

$$\begin{aligned}&g^{u_i} = (1+\sin 2m^{-1}u_i)\,d\mathrm{x}^2 + (1-\sin 2m^{-1}u_i)\,d\mathrm{y}^2, \\ &\text{so by (4.89) } \left\|g^{u_i} - g_{\scriptscriptstyle E} : C^j\left(T^*D_i^{\otimes 2}, g_{\scriptscriptstyle E}\right)\right\| \leq C[N,k,\ell,j]m^2\tau_1.\end{aligned} \tag{4.90}$$

Item (ii) of Lemma 4.42 now yields

$$\left\| m^2 T_i^* g_S - g_E : C^j \left(T^* D_i^{\otimes 2}, g_E \right) \right\| \leq C[N,k,\ell,j] m^2 \tau_1. \tag{4.91}$$

The proofs of (i) and (ii) are now completed by (4.75), (4.76), and the observation that

$$\left\| \widehat{\rho}_i : C^j(D_i, g_E) \right\| + \left\| \widehat{\rho}_i^{-1} : C^j(D_i, g_E) \right\| \leq C[\ell, j]. \tag{4.92}$$

Furthermore, again using (2.3) and the notation of Lemma 4.42,

$$\begin{aligned} &A^{u_i} = -m^{-1} \left(\cos 2m^{-1} u_i \right) \left(d\mathrm{x}^2 - d\mathrm{y}^2 \right), \text{ so by (4.89)}, \\ &\left\| A^{u_i} - m^{-1}(d\mathrm{y}^2 - d\mathrm{x}^2) : C^j \left(T^* D_i^{\otimes 2}, g_E \right) \right\| \leq C[N,k,\ell,j] m \tau_1, \end{aligned} \tag{4.93}$$

while from (4.89) and (4.90)

$$\begin{aligned} C[N,k,\ell,j] m^2 \tau_1 \geq &\left\| D[g^{u_i}]^2 u_i : C^j \left(T^* D_i^{\otimes 2}, g_E \right) \right\| \\ &+ \left\| |du_i|_{g^{u_i}} : C^j \left(D_i, g_E \right) \right\|. \end{aligned} \tag{4.94}$$

Item (iii) of the present proposition is now proved by applying item (i), (4.93), and (4.94) (and (4.89) again) in item (iii) of Lemma 4.42, keeping in mind (4.83). Item (iv) follows in turn.

Finally, from the identity

$$\Delta_{g_E} \operatorname{arcosh} \frac{\mathrm{r}_0}{m \tau_i} = - \frac{m^2 \tau_i^2}{\mathrm{r}_0 \left(\mathrm{r}_0^2 - m^2 \tau_i^2 \right)^{3/2}} \tag{4.95}$$

along with (2.21), the first two estimates of (4.88), definition (4.78), and the fact that $\Delta_{\widehat{\chi}_i} = \widehat{\rho}_i^{-2} \Delta_{g_E}$ (by (4.75) and the two-dimensionality of $\widehat{\mathbb{T}}_i$) we find

$$\begin{aligned} \left\| \Delta_{g_E} u_i - (-1)^{N-i} m \widehat{\rho}_i^2 \mathcal{D}_i T_i^* \underline{w}_i : C^j \left(D_i, g_E \right) \right\| \\ \leq C[N,k,\ell,j] m \tau_1. \end{aligned} \tag{4.96}$$

(Note that without subtracting the dislocation term on the left it would be necessary to allow C on the right-hand side of (4.96) to depend on c, or, if we were to apply the assumption we have used repeatedly that $c \leq m$, to allow the exponent on m to increase.) We now apply item (iv) of Lemma 4.42. In doing so we make use of (4.96), (4.89), (4.90), (4.93), and (4.94); we also take note of Remark 4.45 and of course the facts that $\mathbb{T}$ itself is minimal and $m^2 \tau_1 < 1$. We thereby obtain

$$\left\| H[u_i] - (-1)^{N-i} m \widehat{\rho}_i^2 \mathcal{D}_i T_i^* \underline{w}_i : C^j (D_i, g_E) \right\| \leq C[N,k,\ell,j] m \tau_1, \tag{4.97}$$

and the proof is completed by (4.76) and (4.83). q.e.d.

Decay norms and a global estimate of the mean curvature. As mentioned in Section 1, because the characteristic size τ_1 of the catenoidal waists is so much smaller than the characteristic size m^{-1} of the toral regions, we must allow perturbing functions to be much larger on the toral regions than on the core of the catenoidal regions. For this reason we will weight our norms by powers of the factor $m\rho^{-1}$, which takes the value 1 a maximal distance from the catenoidal regions and is of order $m\tau_1$ at the waists. Specifically, for each $\alpha \in (0,1)$, $\gamma \in [0,\infty)$, and nonnegative integer j, we define the norm

$$\|\cdot\|_{j,\alpha,\gamma} = \|\cdot\|_{C^{j,\alpha,\gamma}(\Sigma)} := \left\| \cdot : C^{j,\alpha}\left(\Sigma, \chi, \frac{m^\gamma}{\rho^\gamma}\right) \right\|, \tag{4.98}$$

(recalling (4.6) and (4.14)) and the corresponding Banach space along with its (closed) $\mathcal{G}$-invariant subspace (recalling (2.26) and (4.4))

$$\begin{aligned} &C^{j,\alpha,\gamma}(\Sigma) := \left\{ u \in C^{j,\alpha}(\Sigma,\chi) \;\middle|\; \|u\|_{j,\alpha,\gamma} < \infty \right\} \text{ and} \\ &C^{j,\alpha,\gamma}_{\mathcal{G}}(\Sigma) := \left\{ u \in C^{j,\alpha,\gamma}(\Sigma) \;\middle|\; \mathfrak{g}.u = u \text{ for all } \mathfrak{g} \in \mathcal{G} \right\}, \end{aligned} \tag{4.99}$$

in accordance with Notation 4.5.

Remark 4.100. Of course, since each initial surface Σ is compact, $C^{j,\alpha,\gamma}(\Sigma)$, $C^{j,\alpha}(\Sigma,\chi)$, and $C^{j,\alpha}_{loc}(\Sigma)$ all refer to the same topological vector space, which we more simply call $C^{j,\alpha}(\Sigma)$ (forgetting the norm structures of the first two spaces and dropping the superfluous subscript of the third).

Definition 4.101 (Continuity in the parameters)**.** If

$$f = f[N,k,\ell,m,\zeta,\xi] \in C^{j,\alpha}(\Sigma[N,k,\ell,m,\zeta,\xi]) \tag{4.102}$$

defines a family of functions on the initial surfaces and we make the usual assumption that $\zeta,\xi \in [-c,c]^{N-1}$ for some $c>0$, we say that f *depends continuously on* (ζ,ξ) if we have continuity of the map

$$\begin{aligned} &[-c,c]^{N-1} \times [-c,c]^{N-1} \to C^{j,\alpha}(\Sigma[N,k,\ell,m,0,0]) \\ &(\zeta,\xi) \to f[N,k,\ell,m,\zeta,\xi] \circ I[N,k,\ell,m](\zeta,\xi,\cdot), \end{aligned} \tag{4.103}$$

where $I = I[N,k,\ell,m]$ is as described in Remark 2.33. Note that this definition does not depend on the particular choice of I.

Similarly, if $\mathcal{A}[\zeta,\xi] = \mathcal{A}[N,k,\ell,m,\zeta,\xi] : C^{J,\alpha}(\Sigma[N,k,\ell,m,\zeta,\xi]) \to C^{j,\alpha}(\Sigma[N,k,\ell,m,\zeta,\xi])$ (or $\mathbb{R}$) is a family of (not necessarily linear) continuous maps, we call the associated map $(u,\zeta,\xi) \mapsto \mathcal{A}[\zeta,\xi][u]$ *continuous* for fixed N, k, ℓ, and m if we have continuity of the map

$$\begin{aligned} &C^{J,\alpha}(\Sigma[N,k,\ell,m,0,0]) \times [-c,c]^{N-1} \times [-c,c]^{N-1} \\ &\qquad \to C^{j,\alpha}(\Sigma[N,k,\ell,m,0,0]) \text{ or } \mathbb{R} \end{aligned} \tag{4.104}$$

given by

$$(4.105)\qquad \begin{aligned} &(u,\zeta,\xi)\mapsto I(\zeta,\xi,\cdot)^*\left(\mathcal{A}[\zeta,\xi]\left(I(\zeta,\xi,\cdot)^{-1^*}u\right)\right)\\ \text{or}\quad &(u,\zeta,\xi)\mapsto \mathcal{A}[\zeta,\xi]\left(I(\zeta,\xi,\cdot)^{-1^*}u\right). \end{aligned}$$

In order to secure acceptable decay estimates for solutions to the linearized problem we will need the following estimate for the initial mean curvature.

Corollary 4.106 (Global weighted estimate of the initial mean curvature)**.** *Given real numbers $c>0$ and $\gamma\in(0,1)$ as well as integers $N\geq 2$ and $\ell\geq k\geq 1$, there exist $C=C[N,k,\ell]>0$ and $m_0=m_0[N,k,\ell,c,\gamma]$ such that for each integer $m>m_0$, each $\zeta,\xi\in[-c,c]^{N-1}$, and each $\alpha\in(0,1)$*

$$(4.107)\qquad \left\|\rho^{-2}H-\sum_{i=2}^{N-1}\mathcal{D}_i\underline{w}_i\right\|_{0,\alpha,\gamma}\leq C\tau_1,$$

using the norm (4.98) and recalling (3.32), (4.8), (4.14), and (4.78). Moreover $\rho^{-2}H$ is $\mathcal{G}$-invariant (recalling (2.26) and (4.4)) and depends continuously, as an element of $C^{0,\alpha,\gamma}(\Sigma)$, on (ζ,ξ) in the sense of Definition 4.101.

Proof. The continuity claim is obvious, as in fact both $\rho\circ I$ and $H\circ I$ are manifestly smooth. The $\mathcal{G}$-invariance of ρ also follows directly from its definition, while that of H follows from the $\mathcal{G}$-invariance of Σ itself, establishing that $\rho^{-2}H$ is $\mathcal{G}$-invariant as well. (Note that in this construction all elements of our symmetry group $\mathcal{G}$ act on all functions under consideration according to the first line of (4.4). Of course, were we to enforce also symmetries reversing the sides of Σ, it would be natural to consider a different action of $\mathcal{G}$ on ρ from that defined by (4.4) on H, since the former represents a true scalar field, while H represents a section of the normal bundle. Specifically, the appropriate action on ρ would be to follow the first line of (4.4) even for elements reversing the sides of Σ. In this case too we would conclude the appropriate $\mathcal{G}$-equivariance of $\rho^{-2}H$.) As for the estimate, from item (v) of Proposition 4.79 and items (iii) and (vi) of Proposition 4.30 we get

$$(4.108)\qquad \left\|\rho^{-2}H-\sum_{i=2}^{N-1}\mathcal{D}_i\underline{w}_i : C^1\left(\Sigma,\chi,m^2\rho^{-2}\tau_1+m^2\tau_1^2\right)\right\| \leq C[N,k,\ell]$$

for some constant $C[N,k,\ell] > 0$ whenever m is sufficiently large in terms of N, k, ℓ, and c, but

$$\begin{aligned}\frac{m^2\rho^{-2}\tau_1 + m^2\tau_1^2}{m^\gamma\rho^{-\gamma}\tau_1} &= \left(\frac{m}{\rho}\right)^{2-\gamma} + m^{2-\gamma}\tau_1\rho^\gamma \\ &\leq \left(\frac{m}{\underline{m}}\right)^{2-\gamma} + m^{2-\gamma}\tau_1^{1-\gamma},\end{aligned} \tag{4.109}$$

using (4.15) for the last inequality, and the estimate now follows from the second item of (3.6). q.e.d.

5. The linearized operator

We continue to write $\iota = \iota[N,k,\ell,m,\zeta,\xi] : \Sigma[N,k,\ell,m,\zeta,\xi] \to \mathbb{S}^3$ for the inclusion map of the initial surface $\Sigma = \Sigma[N,k,\ell,m,\zeta,\xi]$ in $\mathbb{S}^3$ and $\nu = \nu[N,k,\ell,m,\zeta,\xi] : \Sigma \to \iota^*T\mathbb{S}^3$ for the unit normal which points toward C_1 at the points of Σ nearest to C_1 (or equivalently which points upward at the top of Σ as viewed via coordinates obtained through the map Φ defined in (2.2)). Fixing the data $N \geq 2$, $\ell \geq k \geq 1$, and m sufficiently large, we consider deformations of $\iota[N,k,\ell,m,0,0]$ obtained by varying the parameters ζ and ξ and by additionally perturbing the resulting initial surface $\Sigma[N,k,\ell,m,\zeta,\xi]$ in the normal direction by a prescribed function. Specifically we define $\iota[N,k,\ell,m,\zeta,\xi,u] : \Sigma[N,k,\ell,m,\zeta,\xi] \to \mathbb{S}^3$ by

$$\begin{aligned}\iota[u] &= \iota[N,k,\ell,m,\zeta,\xi,u](p) \\ &:= \exp_{\iota[N,k,\ell,m,\zeta,\xi](p)} u(p)\nu[N,k,\ell,m,\zeta,\xi](p),\end{aligned} \tag{5.1}$$

where $\exp : T\mathbb{S}^3 \to \mathbb{S}^3$ is the exponential map for $(\mathbb{S}^3, g_S)$. As asserted in Lemma 4.42, $\iota[N,k,\ell,m,\zeta,\xi,u]$ is an immersion for sufficiently small u. In this case we write $\nu[N,k,\ell,m,\zeta,\xi,u] : \Sigma \to \iota[\zeta,\xi,u]^*T\mathbb{S}^3$ for the unit normal of $\iota[N,k,\ell,m,\zeta,\xi,u]$ whose value at each $p \in \Sigma$ has positive inner product with the vector $\frac{d}{dt}\exp_{\iota[N,k,\ell,m,\zeta,\xi](p)} t\nu[N,k,\ell,m,\zeta,\xi](p)$ and we write $\mathcal{H}[u] = \mathcal{H}[u,\zeta,\xi] = \mathcal{H}[N,k,\ell,m,\zeta,\xi,u] : \Sigma[N,k,\ell,m,\zeta,\xi] \to \mathbb{R}$ for the corresponding mean curvature

$$\begin{aligned}\mathcal{H}[u] = \mathcal{H}[u,\zeta,\xi] &= \mathcal{H}[N,k,\ell,m,\zeta,\xi,u] \\ &:= H[\iota[N,k,\ell,m,\zeta,\xi,u], \nu[N,k,\ell,m,\zeta,\xi,u]],\end{aligned} \tag{5.2}$$

with the notation and conventions introduced just below (4.41).

Our goal is to find $\zeta,\xi \in \mathbb{R}^{N-1}$ and $u \in C^\infty_{\mathcal{G}}(\Sigma)$ (recalling Notation 4.5) solving

$$\mathcal{H}[N,k,\ell,m,\zeta,\xi,u] = 0 \tag{5.3}$$

for each given $N \geq 2$, $\ell \geq k \geq 1$, and m sufficiently large, with u small enough that the resulting minimal surface (the image of $\iota[N,k,\ell,m,\zeta,\xi,u]$) is a small perturbation of the initial surface $\Sigma[N,k,\ell,m,\zeta,\xi]$, so in particular embedded. A major step toward the solution of (5.3) consists in the study of the initial surface's Jacobi operator $\mathcal{L} = \mathcal{L}[N, k, \ell, m, \zeta, \xi]$ defined by

$$\mathcal{L}u = \frac{d}{dt}\bigg|_{t=0} \mathcal{H}[N, k, \ell, m, \zeta, \xi, tu] = \left(\Delta_g + |A|_g^2 + 2\right) u, \tag{5.4}$$

recalling that $g = \iota[N, k, \ell, m, \zeta, \xi]^* g_{S}$. Actually, because of the uniformity afforded by the χ metric (4.9), it is much more convenient to study instead the linear operator

$$\mathcal{L}_\chi = \mathcal{L}_\chi[N, k, \ell, m, \zeta, \xi] := \rho^{-2}\mathcal{L} = \Delta_\chi + \rho^{-2} |A|_g^2 + 2\rho^{-2}, \tag{5.5}$$

which clearly takes $\mathcal{G}$-equivariant functions (as defined by (2.26) and (4.4)) to $\mathcal{G}$-equivariant functions and which, by virtue of the estimates of $\rho^{-2} |A|_g^2$ in Propositions 4.30 and 4.79, defines (for any $\alpha, \gamma \in (0, 1)$) a linear map $\mathcal{L}_\chi : C^{2,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \to C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma)$ bounded independently of m and c.

In this section we construct a likewise bounded right inverse $\mathcal{R}$ to $\mathcal{L}_\chi$, modulo the extended substitute kernel described in Section 1 and formally defined below. We do this by first analyzing $\mathcal{L}_\chi$ "semilocally", meaning on the toral and catenoidal regions individually, and by observing that on each of these regions $\mathcal{L}_\chi$ has a simple limit as $m \to \infty$. Significantly, because adjacent toral and catenoidal regions overlap, when attempting to solve the equation $\mathcal{L}_\chi u = f$ on a toral region $\mathcal{T}[i]$, we may assume that f is supported outside the intersection of $\mathcal{T}[i]$ with the adjoining catenoidal region(s). We will find we can invert these regional limits of $\mathcal{L}_\chi$ (modulo extended substitute kernel in the toral cases) and so produce approximate semilocal inverses to $\mathcal{L}_\chi$, which will be applied iteratively, using decay properties of the solutions they yield, to construct $\mathcal{R}$.

Approximate solutions on the catenoidal regions. Recalling (4.16) and (4.17), we define the operator

$$\widehat{\mathcal{L}}_K := \Delta_{\widehat{\chi}_K} + 2\operatorname{sech}^2 t \tag{5.6}$$

on functions on $\mathbb{K}$. Note that $\widehat{\mathcal{L}}_K$ is simply $\cosh^2 t$ times the Jacobi operator of the standard catenoid (4.18). From items (i) and (v) of Proposition 4.30 we see that (recalling (5.5))

$$\lim_{m\to\infty} {\kappa_i}^* \mathcal{L}_\chi {\kappa_i}^{*-1} = \widehat{\mathcal{L}}_K, \tag{5.7}$$

where the convergence is to be interpreted in the following sense. For any given bounded subset Ω of the cylinder $\mathbb{K}$ the operator on the left-hand side of (5.7) is defined as a map $C^2_{loc}(\Omega) \to C^0_{loc}(\Omega)$ whenever m is taken sufficiently large in terms of the diameter of Ω and $|\zeta|$ (noting $\lim_{m\to\infty} a_i = \infty$ by (4.24)) and its difference from the operator on the right-hand side is a first-order operator $X[m] + f[m]$ on Ω satisfying $\lim_{m\to\infty}\left(\left\|X[m] : C^j(T\Omega, \widehat{\chi}_K)\right\| + \left\|f[m] : C^j(\Omega, \widehat{\chi}_K)\right\|\right) = 0$ for each nonnegative integer j.

Recall that each catenoidal region $\mathcal{K}[i]$ is defined in (4.26) via (4.25) as the image under Φ (2.2) of a certain catenoid in $\mathbb{R}^3$. Of course this last catenoid has an axis of symmetry, a line in $\mathbb{R}^3$ whose intersection with the domain of Φ has image under Φ a quarter great circle in $\mathbb{S}^3$, which circle (at least in this paragraph) we will call *the axis of* $\mathcal{K}[i]$. It follows from (2.25), (2.26), and (4.26) that the subgroup of $\mathcal{G}$ preserving a given catenoidal region $\mathcal{K}[i]$ as a set is isomorphic to the dihedral group D_2 of order 4 (also isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ of course, but we favor the more concise and geometric nomenclature), consisting of (i) the identity element I of $O(4)$, (ii) reflection $\underline{\mathsf{X}}_i$ through the great sphere containing C_2 and the axis of $\mathcal{K}[i]$, (iii) reflection $\underline{\mathsf{Y}}_i$ through the great sphere containing C_1 and the axis of $\mathcal{K}[i]$, and (iv) rotation $\underline{\mathsf{X}}_i\underline{\mathsf{Y}}_i = \underline{\mathsf{Y}}_i\underline{\mathsf{X}}_i$ through angle π (also called reflection) through the axis of $\mathcal{K}[i]$.

Using (2.25) again, we see that κ_i intertwines the above D_2 action on $\mathcal{K}[i]$ with the natural action of the D_2 subgroup of symmetries of $(\mathbb{K}, \widehat{\chi}_K)$

$$\begin{gathered}\mathcal{G}_K := \left\{\widehat{I}_K, \widehat{\underline{\mathsf{X}}}_K, \widehat{\underline{\mathsf{Y}}}_K, \widehat{\underline{\mathsf{X}}}_K\widehat{\underline{\mathsf{Y}}}_K\right\}, \quad \text{where} \\ \widehat{I}_K(t,\theta) := (t,\theta),\ \widehat{\underline{\mathsf{X}}}_K(t,\theta) := (t, \pi-\theta),\ \text{and } \widehat{\underline{\mathsf{Y}}}_K(t,\theta) := (t,-\theta),\end{gathered} \tag{5.8}$$

in the sense that

$$\kappa_i \circ \widehat{\underline{\mathsf{X}}}_K = \underline{\mathsf{X}}_i \circ \kappa_i \qquad \text{and} \qquad \kappa_i \circ \widehat{\underline{\mathsf{Y}}}_K = \underline{\mathsf{Y}}_i \circ \kappa_i \tag{5.9}$$

(and these elements generate the two groups). Since I, $\widehat{\underline{\mathsf{X}}}$, and $\widehat{\underline{\mathsf{Y}}}$ all preserve each side (choice of unit normal) of Σ, the natural action (recalling (4.4)) of any element $\mathfrak{g} \in \mathcal{G}_K$ on a function f on $\mathbb{K}$ is simply $\mathfrak{g}.f = f \circ \mathfrak{g}$.

Next, having in mind (4.98) and using (2.12), (2.14), and (4.27), we also note that on each $\mathbb{K}_{a_i}$

$$C[N,k,\ell]^{-1} m\tau_1 \leq \frac{m{\kappa_i}^*\rho^{-1}}{e^{|t|}} \leq C[N,k,\ell] m\tau_1, \tag{5.10}$$

so that the pullback to $\mathbb{K}_{a_i}$ by κ_i of the weight $m^\gamma\rho^{-\gamma}$ appearing in our global norm (4.98) on Σ is comparable to the weight $m^\gamma\tau_1^\gamma e^{\gamma|t|}$ on $\mathbb{K}$, where obviously the first two factors are constant on $\mathbb{K}$. All the above considerations motivate us to introduce, for each nonnegative integer j and $\alpha \in [0,1)$ and $\gamma \in (0,1)$ the norms

$$\|\cdot\|_{C^{j,\alpha,\gamma}(\mathbb{K})} := \left\|\cdot, C^{j,\alpha}\left(\mathbb{K}, \widehat{\chi}_K, e^{\gamma|t|}\right)\right\| \tag{5.11}$$

(recalling (4.6)) and corresponding Banach spaces of $\mathcal{G}_K$-even functions

$$C^{j,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K}) := \left\{ u \in C^{j,\alpha}(\mathbb{K}, \widehat{\chi}_K) \;\middle|\; \|u\|_{C^{j,\alpha,\gamma}(\mathbb{K})} < \infty \right. \\ \left. \text{and } \ \mathfrak{g}.u = u \text{ for all } \mathfrak{g} \in \mathcal{G}_K \right\}. \tag{5.12}$$

Clearly $\widehat{\mathcal{L}}_K : C^{2,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K}) \to C^{0,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K})$ is bounded (independently of $\alpha, \gamma \in (0,1)$). The following proposition presents a suitable inverse.

Proposition 5.13 (Solutions to the model problem on the catenoid)**.** *There exists a linear map*

$$\widehat{\mathcal{R}}_K : C^{0,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K}) \to C^{2,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K}) \tag{5.14}$$

defined for all $\alpha, \gamma \in (0,1)$*, and, given any* $\alpha, \gamma \in (0,1)$*, there exists a constant* $C = C[\alpha,\gamma] > 0$ *such that whenever* $f \in C^{0,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K})$*,*

$$\widehat{\mathcal{L}}_K \widehat{\mathcal{R}}_K f = f \qquad \text{and} \qquad \left\| \widehat{\mathcal{R}}_K f \right\|_{C^{2,\alpha,\gamma}(\mathbb{K})} \le C \, \|f\|_{C^{0,\alpha,\gamma}(\mathbb{K})}, \tag{5.15}$$

recalling (4.16)*,* (5.6)*,* (5.8)*,* (5.11)*, and* (5.12)*.*

Proof. Let $f \in C^{0,\alpha,\gamma}_{\mathcal{G}_K}(\mathbb{K})$ for some $\alpha, \gamma \in (0,1)$. For each nonnegative integer n we define the functions $f^{\pm}_n : \mathbb{R} \to \mathbb{R}$ by

$$f^+_n(t) := \int_0^{2\pi} f(t,\theta) \cos n\theta \, d\theta \qquad \text{and} \\ f^-_n(t) := \int_0^{2\pi} f(t,\theta) \sin n\theta \, d\theta, \tag{5.16}$$

but f is $\mathcal{G}_K$-even, so by (5.8) and (5.12) $f^-_n(t) \equiv 0$ for every n and $f^+_n(t) \equiv 0$ for every odd n, so that

$$f(t,\theta) = \frac{1}{2\pi} f^+_0(t) + \frac{1}{\pi} \sum_{n=1}^{\infty} f^+_{2n}(t) \cos 2n\theta, \tag{5.17}$$

at least distributionally. From the factorizations

$$\partial_t^2 + 2\operatorname{sech}^2 t - 1 = (\partial_t - \tanh t)(\partial_t + \tanh t) \qquad \text{and} \\ \partial_t^2 - 1 = (\partial_t + \tanh t)(\partial_t - \tanh t) \tag{5.18}$$

we find that for $n \ge 2$ the kernel (without any restriction on the rate of growth) of $\partial_t^2 + 2\operatorname{sech}^2 t - n^2$ is spanned by the functions $(\partial_t - \tanh t)e^{\pm nt} = (\pm n - \tanh t)e^{\pm nt}$, while for $n = 0$ it is spanned by the functions $-(\partial_t - \tanh t)1 = \tanh t$ and $(\partial_t - \tanh t)t = 1 - t\tanh t$ (the Jacobi fields on the catenoid (4.18) induced respectively by vertical translation and dilations about the origin), and for $n = 1$ (not needed for this construction) the kernel is spanned by the functions $(\partial_t - \tanh t)\sinh t = \operatorname{sech} t$ and $(\partial_t - \tanh t)t\sinh t = \sinh t + t\operatorname{sech} t$ (which, multiplied by linear combinations of $\cos\theta$ and $\sin\theta$, respectively generate horizontal translations and rotations about horizontal axes through the center of (4.18)).

It follows (and is straightforward to check directly) that if for each nonnegative integer $n \neq 1$ we define the function $u_n : \mathbb{R} \to \mathbb{R}$ by

$$u_0(t) := \int_0^t \left[(t-s)\tanh s \tanh t + (\tanh t - \tanh s)\right] f_n(s)\, ds$$

and for $n > 1$

$$\begin{aligned} u_n(t) := {} & \frac{n+\tanh t}{2n(1-n^2)} e^{-nt} \int_{-\infty}^t (n - \tanh s) e^{ns} f_n(s)\, ds \\ & + \frac{n - \tanh t}{2n(1-n^2)} e^{nt} \int_t^\infty (n + \tanh s) e^{-ns} f_n(s)\, ds, \end{aligned} \tag{5.19}$$

then u_n solves $\left(\partial_t^2 + 2\operatorname{sech}^2 t - n^2\right) u_n = f_n$ with $u_0(0) = \dot{u}_0(0) = 0$ and u_n bounded whenever f_n is compactly supported and $n > 1$. Therefore the distribution

$$u := \frac{1}{2\pi} u_0 + \frac{1}{\pi} \sum_{n=1}^\infty u_{2n} \tag{5.20}$$

solves $\widehat{\mathcal{L}}_K u = f$, at least in the distributional sense, and is even (also as a distribution) under the reflections $\underline{\widehat{\mathsf{X}}}_K$ and $\underline{\widehat{\mathsf{Y}}}_K$ (defined in (5.8)). It is elementary to verify from (5.19) that

$$\begin{aligned} |u_n(t)| &\leq \frac{C[\gamma]}{n^2+1} \|f\|_{C^{0,\alpha,\gamma}(\mathbb{K})} e^{\gamma|t|}, \text{ so} \\ \|u\|_{C^{0,0,\gamma}(\mathbb{K})} &\leq C[\gamma] \|f\|_{C^{0,\alpha,\gamma}(\mathbb{K})} \end{aligned} \tag{5.21}$$

for some constant $C[\gamma]$ independent of the data f. Standard elliptic theory, using in particular the Schauder estimates, then implies that in fact u is a classical solution satisfying

$$\|u\|_{C^{2,\alpha,\gamma}(\mathbb{K})} \leq C[\alpha,\gamma] \|f\|_{C^{0,\alpha,\gamma}(\mathbb{K})} \tag{5.22}$$

for some constant $C[\alpha,\gamma] > 0$ independent of the data f, and we have already observed that u is $\mathcal{G}_K$-even. Taking $\widehat{\mathcal{R}}_K f := u$ thus concludes the proof. q.e.d.

Approximate solutions on the toral regions. Recalling (4.71) and (4.75), note that both $\widehat{\mathbb{T}}_i$ and $\widehat{\chi}_i$ are, for all i, independent of m and the ζ, ξ parameters. Recalling also (4.70) and (5.5), from items (i) and (ii) of Proposition 4.79 we see that on $\mathbb{T}_i$

$$\lim_{m\to\infty} T_i^* \mathcal{L}_\chi T_i^{*-1} = \Delta_{\widehat{\chi}_i}, \tag{5.23}$$

where the convergence is to be interpreted along the lines described for (5.7), using in this case, in addition to Proposition 4.79, the fact that $\lim_{m\to\infty} m\sqrt{\tau_i} = 0$, as follows from the second line of (3.6). Note additionally that, by (2.25), (2.26), (2.30), and (4.72), the pullback by T_i of any $\mathcal{G}$-invariant function on Σ to $\mathbb{T}_i$ must satisfy periodic boundary conditions on the rectangular part of the boundary and must moreover respect a D_2 group of symmetries.

Specifically, for $1 \le i \le N$, we define the quotients

$$(5.24)\qquad \begin{gathered}(\mathbb{T}_i/\sim) \subset \widehat{\mathbb{T}}_i/\sim, \qquad \text{where} \\ (\mathrm{x},\mathrm{y}) \sim (\mathrm{x}',\mathrm{y}') \Leftrightarrow (\mathrm{x}-\mathrm{x}',\mathrm{y}-\mathrm{y}') \in \frac{\sqrt{2}\pi}{k}\mathbb{Z} \times \frac{\sqrt{2}\pi}{\ell}\mathbb{Z},\end{gathered}$$

so that $\mathbb{T}_i/\sim$ (or $\widehat{\mathbb{T}}_i/\sim$) is a $\sqrt{2}\pi/k \times \sqrt{2}\pi/\ell$ torus with one disc (or point) deleted if $i \in \{1, N\}$ and two otherwise. We also define the D_2 subgroup of symmetries of $\widehat{\mathbb{T}}_i/\sim$

$$(5.25)\qquad \begin{aligned} &\mathcal{G}_{T_i} := \left\{ \widehat{I}_T, \underline{\widehat{\mathsf{X}}}_{T_i}, \underline{\widehat{\mathsf{Y}}}_{T_i}, \underline{\widehat{\mathsf{X}}}_{T_i}\underline{\widehat{\mathsf{Y}}}_{T_i} \right\}, \qquad \text{where} \\ &\widehat{I}_T(\mathrm{x},\mathrm{y}) := (\mathrm{x},\mathrm{y}), \\ &\underline{\widehat{\mathsf{X}}}_{T_i}(\mathrm{x},\mathrm{y}) := \begin{cases} (-\mathrm{x},\mathrm{y}) \text{ if } i \in \{1,N\} \\ \left(\frac{\pi}{\sqrt{2}k} - \mathrm{x}, \mathrm{y}\right) \text{ if } 1 < i < N, \end{cases} \qquad \text{and} \\ &\underline{\widehat{\mathsf{Y}}}_{T_i}(\mathrm{x},\mathrm{y}) := \begin{cases} (\mathrm{x},-\mathrm{y}) \text{ if } i \in \{1,N\} \\ \left(\mathrm{x}, \frac{\pi}{\sqrt{2}\ell} - \mathrm{y}\right) \text{ if } 1 < i < N \end{cases} \end{aligned}$$

(using coordinates on the universal cover of $\widehat{\mathbb{T}}_i/\sim$ to define the symmetries). Obviously $\mathcal{G}_{T_i}$ preserves both $\widehat{\mathbb{T}}_i/\sim$ and $\mathbb{T}_i/\sim$.

Remark 5.26. Recalling Notation 4.5, it now follows from (2.25), (2.26), (2.30), (4.25), (4.26), (4.72), (4.73), (5.8), and (5.25) that, for any $\alpha \in [0, 1)$ and nonnegative integer j, a function

$$(5.27)\qquad f \in C^{j,\alpha}_{loc}\left(\bigcup_{i=1}^{N-1} \mathcal{K}[i] \cup \bigcup_{i=1}^{N} \mathcal{T}[i] \right)$$

extends (uniquely) to a function in $C^{j,\alpha}_{loc,\mathcal{G}}(\Sigma)$ if and only if ${\kappa_i}^* f \in C^{j,\alpha}_{loc,\mathcal{G}_K}(\mathbb{K}_{a_i})$ for each $1 \le i \le N-1$ and $T_i^* f$ descends to a function in $C^{j,\alpha}_{loc,\mathcal{G}_{T_i}}(\mathbb{T}_i/\sim)$ for each $1 \le i \le N$.

Motivated also by (4.76) and definition (4.98), we are led to define, for any nonnegative integer j, $\alpha \in [0, 1)$, and $\gamma \in (0, \infty)$, the norms

$$(5.28)\qquad \|\cdot\|_{C^{j,\alpha,\gamma}(\widehat{\mathbb{T}}_i/\sim)} := \left\| \cdot : C^{j,\alpha}\left(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i, \widehat{\rho}_i^{\,-\gamma}\right) \right\|$$

(recalling (4.6)) and the corresponding Banach spaces of $\mathcal{G}_{T_i}$-even functions

$$(5.29)\qquad \begin{aligned} C^{j,\alpha,\gamma}_{\mathcal{G}_{T_i}}\left(\widehat{\mathbb{T}}_i/\sim\right) := \Big\{ u \in C^{j,\alpha}\left(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i\right) \;\Big|\; &\|u\|_{C^{j,\alpha,\gamma}(\widehat{\mathbb{T}}_i/\sim)} < \infty \\ &\text{and } \mathfrak{g}.u = u \text{ for all } \mathfrak{g} \in \mathcal{G}_{T_i} \Big\}. \end{aligned}$$

Clearly $\Delta_{\widehat{\chi}_i} : C^{2,\alpha,\gamma}_{\mathcal{G}_{T_i}}\left(\widehat{\mathbb{T}}_i/\sim\right) \to C^{0,\alpha,\gamma}_{\mathcal{G}_{T_i}}\left(\widehat{\mathbb{T}}_i/\sim\right)$ is bounded (independently of $\alpha, \gamma \in (0,1)$). Proposition 5.31 below presents a suitable inverse, modulo the extended substitute kernel and with a support condition on the source function. The support assumption (expressed below using the function d_i from (4.75) and writing $\operatorname{spt} f$ for the support of a function f) we can afford to make because in practice we will apply Proposition 5.13 before applying Proposition 5.31. The extended substitute kernel we formally define right now. We first recall the definition (4.78) of $\underline{w}_i$, remembering in particular that $\underline{w}_1$ and $\underline{w}_N$ vanish identically, and for $1 \leq i \leq N$ we introduce $w_i \in C^\infty_{\mathcal{G}}(\Sigma)$ defined by

$$\begin{aligned} T_i^* w_i &:= \psi\left[\frac{1}{10\ell}, \frac{1}{5\ell}\right] \circ d_i \text{ and} \\ w_i|_{\Sigma \setminus \mathcal{G}\mathcal{T}[i]} &:= 0 \text{ for } 1 \leq i \leq N, \end{aligned} \tag{5.30}$$

recalling the function d_i from (4.75). Finally we define the *extended substitute kernel* to be the linear span in $C^\infty(\Sigma)$ of $\{w_i, \underline{w}_i\}_{i=1}^N$.

Proposition 5.31 (Solutions to the model problems on the torus)**.** *Let $\ell \geq k \in \mathbb{Z} \cap [1,\infty)$ and $i \in \mathbb{Z} \cap [1,N]$. There exists a linear map*

$$\begin{aligned} &\widehat{\mathcal{R}}_{T_i} = \widehat{\mathcal{R}}_{T_i}[k,\ell] : \\ &\quad \left\{ f \in C^{0,\alpha}_{\mathcal{G}_{T_i}}\left(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i\right) \;\middle|\; \operatorname{spt} f \subset \left\{ d_i > \frac{1}{20\ell} \right\} \right\} \\ &\qquad \to C^{2,\alpha,2}_{\mathcal{G}_{T_i}}\left(\widehat{\mathbb{T}}_i/\sim\right) \times \mathbb{R} \times \mathbb{R} \end{aligned} \tag{5.32}$$

defined for all $\alpha \in (0,1)$, and, given any $\alpha \in (0,1)$, there exists a constant $C = C[k,\ell,\alpha] > 0$ such that whenever f belongs to the domain of $\widehat{\mathcal{R}}_{T_i}$ above and $(u, \mu, \underline{\mu}) = \widehat{\mathcal{R}}_{T_i} f$, then

$$\begin{aligned} &\Delta_{\widehat{\chi}_i} u = f + \mu T_i^* w_i + \underline{\mu} T_i^* \underline{w}_i \text{ and} \\ &\|u\|_{C^{2,\alpha,2}\left(\widehat{\mathbb{T}}_i/\sim\right)} + |\mu| + |\underline{\mu}| \leq C \left\| f : C^{0,\alpha}\left(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i\right) \right\|, \end{aligned} \tag{5.33}$$

recalling (4.71), (4.75), (4.78), (5.24), (5.25), (5.28), (5.29), *and* (5.30).

Proof. Suppose $f \in C^{0,\alpha}_{\mathcal{G}_{T_i}}(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i)$ has support contained in the set $U := \{1/d_i < 20\ell\}$. We intend to apply a conformal change of metric and attack the corresponding problem on the flat torus $(\overline{\mathbb{T}}_i, g_E)$, where $\overline{\mathbb{T}}_i$ is simply $\widehat{\mathbb{T}}_i/\sim$ with the missing point(s) filled in and $g_E = d\mathrm{x}^2 + d\mathrm{y}^2$ is the standard flat metric. By definition (4.75) (and because $\widehat{\mathbb{T}}_i$ is two-dimensional) the equation $\Delta_{\widehat{\chi}_i} u = f$ on $\widehat{\mathbb{T}}_i/\sim$ is equivalent to $\Delta_{g_E} u = \widehat{\rho}_i^2 f$. Note that $\mathcal{G}_{T_i}$ acts by isometries on $(\overline{\mathbb{T}}_i, g_E)$ in the obvious way. Clearly the function $\widehat{\rho}_i$ defined in (4.75) descends to a function (which we give the same name) in $C^\infty_{\mathcal{G}_{T_i}}(\widehat{\mathbb{T}}_i/\sim)$ and clearly

$$\big\|\widehat{\rho}_i|_U : C^j(U,\widehat{\chi}_i)\big\| \le C[\ell,j] \tag{5.34}$$

for some constant $C[\ell,j]$. Consequently

$$\big\|\widehat{\rho}_i^2 f : C^{0,\alpha}(\overline{\mathbb{T}}_i, g_E)\big\| \le C[\ell]\,\Big\|f : C^{0,\alpha}(\widehat{\mathbb{T}}_i/\sim,\widehat{\chi}_i)\Big\| \tag{5.35}$$

(where we have trivially extended $\widehat{\rho}_i^2 f$ to a function of the same name on $\overline{\mathbb{T}}_i$, simply by requiring it to vanish at the filled in point(s)).

Of course the equation $\Delta_{g_E} u = \widehat{\rho}_i^2 f$ has a solution on $\overline{\mathbb{T}}_i$ if and only if the right-hand side has vanishing integral over $\overline{\mathbb{T}}_i$, which we do not assume. Accordingly we would like to permit ourselves the freedom of adding a constant to the right-hand side. Soon though (at the end of this section) we intend to transfer the solution from this model problem to the initial surface, so we want to confine any modification of the right-hand side to the toral region in question, avoiding any interference on the adjoining catenoidal regions. Therefore we will we use the cutoff function $T_i^* w_i$ in (5.30) in lieu of the constant function 1 for the purpose of adjusting the right-hand side to make it orthogonal to the kernel.

More precisely we note that $T_i^* w_i$ descends smoothly to $\overline{\mathbb{T}}_i$ and we define

$$\mu := -\frac{\int_{\overline{\mathbb{T}}_i} \widehat{\rho}_i^2 f \,d\mathrm{x}\,d\mathrm{y}}{\int_{\overline{\mathbb{T}}_i} \widehat{\rho}_i^2 T_i^* w_i \,d\mathrm{x}\,d\mathrm{y}}, \tag{5.36}$$

so that $\int_{\overline{\mathbb{T}}_i} \widehat{\rho}_i^2\,(f+\mu T_i^* w_i)\,d\mathrm{x}\,d\mathrm{y} = 0$. Consequently there is a unique function $u_0 : \overline{\mathbb{T}}_i \to \mathbb{R}$ solving

$$\Delta_{g_E} u_0 = \widehat{\rho}_i^2 (f+\mu T_i^* w_i) \tag{5.37}$$

and satisfying $\int_{\overline{\mathbb{T}}_i} u_0\,d\mathrm{x}\,d\mathrm{y} = 0$; in particular u_0 is necessarily $\mathcal{G}_{T_i}$-invariant. Note also that by (4.75) $\widehat{\rho}_i \ge 1$ on $\widehat{\mathbb{T}}_i$ and that by (4.71) and (5.30) $w = 1$ on a region of positive g_E-area (depending on just k and ℓ), while of course $\overline{\mathbb{T}}_i$ itself has area $\frac{2\pi^2}{k\ell}$; it then follows from (5.35) and (5.36) that

$$|\mu| \le C[k,\ell]\,\Big\|f : C^{0,\alpha}(\widehat{\mathbb{T}}_i/\sim,\widehat{\chi}_i)\Big\| \tag{5.38}$$

for some constant $C[k,\ell] > 0$. The classical global Schauder estimates applied to (5.37) imply in particular that

$$\begin{aligned}\big\|u_0 : C^0(\overline{\mathbb{T}}_i)\big\| &\le C[k,\ell]\,\big\|\widehat{\rho}_i^2(f+\mu T_i^* w_i) : C^{0,\alpha}(\overline{\mathbb{T}}_i, g_E)\big\| \\ &\le C[k,\ell]\,\Big\|f : C^{0,\alpha}(\widehat{\mathbb{T}}_i/\sim,\widehat{\chi}_i)\Big\|\end{aligned} \tag{5.39}$$

(for a possibly larger constant $C[k,\ell]$ than above), where for the first inequality we have again used the fact that $\overline{\mathbb{T}}_i$ is just a flat $\sqrt{2}\pi/k \times \sqrt{2}\pi/\ell$ torus and for the second we have used (5.35) and (5.38).

We still need to arrange the rapid decay of our solution toward the point(s) in $\overline{\mathbb{T}}_i$ missing from $\widehat{\mathbb{T}}_i$. To this end we first observe that, because both f (by assumption) and w (by definition (5.30)) have support contained in $U = \{1/d_i < 20\ell\}$, the solution u_0 to (5.37) is harmonic on the set $\overline{\mathbb{T}}_i \backslash U = \left\{d_i \leq \frac{1}{20\ell}\right\}$, where, as we see from (4.75), $d_i = 1/\widehat{\rho}_i$. For $i \in \{1, N\}$ this set has one component—the closed disc of radius $\frac{1}{20\ell}$ and center $p_0 := (0,0)$—while otherwise it has two components—the closed discs of radius $\frac{1}{20\ell}$ and centers $p_\pm := \left(\frac{\pi}{2\sqrt{2}k}, \frac{\pi}{2\sqrt{2}\ell}\right) = \pm\left(\frac{mX}{2}, \frac{mY}{2}\right)$, recalling (2.28). Now we define $\underline{\mu} \in \mathbb{R}$ by

$$(5.40) \qquad (-1)^{N-i}\underline{\mu} := \begin{cases} 0 \quad \text{if} \quad i \in \{1, N\} \\ \frac{1}{2}[u_0(p_-) - u_0(p_+)] \quad \text{if} \quad i \notin \{1, N\}, \end{cases}$$

so that by (5.39)

$$(5.41) \qquad \left|\underline{\mu}\right| \leq C[k, \ell] \left\| f : C^{0,\alpha}(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i) \right\|.$$

Recalling (4.77) and noting that v descends smoothly to $\overline{\mathbb{T}}_i$, we also define $u : \overline{\mathbb{T}}_i \to \mathbb{R}$ by

$$(5.42) \quad u := \begin{cases} u_0 - u_0(p_0) \quad \text{if} \quad i \in \{1, N\} \\ u_0 - \frac{1}{2}[u_0(p_-) + u_0(p_+)] + (-1)^{N-i}\underline{\mu} v_i \quad \text{otherwise} \end{cases}$$

(where we include the alternating signs because of the one present in definition (4.78) of $\underline{w}_i$, which in turn we included to account for the alternating direction of the unit normal on the toral regions). Thus by (5.37) and (4.78)

$$(5.43) \qquad \Delta_{\widehat{\chi}_i} u_0 = f + \mu T_i^* w_i + \underline{\mu} T_i^* \underline{w}_i.$$

Note that v_i is constant on each component of $\overline{\mathbb{T}}_i \backslash U$, so, like u_0, the function u is harmonic on $\overline{\mathbb{T}}_i \backslash U$. By classical harmonic function theory

$$(5.44) \qquad \left\|u : C^2(\overline{\mathbb{T}}_i \backslash U, g_E)\right\| \leq C[\ell] \left\|u : C^0(\partial U, g_E)\right\|$$

for some constant $C[\ell] > 0$. On the other hand, since $v_i(p_\pm) = \pm 1$, we have $u(p_0) = 0$ if $i \in \{1, N\}$ and $u(p_\pm) = 0$ otherwise. Moreover, u is $\mathcal{G}_{T_i}$-invariant (because u_0, v, and the constants are), so, recalling (5.25), both first partial derivatives of u also vanish at p_0 if $i \in \{1, N\}$ and at both points $p_\pm$ otherwise. Using Taylor's theorem and (5.44), we therefore obtain

$$(5.45) \qquad \begin{aligned} \left\|u : C^0\left(\overline{\mathbb{T}}_i \backslash U, g_E, {d_i}^2\right)\right\| &\leq C[\ell] \left\|u : C^0(\partial U, g_E)\right\| \\ &\leq C[k, \ell] \left\| f : C^{0,\alpha}(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i) \right\|, \end{aligned}$$

where we recall (4.6) and for the last inequality we use (5.39). As already observed $\widehat{\rho_i}^{-1} = d_i$ on $\overline{\mathbb{T}}_i \backslash U$, while on U it is bounded below by $\frac{1}{20\ell}$, so it now follows from (4.6), (5.28) (5.43), (5.45), and the standard local Schauder estimates together with the bounded geometry of $(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i)$ that

$$\|u\|_{C^{2,\alpha,2}(\widehat{\mathbb{T}}_i/\sim)} \le C[k,\ell,\alpha] \left\| f : C^{0,\alpha}(\widehat{\mathbb{T}}_i/\sim, \widehat{\chi}_i) \right\|, \tag{5.46}$$

which, along with (5.38), (5.41), (5.43), and the already observed $\mathcal{G}_{T_i}$-invariance of u, concludes the proof. q.e.d.

Exact global solutions. Now we use Propositions 5.13 and 5.31 to construct global solutions to the linearized problem on each initial surface, modulo extended substitute kernel.

Proposition 5.47 (Global solutions to the linearized problem)**.** *Given a real number $c > 0$ and integers $N \ge 2$ and $\ell \ge k \ge 1$, there exists $m_0 = m_0[N,k,\ell,c] > 0$ such that whenever $\zeta, \xi \in [-c,c]^{N-1}$ and $m > m_0$, there is a linear map*

$$\begin{aligned} \mathcal{R} = \mathcal{R}[N,k,\ell,m,\zeta,\xi] : C^{0,\alpha,\gamma}_{\mathcal{G}[k,\ell,m]}(\Sigma[N,k,\ell,m,\zeta,\xi]) \\ \to C^{2,\alpha,\gamma}_{\mathcal{G}[k,\ell,m]}(\Sigma[N,k,\ell,m,\zeta,\xi]) \times \mathbb{R}^N \times \mathbb{R}^{N-2} \end{aligned} \tag{5.48}$$

(recalling (4.99)) defined for all $\alpha, \gamma \in (0,1)$, and, given $\alpha, \gamma \in (0,1)$, there is a constant $C = C[N,k,\ell,\alpha,\gamma] > 0$ such that whenever $f \in C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma)$ and $(u, (\mu_1, \cdots, \mu_N), (\underline{\mu}_2, \cdots, \underline{\mu}_{N-1})) = \mathcal{R}f$, then

$$\begin{aligned} &\mathcal{L}_\chi u = f + \sum_{i=1}^{N} \mu_i w_i + \sum_{i=2}^{N-1} \underline{\mu}_i \underline{w}_i \quad \textit{and} \\ &\|u\|_{2,\alpha,\gamma} + \sum_{i=1}^{N} |\mu_i| + \sum_{i=2}^{N-1} \left|\underline{\mu}_i\right| \le \|f\|_{0,\alpha,\gamma} \end{aligned} \tag{5.49}$$

(recalling (4.98) and (5.5)); moreover, for any fixed N, k, ℓ, and m, the map

$$(f,\zeta,\xi) \mapsto \mathcal{R}[N,k,\ell,m,\zeta,\xi]f \;\; \textit{is continuous} \tag{5.50}$$

in the sense of Definition 4.101.

Proof. Let $c > 0$, $\alpha, \gamma \in (0,1)$, $N \in \mathbb{Z} \cap [2,\infty)$, $\ell \ge k \in \mathbb{Z} \cap [1,\infty)$, $\zeta, \xi \in [-c,c]^{N-1}$, and $m \in \mathbb{Z} \cap [m_0, \infty)$, where m_0 is at least as large as the maximum of the homonymous quantities appearing in Propositions 4.30, 4.79, 5.13, and 5.31 and is subject to an additional lower bound imposed at the end of the proof. Recalling (4.26), we start by defining, for $1 \le i \le N-1$, the linear maps

$$\Psi_{\mathcal{K}[i]} : C(\mathcal{K}[i]) \to C(\Sigma) \tag{5.51}$$

so that $\Psi_{\mathcal{K}[i]}f$ is the unique $\mathcal{G}$-equivariant function which vanishes outside $\mathcal{G}\mathcal{K}[i]$ and which satisfies

$$\begin{aligned}\left(\Psi_{\mathcal{K}[i]}f\right)\big|_{\mathcal{K}[i]} &:= \kappa_i^{-1*}\left[\left(\psi\left[a_i, a_i - 1/2\right] \circ |t|\right) \cdot \kappa_i{}^* f\right] \\ &= \left(\psi\left[a_i, a_i - 1/2\right] \circ \left|t \circ \kappa_i^{-1}\right|\right) \cdot f,\end{aligned} \tag{5.52}$$

recalling (2.4), (4.24), and (4.25). Note that by (4.17) the j^{th} $\widehat{\chi}_K$ covariant derivative of $\psi\left[a_i, a_i - 1\right] \circ |t|$ is uniformly $\widehat{\chi}_K$-bounded on $\mathbb{K}_{a_i}$ by a constant depending on just j. Recalling Proposition 5.13, we also define

$$\begin{aligned}&\widetilde{\mathcal{R}}_K : C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \to C^{2,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \text{ by} \\ &\widetilde{\mathcal{R}}_K f := \sum_{i=1}^{N-1} \Psi_{\mathcal{K}[i]} v_{\mathcal{K}[i]}, \quad \text{with} \\ &v_{\mathcal{K}[i]} := \kappa_i^{-1*}\left(\widehat{\mathcal{R}}_K\left(\kappa_i{}^*\Psi_{\mathcal{K}[i]}(f|_{\mathcal{K}[i]})\right)\right),\end{aligned} \tag{5.53}$$

where $\kappa_i{}^*\Psi_{\mathcal{K}[i]}(f|_{\mathcal{K}[i]})$ is trivially (and smoothly) extended from $\mathbb{K}_{a_i}$ to $\mathbb{K}$ so that it vanishes outside $\mathbb{K}_{a_i}$, recalling (4.16) and (4.22). Then

$$\begin{aligned}\mathcal{L}_\chi \widetilde{\mathcal{R}}_K f = \sum_{i=1}^{N-1} &\left(\left[\mathcal{L}_\chi, \Psi_{\mathcal{K}[i]}\right] v_{\mathcal{K}[i]}\right. \\ &\left.+\Psi_{\mathcal{K}[i]}\kappa_i^{-1*}\left(\kappa_i{}^*\mathcal{L}_\chi\kappa_i^{-1*} - \widehat{\mathcal{L}}_K\right)\kappa_i{}^* v_{\mathcal{K}[i]} + \Psi^2_{\mathcal{K}[i]} f|_{\mathcal{K}[i]}\right),\end{aligned} \tag{5.54}$$

where in the first term the brackets indicate the commutator of the operators they enclose, in the second term we recall (5.6), and in the last term we make use of Proposition 5.13.

We will absorb the "cutoff error" in (5.54), present in the first and third terms, into the right-hand side when solving on the toral regions in the next step. More precisely, for any given $f \in C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma)$ we define

$$f_T := f - \sum_{i=1}^{N-1} \Psi^2_{\mathcal{K}[i]} f|_{\mathcal{K}[i]} - \sum_{i=1}^{N-1} \left[\mathcal{L}_\chi, \Psi_{\mathcal{K}[i]}\right] v_{\mathcal{K}[i]}, \tag{5.55}$$

where each $v_{\mathcal{K}[i]}$ is defined (for the given f) in (5.53). Note that f_T is $\mathcal{G}$-equivariant and has support contained in $\mathcal{G}\left(\bigcup_{i=1}^N \mathcal{T}[i]\right)$. In fact, since

$$\begin{aligned}\tau_i \cosh\left(a_i - \frac{1}{2}\right) &= \tau_i \cosh a_i \left(\cosh\frac{1}{2} - \tanh a_i \sinh\frac{1}{2}\right) \\ &\geq e^{-1/2}\tau_i \cosh a_i > \frac{1}{20\ell m}\end{aligned} \tag{5.56}$$

(using (4.24) for the final inequality), we have, recalling (4.72),

$$\operatorname{spt} T_i{}^* f_T \subset \left\{d_i > \frac{1}{20\ell}\right\}. \tag{5.57}$$

Next, recalling (4.73), for $1 \le i \le N$ we now define the linear maps

$$\Psi_{\mathcal{T}[i]} : C(\mathcal{T}[i]) \to C(\Sigma) \tag{5.58}$$

so that $\Psi_{\mathcal{T}[i]} f$ is the unique $\mathcal{G}$-equivariant function on Σ vanishing outside $\mathcal{G}\mathcal{T}[i]$ and satisfying

$$\begin{aligned}\left(\Psi_{\mathcal{T}[i]} f\right)\big|_{\mathcal{T}[i]} &:= T_i^{-1*}\left[\left(\psi\left[\ln m\tau_1^{1/3}, \ln m\tau_1^{1/6}\right] \circ \ln d_i\right) \cdot T_i{}^* f\right] \\ &= \left(\psi\left[\ln m\tau_1^{1/3}, \ln m\tau_1^{1/6}\right] \circ \ln\left(d_i \circ T_i{}^{-1}\right)\right) \cdot f\end{aligned} \tag{5.59}$$

for which we recall (4.72) and (4.75). Note that by (2.12) and (2.14) we have $\sqrt{\tau_i} < \tau_1^{1/3} < \tau_1^{1/6}$ for $1 \le i \le N-1$ and that moreover by (4.75) all $\widehat{\chi}_i$ covariant derivatives of $\psi\left[\ln m\tau_1^{1/3}, \ln m\tau_1^{1/6}\right] \circ \ln d_i$ are uniformly $\widehat{\chi}_i$-bounded on $\mathbb{T}_i$.

Now, recalling Proposition 5.31 and (5.55) just above, we also define for $1 \le i \le N$ the maps

$$\begin{aligned}&\widetilde{\mathcal{R}}_{\mathcal{T}[i]} : C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \to C^{2,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \times \mathbb{R} \times \mathbb{R} \text{ by} \\ &\widetilde{\mathcal{R}}_{\mathcal{T}[i]} f := \left(\Psi_{\mathcal{T}[i]} v_{\mathcal{T}[i]}, \mu, \underline{\mu}\right) \quad \text{with} \\ &v_{\mathcal{T}[i]} := T_i^{-1*} \widehat{v}_{\mathcal{T}[i]} \quad \text{and} \quad \left(\widehat{v}_{\mathcal{T}[i]}, \mu, \underline{\mu}\right) := \widehat{\mathcal{R}}_{T_i}\left(T_i{}^* f_T\right).\end{aligned} \tag{5.60}$$

Here we are implicitly regarding $T_i{}^* f_T$ as a function on $\widehat{\mathbb{T}}_i/\!\sim$ (possible because f_T is $\mathcal{G}$-equivariant) after extending it to a function on $\widehat{\mathbb{T}}_i$ which simply vanishes outside $\mathbb{T}_i$, and moreover we see that (5.57) ensures that this function truly belongs to the domain of $\widehat{\mathcal{R}}_{T_i}$. It now follows by Proposition 5.31 that if $(u_{\mathcal{T}[i]}, \mu, \underline{\mu}) = \widetilde{\mathcal{R}}_{\mathcal{T}[i]} f$, then

$$\begin{aligned}\mathcal{L}_\chi u_{\mathcal{T}[i]} &= \left[\mathcal{L}_\chi, \Psi_{\mathcal{T}[i]}\right] v_{\mathcal{T}[i]} + \Psi_{\mathcal{T}[i]} f_T|_{\mathcal{T}[i]} + \mu\, w_i + \underline{\mu}\, \underline{w}_i \\ &\quad + \Psi_{\mathcal{T}[i]} T_i^{-1*}\left(T_i{}^* \mathcal{L}_\chi T_i^{-1*} - \Delta_{\widehat{\chi}_i}\right) T_i{}^* v_{\mathcal{T}[i]}.\end{aligned} \tag{5.61}$$

Next we define the approximate solution operator

$$\begin{aligned}&\widetilde{\mathcal{R}} : C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \to C^{2,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \times \mathbb{R}^N \times \mathbb{R}^{N-2} \text{ by} \\ &\widetilde{\mathcal{R}} f := \left(\widetilde{\mathcal{R}}_K f + \sum_{i=1}^N u_{\mathcal{T}[i]}, (\mu_1, \dots, \mu_N), \left(\underline{\mu}_2, \dots, \underline{\mu}_{N-1}\right)\right) \\ &\text{with } \left(u_{\mathcal{T}[i]}, \mu_i, \underline{\mu}_i\right) := \widetilde{\mathcal{R}}_{\mathcal{T}[i]} f,\end{aligned} \tag{5.62}$$

where $\widetilde{\mathcal{R}}_K$ and $\widetilde{\mathcal{R}}_{\mathcal{T}[i]}$ are defined in (5.53) and (5.60) above and where from the output of $\widetilde{\mathcal{R}}$ we are simply omitting $\underline{\mu}_1 = \underline{\mu}_N = 0$, as indicated. Clearly $\widetilde{\mathcal{R}}$ (from its definition and Propositions 5.13 and 5.31) is bounded independently of c and m. Moreover, the map $(f, \zeta, \xi) \mapsto \widetilde{\mathcal{R}}[N, k, \ell, m, \zeta, \xi] f$ is manifestly continuous (in the sense of Definition

4.101), since all the operators (including $\mathcal{L}_\chi$ itself) on Σ used to construct it clearly enjoy this continuous dependence themselves, while the maps $\widehat{\mathcal{R}}_K$ and $\widehat{\mathcal{R}}_{T_i}$ are of course independent of the parameters. Defining also the operator

$$\widetilde{\mathcal{L}} : C^{2,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \times \mathbb{R}^N \times \mathbb{R}^{N-2} \to C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma) \quad \text{by}$$

$$\begin{aligned} &\widetilde{\mathcal{L}}\left(u, (\mu_1, \dots, \mu_N), \left(\underline{\mu}_2, \dots, \underline{\mu}_{N-1}\right)\right) \\ &\qquad := \mathcal{L}_\chi u - \sum_{i=1}^N \mu_i w_i - \sum_{i=2}^{N-1} \underline{\mu}_i \underline{w}_i \end{aligned} \tag{5.63}$$

and using (5.54), (5.55), (5.61), and the definitions of $\Psi_{\mathcal{K}[i]}$ and $\Psi_{\mathcal{T}[i]}$ above, we find that for any $f \in C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma)$

$$\begin{aligned} \widetilde{\mathcal{L}}\widetilde{\mathcal{R}} f - f = &\sum_{i=1}^{N-1} \kappa_i^{-1*} \left(\kappa_i{}^* \mathcal{L}_\chi \kappa_i^{-1*} - \widehat{\mathcal{L}}_K\right) \kappa_i{}^* v_{\mathcal{K}[i]} \\ &+ \sum_{i=1}^{N} T_i^{-1*} \left(T_i{}^* \mathcal{L}_\chi T_i^{-1*} - \Delta_{\widehat{\chi}_i}\right) T_i{}^* v_{\mathcal{T}[i]} \\ &+ \sum_{i=1}^{N} \left[\mathcal{L}_\chi, \Psi_{\mathcal{T}[i]}\right] v_{\mathcal{T}[i]}, \end{aligned} \tag{5.64}$$

where $v_{\mathcal{K}[i]}$ and $v_{\mathcal{T}[i]}$ are defined in (5.53) and (5.60).

From (5.5), items (i) and (v) of Proposition 4.30, items (i) and (iv) of Proposition 4.79, Propositions 5.13 and 5.31, and the definitions of $\Psi_{\mathcal{K}[i]}$ and $\Psi_{\mathcal{T}[i]}$ above we find that the first two sums in (5.64) have $C^{0,\alpha,\gamma}$ norm bounded by m^{-2} times some constant $C = C[N, k, \ell, \alpha, \gamma] > 0$ times $\|f\|_{0,\alpha,\gamma}$. As for the commutator terms, note that each commutator $\left[\mathcal{L}_\chi, \Psi_{\mathcal{T}[i]}\right]$ itself has support contained in $T_i\left(\left\{d_i \leq m\tau^{1/6}\right\}\right)$, but by Proposition 5.31 and the definition of $v_{\mathcal{T}[i]}$ in (5.60) we know that

$$\begin{aligned} &\left\| v_{\mathcal{T}[i]} : C^{0,\alpha}\left(T_i\left(\left\{d_i \leq m\tau_1^{1/6}\right\}\right), \chi, \frac{m^\gamma}{\rho^\gamma}\right)\right\| \\ &\qquad \leq C\, \|f\|_{0,\alpha,\gamma}\, m^{2-\gamma} \tau_1^{1/3-\gamma/6} \end{aligned} \tag{5.65}$$

(for a possibly larger $C = C[N, k, \ell, \alpha, \gamma]$ than above). Thus (making use of line 2 of (3.6)) we may take m large enough (in terms of C) so that $\widetilde{\mathcal{L}}\widetilde{\mathcal{R}}$ is a small perturbation of the identity operator on $C^{0,\alpha,\gamma}_{\mathcal{G}}(\Sigma)$ and consequently invertible. Taking $\mathcal{R} := \widetilde{\mathcal{R}}\left(\widetilde{\mathcal{L}}\widetilde{\mathcal{R}}\right)^{-1}$ concludes the proof. q.e.d.

As an immediate application we obtain the first-order correction of the initial surface toward minimality.

Corollary 5.66 (The solution to first order). *Given $c>0$, $\alpha,\gamma\in(0,1)$, and integers $N\geq 2$ and $\ell\geq k\geq 1$, there exist real numbers $m_0=m_0[N,k,\ell,c,\gamma]>0$ and $C=C[N,k,\ell]>0$ such that whenever $\zeta,\xi\in[-c,c]^{N-1}$, $m>m_0$, and*

$$(5.67)\qquad \begin{aligned}&\left(u_1,(\lambda_1,\dots,\lambda_N),(\underline{\lambda}_2,\dots,\underline{\lambda}_{N-1})\right)\\ &:=-\mathcal{R}\left(\rho^{-2}H-\sum_{i=2}^{N-1}\mathcal{D}_i\underline{w}_i\right)\end{aligned}$$

(recalling (3.32), (4.8), (4.14), (4.78), and Proposition 5.47), then

$$(5.68)\qquad \|u_1\|_{2,\alpha,\gamma}+\sum_{i=1}^{N}|\lambda_i|+\sum_{i=2}^{N-1}|\underline{\lambda}_i|\leq C\tau_1$$

(recalling (4.98)); moreover, $\lambda_1,\dots,\lambda_N$ and $\underline{\lambda}_2,\dots,\underline{\lambda}_{N-2}$ all depend continuously on (ζ,ξ), as does u_1 (in the sense of Definition 4.101).

Proof. All the claims follow directly from Corollary 4.106 and Proposition 5.47, with the obvious supplemental facts that $\mathcal{D}_i$ is continuous in the parameters and, in the sense of Definition 4.101, $\underline{w}_i$ is too. q.e.d.

6. The main theorem

The nonlinear terms. Recall (4.8), (5.2) and (5.4). We will need the following estimate for the nonlinear contribution

$$(6.1)\qquad \begin{aligned}\mathcal{Q}[u]&=\mathcal{Q}[N,k,\ell,m,\zeta,\xi,u]\\ &:=\mathcal{H}[N,k,\ell,m,\zeta,\xi,u]-H-\mathcal{L}[N,k,\ell,m,\zeta,\xi]u\end{aligned}$$

that the deforming function u makes to the mean curvature. (Of course $H=\mathcal{H}[N,k,\ell,m,\zeta,\xi,0]$.)

Lemma 6.2 (The nonlinear terms). *Given $C_u,c>0$, $\alpha,\gamma\in(0,1)$, and integers $N\geq 2$ and $\ell\geq k\geq 1$, there exists*

$$m_0:=m_0[N,k,\ell,m,C_u,c]>0$$

such that (recalling (2.13), (4.14), (4.98), and (6.1)) $\mathcal{Q}[N,k,\ell,m,\zeta,\xi,u]$ is well-defined and

$$(6.3)\qquad \left\|\rho^{-2}\mathcal{Q}[N,k,\ell,m,\zeta,\xi,u]\right\|_{0,\alpha,\gamma}\leq\underline{\tau}_1^{1+\gamma/2}$$

whenever $m>m_0$, $\zeta,\xi\in[-c,c]^{N-1}$, and $u\in C^{2,\alpha}(\Sigma,\chi)$ satisfies $\|u\|_{2,\alpha,\gamma}\leq C_u\tau_1$; furthermore, for each fixed N, k, ℓ, and $m>m_0$, the map $(u,\zeta,\xi)\mapsto\mathcal{Q}[N,k,\ell,m,\zeta,\xi,u]$ is continuous (in the sense of Definition 4.101).

Proof. That $\mathcal{Q}[u]$ is defined at all will be clear from Lemma 4.42 in conjunction with the estimates below (which show that $\iota[u]$ (5.1) is an immersion and $\mathcal{H}[u]$ is defined). The continuity follows immediately from the smooth dependence (Remark 2.33) of the initial surfaces on the parameters and from definition (5.2). To make the estimate we will apply Lemma 4.42 to the embedding $\iota : \Sigma \to \mathbb{S}^3$ of the initial surface into $(\mathbb{S}^3, g_S)$, as $\mathcal{Q}[u]$ can then be read off from item (iv) of the lemma. First we observe, recalling (4.6), (4.7), (4.8), and (4.14), that by (4.9) and Propositions 4.30 and 4.79

$$\begin{aligned} C[j] \geq &\left\| g : C^j\left(T^*\Sigma^{\otimes 2}, \chi, \rho^{-2}\right)\right\| + \left\| g^{-1} : C^j\left(T\Sigma^{\otimes 2}, \chi, \rho^2\right)\right\| \\ &+ \left\| A : C^j\left(T^*\Sigma^{\otimes 2}, \chi, \tau_1 + \rho^{-2}\right)\right\|. \end{aligned} \tag{6.4}$$

Now, using the notation of Lemma 4.42, we can apply its system (i) to estimate g^t, g_t, and A^t.

Actually the estimates become more transparent if we first rescale the system: we set

$$\widetilde{g}^s := \rho^2 g^{s/\rho(\cdot)}, \qquad \widetilde{g}_s := \rho^{-2} g_{s/\rho(\cdot)}, \qquad \text{and} \qquad \widetilde{A}^s := \rho A^{s/\rho(\cdot)}, \tag{6.5}$$

so that by item (i) of Lemma 4.42 and Remark 4.45

$$\partial_s \widetilde{g}^s_{\alpha\beta} = -2\widetilde{A}^s_{\alpha\beta} \qquad \text{and} \qquad \partial_s \widetilde{A}^s_{\alpha\beta} = \rho^{-2}\widetilde{g}^s_{\alpha\beta} - \widetilde{g}_s^{\gamma\delta}\widetilde{A}^s_{\alpha\gamma}\widetilde{A}^s_{\beta\delta} \tag{6.6}$$

and by (6.4), (4.15), item (ii) of Proposition 4.30, and item (ii) of Proposition 4.79

$$\begin{aligned} C[j] \geq &\left\| \widetilde{g}^0 : C^j\left(T^*\Sigma^{\otimes 2}, \chi\right)\right\| + \left\| \widetilde{g}_0 : C^j\left(T\Sigma^{\otimes 2}, \chi\right)\right\| \\ &+ \left\| \widetilde{A}^0 : C^j\left(T^*\Sigma^{\otimes 2}, \chi\right)\right\| + \left\| \rho^{-2} : C^j(\Sigma, \chi)\right\|. \end{aligned} \tag{6.7}$$

It follows from the system (6.6) and the estimates (6.7) on the initial conditions and coefficients that there exists some $\epsilon > 0$ such that the solution to the system exists at all points $p \in \Sigma$ whenever $|s| < \epsilon$ and moreover for any nonnegative integers i and j there exists a constant $C[i,j]$ such that whenever $|s| \leq \epsilon/2$

$$\begin{aligned} C[i,j] \geq &\left\| \partial_s{}^i \widetilde{g}^s : C^j\left(T^*\Sigma^{\otimes 2}, \chi\right)\right\| \\ &+ \left\| \partial_s{}^i \widetilde{g}_s : C^j\left(T\Sigma^{\otimes 2}, \chi\right)\right\| + \left\| \partial_s{}^i \widetilde{A}^s : C^j\left(T^*\Sigma^{\otimes 2}, \chi\right)\right\|. \end{aligned} \tag{6.8}$$

Now let $C_u > 0$ and $u \in C^{2,\alpha,\gamma}_{\mathcal{G}}(\Sigma)$ with $\|u\|_{2,\alpha,\gamma} \leq C_u \tau_1$. By (4.6) and (4.98)

$$\left\| \rho u : C^{2,\alpha}\left(\Sigma, \chi, m^\gamma \tau_1 \rho^{1-\gamma}\right)\right\| \leq C_u, \tag{6.9}$$

so in particular by (4.15) and line 2 of (3.6)

$$\left\| \rho u : C^{2,\alpha}(\Sigma, \chi)\right\| \leq m^\gamma \tau_1^\gamma \leq \frac{\epsilon}{2}, \tag{6.10}$$

provided m is chosen large enough (in terms of $\epsilon > 0$, c, and C_u). Consequently we can apply the estimates (6.8) along with the definitions

(6.5) to conclude that for all $t \in [0,1]$

(6.11)
$$C \geq \left\| g^{tu} : C^{2,\alpha}\left(T^*\Sigma^{\otimes 2}, \chi, \rho^{-2}\right)\right\| + \left\| g^{tu} : C^{2,\alpha}\left(T\Sigma^{\otimes 2}, \chi, \rho^{2}\right)\right\| + \left\| A^{tu} : C^{2,\alpha}\left(T^*\Sigma^{\otimes 2}, \chi, \rho^{-1}\right)\right\|$$

for some constant $C = C[N,k,\ell] > 0$ whenever $m > m_0$ for some $m_0 = m_0[N,k,\ell,c] > 0$. Thus we also have

$$C[N,k,\ell,C_u] \geq \left\| H^u : C^{2,\alpha}(\Sigma,\chi,\rho)\right\| + \left\| g^u - g : C^{2,\alpha}\left(T^*\Sigma^{\otimes 2}, \chi, m^\gamma \tau_1 \rho^{-1-\gamma}\right)\right\|, \tag{6.12}$$

using (i) of Lemma 4.42 to estimate the second norm.

Since $\chi = \rho^2 g$,

$$\left\| D\left[T^*\Sigma, g\right] - D\left[T^*\Sigma, \chi\right] : C^j\left(T^*\Sigma^{\otimes 2}, \chi\right)\right\| \leq C[j], \tag{6.13}$$

so, using also $\|u\|_{2,\alpha,\gamma} \leq C_u \tau_1$, (6.11), and the estimate for the second term of (6.12),

$$C \geq \left\| D[g_u] A^{tu} : C^{1,\alpha}\left(T^*\Sigma^{\otimes 3}, \chi, \rho^{-1}\right)\right\| + \left\| D[g^u]^2 u : C^{0,\alpha}\left(T^*\Sigma^{\otimes 2}, \chi, \tau_1\right)\right\| \tag{6.14}$$

for another constant $C = C[N,k,\ell,C_u] > 0$, whenever $\zeta, \xi \in [-c,c]^{N-1}$ and $m > m_0$ for some $m_0 = m_0[N,k,\ell,c,C_u] > 0$. Applying (6.11), (6.14), and $\|u\|_{2,\alpha,\gamma} \leq C_u \tau_1$ (as well as Remark 4.45) in item (iv) of Lemma 4.42 and then using (2.14) and (4.15), for each $p \in \Sigma$ we obtain

$$\begin{aligned} \frac{\left\| \rho^{-2} \mathcal{Q}[u] : C^{0,\alpha}(B[p,1,\chi],\chi)\right\|}{m^\gamma \rho^{-\gamma}} &\leq C\tau_1^2 m^\gamma \rho(p)^{1-\gamma} \\ &= C\tau_1^{1+\gamma/2} m^\gamma \tau_1^{1-\gamma/2} \tau_1^{\gamma-1} \\ &\leq Ce^{4c} \underline{\tau}_1^{1+\gamma/2} m^\gamma \tau_1^{\gamma/2}, \end{aligned} \tag{6.15}$$

where $B[p,1,\chi]$ is the χ metric ball of center p and radius 1 and $C = C[N,k,\ell,C_u] > 0$ is yet another constant, whenever $\zeta, \xi \in [-c,c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, and c. The proof is now concluded by invoking line 2 of (3.6). q.e.d.

Forces through the perturbed surface. Recall, in addition to (5.1) and (5.2), the perturbed unit normal $\nu[u] = \nu[N,k,\ell,m,\zeta,\xi,u]$ defined just after (5.1) for sufficiently small u. For such u and for each integer $i \in [1,N]$ we define the force

$$\begin{aligned} \widetilde{\mathcal{F}}_i &= \widetilde{\mathcal{F}}_i[N,k,\ell,m,\zeta,\xi,u] \\ &:= \int_{\Omega_i} \mathcal{H}[u]\left(g_S \circ \iota[u]\right)(K \circ \iota[u], \nu[u]) \sqrt{|\iota[u]^* g_S|}, \end{aligned} \tag{6.16}$$

the perturbation by u of (3.2), where $\sqrt{|\iota[u]^* g_S|}$ is the area form induced by $\iota[u]$ and g_S. We will need the following estimate for $\widetilde{\mathcal{F}}_i$.

Lemma 6.17 (Estimates of the perturbations to the forces)**.** *Given $C_u, c > 0$, $\alpha, \gamma \in (0,1)$, and integers $N \geq 2$ and $\ell \geq k \geq 1$, there exist real numbers $\widetilde{c} := \widetilde{c}[N,k,\ell,C_u] > 0$ and $m_0 := m_0[N,k,\ell,m,C_u,c] > 0$ such that (recalling* (3.2) *and* (6.16)*)*

$$\left|\widetilde{\mathcal{F}}_i[N,k,\ell,m,\zeta,\xi,u] - \mathcal{F}_i[N,k,\ell,m,\zeta,\xi]\right| \leq \widetilde{c} m^{-2}\tau_1 \tag{6.18}$$

whenever $1 \leq i \leq N$, $m > m_0$, $\zeta, \xi \in [-c,c]^{N-1}$, and $u \in C^{2,\alpha}(\Sigma,\chi)$ satisfies $\|u\|_{2,\alpha,\gamma} \leq C_u\tau_1$; furthermore, for each fixed i, N, k, ℓ, and $m > m_0$, the map $(u,\zeta,\xi) \mapsto \widetilde{\mathcal{F}}_i[N,k,\ell,m,\zeta,\xi,u]$ is continuous (in the sense of Definition 4.101).

We emphasize that in the statement of Lemma 6.17 $\widetilde{c}$ does not depend on c or m.

Proof. The continuity is clear from the smooth dependence of the initial surfaces on the ζ, ξ parameters and from definitions (5.1) and (5.2). Turning to the estimate, obviously

$$\begin{aligned} \widetilde{\mathcal{F}}_i - \mathcal{F}_i = \int_{\Omega_i} \mathcal{H}[u]\,(g_S \circ \iota[u])(K \circ \iota[u], \nu[u])\,\left[\sqrt{|g[u]|} - \sqrt{|g|}\right] \\ + \int_{\Omega_i} \mathcal{H}[u]\,\left[(g_S \circ \iota[u])(K \circ \iota[u], \nu[u]) - (g_S \circ \iota)(K \circ \iota, \nu)\right]\,\sqrt{|g|} \\ + \int_{\Omega_i} \left[\mathcal{H}[u] - H\right]\,(g_S \circ \iota)(K \circ \iota, \nu)\,\sqrt{|g|}, \end{aligned} \tag{6.19}$$

using the notation of Lemma 4.42.

From (5.2), (5.5), Corollary 4.106, Lemma 6.2, (2.14), (3.32), and the assumption that $\|u\|_{2,\alpha,\gamma} \leq C_u\tau_1$

$$\left\|\rho^{-2}\mathcal{H}[u]\right\|_{0,\alpha,\gamma} = \left\|\rho^{-2}H + \mathcal{L}_\chi u + \rho^{-2}\mathcal{Q}[u]\right\|_{0,\alpha,\gamma} \leq m\tau_1 \tag{6.20}$$

whenever $\zeta, \xi \in [-c,c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, C_u, and c. By (3.1)

$$|(g_S \circ \iota[u])(K \circ \iota[u], \nu[u])| + |(g_S \circ \iota)(K \circ \iota, \nu)| \leq 2 \tag{6.21}$$

and, using also (4.64), the proof of Lemma 6.2 (particularly (6.11)), and again the assumption $\|u\|_{2,\alpha,\gamma} \leq K\tau_1$,

$$\begin{aligned} C[N,k,\ell,C_u]\tau_1 \geq \big\|(g_S \circ \iota[u])(K \circ \iota[u], \nu[u]) \\ -(g_S \circ \iota)(K \circ \iota, \nu) : C^{1,\alpha}\left(\Sigma, \chi, m^\gamma \rho^{1-\gamma}\right)\big\|. \end{aligned} \tag{6.22}$$

By item (ii) of Lemma 4.42 and the proof of Lemma 6.2 (particularly the estimate of the second term in 6.12)

$$\left\| \sqrt{|g[u]|} - \sqrt{|g|} : C^{1,\alpha}\left(\Sigma, \chi, m^\gamma \rho^{-1-\gamma}\right) \right\| \leq C[N,k,\ell,C_u]\tau_1. \tag{6.23}$$

Finally, for the χ area $|\Omega_i|_\chi$ of Ω_i we have, whenever $\zeta, \xi \in [-c,c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, and c, the estimate

$$\begin{aligned} |\Omega_i|_\chi &\leq |\mathcal{T}[i] \backslash (\mathcal{K}[i-1] \cup \mathcal{K}[i])|_\chi + |\mathcal{K}[i-1]|_\chi + |\mathcal{K}[i]|_\chi \\ &= \int_{-\frac{\pi}{\sqrt{2}\ell}}^{\frac{\pi}{\sqrt{2}\ell}} \int_{-\frac{\pi}{\sqrt{2}k}}^{\frac{\pi}{\sqrt{2}k}} \left(1 + Cm^2\tau_1\right)\, d\mathrm{x}\, d\mathrm{y} \\ &\quad + 4\int_0^{2\pi} \int_0^{m^2+2c} \left(1 + Cm^2\tau_1\right)\, dt\, d\theta \leq 200m^2, \end{aligned} \tag{6.24}$$

recalling (2.29), (4.26), and (4.73), understanding $\mathcal{K}[0] = \mathcal{K}[N] = \emptyset$, and using (2.12), (4.24) and Propositions 4.30 and 4.79, which supply the constant $C = C[N,k,\ell]$.

It now follows from the estimates of the previous paragraph (and (4.15) and line 2 of (3.6)) that, whenever $\zeta, \xi \in [-c,c]^{N-1}$ and m is sufficiently large in terms of N, k, ℓ, c, and C_u,

$$\begin{aligned} &\left| \int_{\Omega_i} \mathcal{H}[u]\, (g_S \circ \iota[u])(K \circ \iota[u], \nu[u])\, \left[\sqrt{|g[u]|} - \sqrt{|g|}\right] \right| \\ &\qquad \leq Cm^{3+2\gamma}\tau_1^2 \left\|\rho^{1-2\gamma}\right\|_{C^0(\Sigma)} \leq m^{-2}\tau_1 \qquad \text{and} \\ &\left| \int_{\Omega_i} \mathcal{H}[u]\, \left[(g_S \circ \iota[u])(K \circ \iota[u], \nu[u]) - (g_S \circ \iota)(K \circ \iota, \nu)\right]\, \sqrt{|g|} \right| \\ &\qquad \leq Cm^{3+2\gamma}\tau_1^2 \left\|\rho^{1-2\gamma}\right\|_{C^0(\Sigma)} \leq m^{-2}\tau_1 \end{aligned} \tag{6.25}$$

(regardless of the sign of $1 - 2\gamma$). Furthermore, using also Lemma 6.2 (as well as (2.14)),

$$\left| \int_{\Omega_i} \mathcal{Q}[u]\, (g_S \circ \iota)(K \circ \iota, \nu)\, \sqrt{|g|} \right| \leq Cm^2 \underline{\tau}_1^{1+\gamma/2} \leq m^{-2}\tau_1, \tag{6.26}$$

again for m sufficiently large in terms of N, k, ℓ, c, and C_u. Therefore

$$\begin{aligned} &\left| \int_{\Omega_i} \left[\mathcal{H}[u] - H\right]\, (g_S \circ \iota)(K \circ \iota, \nu)\, \sqrt{|g|} \right| \leq \\ &\qquad \left| \int_{\Omega_i} (g_S \circ \iota)(K \circ \iota, \nu)\, \mathcal{L}u\, \sqrt{|g|} \right| + m^{-2}\tau_1, \end{aligned} \tag{6.27}$$

recalling (5.4), but K is Killing, so integration by parts (specifically Green's identity) yields

$$\begin{aligned}\int_{\Omega_i} (g_S \circ \iota)(K \circ \iota, \nu)\, \mathcal{L}u \sqrt{|g|} = &- \int_{\Omega_i} u\,(g_S \circ \iota)(K \circ \iota, \iota_* \nabla_g H) \sqrt{|g|} \\ &+ \int_{\partial\Omega_i} [(g_S \circ \iota)(K \circ \iota, \nu)(\eta u) \\ &\quad - u\,(\eta\,[(g_S \circ \iota)(K \circ \iota, \nu)])] \sqrt{|g|},\end{aligned} \tag{6.28}$$

where η is the outward conormal on Ω_i induced by g (and acts on functions as a derivation).

Using (3.1), (3.32), (4.78), (4.108), and (6.24), it follows that

$$\begin{aligned}&\left|\int_{\Omega_i} u\,(g_S \circ \iota)(K \circ \iota, \iota_* \nabla_g H) \sqrt{|g|}\right| \\ &\leq C C_u m^{2+\gamma} \tau_1 \left(\left\|m^2 \rho^{-1-\gamma} \tau_1 + m^2 \rho^{1-\gamma} \tau_1^2\right\|_0 + c m^{1-\gamma} \tau_1\right) \\ &\leq m^{-2} \tau_1\end{aligned} \tag{6.29}$$

for m sufficiently large in terms of N, k, ℓ, C_u, and c. Turning to the boundary term, as in the computation following (3.3), $\partial\Omega_i$ has one or two circular components (catenoidal waists) and a single rectangular component. Suppose $S := \kappa_i(\{t = 0\})$ or $S := \kappa_{i-1}(\{t = 0\})$ is a circular component and $T := \partial\Omega_i \backslash [\kappa_{i-1}(\{t = 0\}) \cup \kappa_i(\{t = 0\})]$ is the rectangular component. By (2.2), (2.29), and (3.1)

$$\begin{aligned}C \geq &m^2 \left\|(g_S \circ \iota)(K \circ \iota, \nu) - 1 : C^0(T)\right\| \\ &+ m \left\|\eta\,[(g_S \circ \iota)(K \circ \iota, \nu)] : C^0(T)\right\|\end{aligned} \tag{6.30}$$

for some constant $C = C[N, k, \ell] > 0$ and obviously T has g length $|T|\,|_{g|T} \leq C m^{-1}$ and $\left\|u : C^0(T)\right\| \leq C_u \tau_1$, so

$$\left|\int_T u\,(\eta\,[(g_S \circ \iota)(K \circ \iota, \nu)])\, \sqrt{|g|_T|}\right| \leq C m^{-2} \tau_1 \tag{6.31}$$

for another constant $C = C[N, k, \ell, C_u] > 0$, while $\left\|\eta u : C^0(T)\right\| \leq C_u m \tau_1$, so

$$\begin{aligned}&\left|\int_T [(g_S \circ \iota)(K \circ \iota, \nu)]\,(\eta u) \sqrt{|g|_T|}\right| \\ &\leq C m^{-2} \tau_1 + \left|\int_T (\eta u) \sqrt{|g|_T|}\right| \leq C m^{-2} \tau_1,\end{aligned} \tag{6.32}$$

where for the first inequality we have used (6.30) and for the second we have used the fact that, because u is $\mathcal{G}$-equivariant, it satisfies periodic boundary conditions on T and accordingly $\int_T (\eta u) \sqrt{|g|_T|} = 0$.

On the other hand, on S we have

$$\begin{aligned} &\left\| u : C^0(S) \right\| \leq C_u m^\gamma \tau_1^{1+\gamma}, \\ &\left\| \eta u : C^0(S) \right\| \leq C_u m^\gamma \tau_1^\gamma, \text{ and} \\ &\left\| \eta \left[(g_S \circ \iota)(K \circ \iota, \nu) \right] : C^0(S) \right\| \\ &\quad \leq \sup_S \left(\left| D[g_S] K \right|_{g_S} + |K|_{g_S} |A|_g \right) \leq C \tau_1^{-1} \end{aligned} \tag{6.33}$$

for some constant $C = C[N, k, \ell] > 0$, having used item (v) of Proposition 4.30 for the last inequality, and S has g length $|S|_{g|_S} \leq C\tau_1$, so

$$\begin{aligned} m^{-2}\tau_1 &\geq C[N, k, \ell, C_u] m^\gamma \tau_1^{1+\gamma} \\ &\geq \int_S \left(\left| u \left(\eta \left[(g_S \circ \iota)(K \circ \iota, \nu) \right] \right) \right| \right. \\ &\qquad \left. + \left| \left[(g_S \circ \iota)(K \circ \iota, \nu) \right] (\eta u) \right| \right) \sqrt{|g|_S|}, \end{aligned} \tag{6.34}$$

provided m is sufficiently large in terms of N, k, ℓ, c, and C_u, yet again using line 2 of (3.6) for the last inequality. The proof is now completed by combining (6.19), (6.25), (6.27), (6.28), (6.29), (6.31), (6.32), and (6.34). q.e.d.

Explicitly defined diffeomorphisms between initial surfaces. Recall Remark 2.33 and Definition 4.101. Throughout the construction we have made use of the existence of maps $I[N, k, \ell, m]$ as in Remark 2.33 in order to identify function spaces defined on initial surfaces with identical data N, k, ℓ, and m but different ζ, ξ parameter values. So far we have made these identifications merely so as to articulate certain continuity properties, which do not depend on the choice of I. In the proof of the main theorem, however, we will need bounds for the norms of the corresponding identification maps between our normed function spaces, and so we now explicitly define diffeomorphisms between the initial surfaces. We define these diffeomorphisms as compromises between natural identifications on the various standard regions. More precisely, recalling (4.24), (4.25), (4.26), (4.72), and (4.73), for any given data N, k, ℓ, m, and ζ, ξ we start by defining, for $1 \leq i \leq N - 1$,

$$\begin{aligned} a_i &:= a_i[N, k, \ell, m, \zeta, \xi] \quad \text{and} \quad \underline{a}_i := a_i[N, k, \ell, m, 0, 0], \\ \mathcal{K}[i] &:= \mathcal{K}[i; N, k, \ell, m, \zeta, \xi] \quad \text{and} \quad \underline{\mathcal{K}}[i] := \mathcal{K}[i; N, k, \ell, m, 0, 0], \\ \kappa_i &:= \kappa_i[N, k, \ell, m, \zeta, \xi] \quad \text{and} \quad \underline{\kappa}_i := \kappa_i[N, k, \ell, m, 0, 0], \end{aligned} \tag{6.35}$$

and also, for $1 \leq i \leq N$,

$$\begin{aligned} \mathcal{T}[i] &:= \mathcal{T}[i; N, k, \ell, m, \zeta, \xi] \quad \text{and} \quad \underline{\mathcal{T}}[i] := \mathcal{T}[i; N, k, \ell, m, 0, 0], \\ T_i &:= T_i[N, k, \ell, m, \zeta, \xi] \quad \text{and} \quad \underline{T}_i := T_i[N, k, \ell, m, 0, 0]. \end{aligned} \tag{6.36}$$

We observe that the map

$$T_i \circ \underline{T}_i^{-1}\big|_{\underline{\mathcal{T}}[i]\setminus(\underline{\mathcal{K}}[i-1]\cup\underline{\mathcal{K}}[i])} : \\ \underline{\mathcal{T}}[i]\setminus(\underline{\mathcal{K}}[i-1]\cup\underline{\mathcal{K}}[i]) \to \mathcal{T}[i]\setminus(\mathcal{K}[i-1]\cup\mathcal{K}[i]), \tag{6.37}$$

(understanding $\mathcal{K}[0]=\mathcal{K}[N]=\emptyset$) is a well-defined diffeomorphism. We also observe (recalling (4.22)) that whenever $\mathcal{T}[i]\cap\mathcal{K}[j]\neq\emptyset$, the map $T_i\circ\underline{T}_i^{-1}\circ\underline{\kappa}_j$ is well-defined on the component of $\mathbb{K}_{\underline{a}_j}\setminus\mathbb{K}_{\underline{a}_j-1}$ whose image under $\underline{\kappa}_j$ lies in $\underline{\mathcal{T}}[i]$ and that on this set $T_i\circ\underline{T}_i^{-1}\circ\underline{\kappa}_j$ has image contained in $\mathcal{K}[j]$ and moreover satisfies

$$\left(T_i\circ\underline{T}_i^{-1}\circ\underline{\kappa}_j\right)(\underline{t},\theta)=\kappa_j\left((\operatorname{sgn}\underline{t})\operatorname{arcosh}\left[\frac{\underline{\tau}_j}{\tau_j}\cosh\underline{t}\right],\theta\right), \tag{6.38}$$

where $\operatorname{sgn}:\mathbb{R}\to\mathbb{R}$ takes the value 1 when its argument is nonnegative and -1 otherwise. Note that (using the identity (2.20))

$$\operatorname{arcosh}\left(\frac{\underline{\tau}_j}{\tau_j}\cosh\underline{t}\right)=|\underline{t}|+\ln\frac{\underline{\tau}_j}{2\tau_j}+\ln\left(1+e^{-2|\underline{t}|}\right) \\ +\ln\left(1+\sqrt{1-\tau_j^2\underline{\tau}_j^{-2}\operatorname{sech}^2\underline{t}}\right). \tag{6.39}$$

So motivated, for $1\leq j\leq N-1$ we define the function $\widetilde{t}_j:\mathbb{R}\to\mathbb{R}$ by

$$\widetilde{t}_j(\underline{t}):=\frac{a_j}{\underline{a}_j}\underline{t}\cdot\psi\left[\underline{a}_j,\underline{a}_j-1\right](|\underline{t}|) \\ +(\operatorname{sgn}\underline{t})\operatorname{arcosh}\left(\frac{\underline{\tau}_j}{\tau_j}\cosh\underline{t}\right)\cdot\psi\left[\underline{a}_j-1,\underline{a}_j\right](|\underline{t}|). \tag{6.40}$$

Using

$$\frac{d}{d\underline{t}}\operatorname{arcosh}\left(\frac{\underline{\tau}_j}{\tau_j}\cosh\underline{t}\right)=\frac{\tanh\underline{t}}{\sqrt{\underline{\tau}_j^2\cosh^2\underline{t}-\tau_j^2}}\,\underline{\tau}_j\cosh\underline{t}, \\ \operatorname{arcosh}\left(\frac{\underline{\tau}_j}{\tau_j}\cosh\pm\underline{a}_j\right)=\pm a_j, \\ \text{and (by (4.24))}\quad\left|\frac{a_j}{\underline{a}_j}-1\right|\leq\frac{2c}{m}, \tag{6.41}$$

along with (2.14) and (4.24), we see that by taking m sufficiently large in terms of c we can guarantee that $\widetilde{t}$ takes $[-\underline{a}_j,\underline{a}_j]$ monotonically onto $[-a_j,a_j]$. Away from the ends of $[-\underline{a}_j,\underline{a}_j]$ this reparametrization is simply multiplication by $a_j/\underline{a}_j\approx 1$, while close to the ends it almost agrees with the map $\underline{t}\mapsto\underline{t}+(\operatorname{sgn}\underline{t})(a_j-\underline{a}_j)$.

We can now define the diffeomorphism

$$P[\zeta,\xi]=P[N,k,\ell,m,\zeta,\xi]: \\ \Sigma[N,k,\ell,m,0,0]\to\Sigma[N,k,\ell,m,\zeta,\xi] \tag{6.42}$$

by requiring that

(i) P commute with the action of $\mathcal{G}$ (recalling (2.26)),

(ii) for $1 \leq i \leq N$

$$P[\zeta,\xi]|_{\underline{\mathcal{T}}[i]\setminus(\underline{\mathcal{K}}[i-1]\cup\underline{\mathcal{K}}[i])} := T_i \circ \underline{T}_i^{-1}, \tag{6.43}$$

(iii) and for $1 \leq i \leq N-1$

$$\left(P[\zeta,\xi]\circ\underline{\kappa}_i\right)(\underline{t},\theta) := \kappa_i\left(\widetilde{t}(\underline{t}),\theta\right) \text{ for all } (\underline{t},\theta)\in\mathbb{K}_{\underline{a}_i}$$

(continuing to understand $\mathcal{K}[0] = \mathcal{K}[N] = \emptyset$). We define in turn the map

$$\mathcal{P} = \mathcal{P}[\zeta,\xi] = \mathcal{P}[N,k,\ell,m,\zeta,\xi] := P[N,k,\ell,m,\zeta,\xi]^* \tag{6.44}$$

taking functions on $\Sigma[N,k,\ell,m,\zeta,\xi]$ to functions on $\Sigma[N,k,\ell,m,0,0]$. Clearly the map $I = I[N,k,\ell] : \mathbb{R}^{N-1}\times\mathbb{R}^{N-1}\times\Sigma[N,k,\ell,0,0]\to\mathbb{S}^3$ defined by $I(\zeta,\xi,\cdot) := \iota[N,k,\ell,m,\zeta,\xi]\circ P[N,k,\ell,m,\zeta,\xi]$ satisfies the properties specified in Remark 2.33. Last we have the following estimate.

Lemma 6.45 (Bound for $\mathcal{P}$ and its inverse)**.** *Given real numbers $\alpha,\gamma\in(0,1)$ and $c>0$ as well as integers $N\geq 2$ and $\ell\geq k\geq 1$, there exist real numbers $C = C[N,k,\ell,\alpha,\gamma]>0$ and $m_0 = m_0[N,k,\ell,c]>0$ such that whenever $\zeta,\xi\in[-c,c]^{N-1}$ and $m>m_0$ we have (recalling (4.98)) the estimates*

$$\begin{aligned}
&\|\mathcal{P}[\zeta,\xi]u\|_{C^{2,\alpha,\gamma}(\Sigma[N,k,\ell,m,0,0])}\\
&\qquad\leq Ce^{2c}\|u\|_{C^{2,\alpha,\gamma}(\Sigma[N,k,\ell,m,\zeta,\xi])} \quad \textit{and}\\
&\left\|\mathcal{P}[\zeta,\xi]^{-1}v\right\|_{C^{2,\alpha,\gamma}(\Sigma[N,k,\ell,m,\zeta,\xi])}\\
&\qquad\leq Ce^{2c}\|v\|_{C^{2,\alpha,\gamma}(\Sigma[N,k,\ell,m,0,0])}\,.
\end{aligned} \tag{6.46}$$

Proof. Let $u\in C^{2,\alpha,\gamma}(\Sigma[N,k,\ell,m,\zeta,\xi])$. By (4.70), (4.72), and (6.35)

$$T_i^{-1}(\mathcal{T}[i]\setminus(\mathcal{K}[i-1]\cup\mathcal{K}[i])) = \underline{T}_i^{-1}(\underline{\mathcal{T}}[i]\setminus(\underline{\mathcal{K}}[i-1]\cup\underline{\mathcal{K}}[i])), \tag{6.47}$$

and by (6.43) and (6.44)

$$\begin{aligned}
&\left(T_i^* u\right)(\mathrm{x},\mathrm{y}) = \left(\underline{T}_i^*\left(\mathcal{P}u\right)\right)(\mathrm{x},\mathrm{y})\\
&\quad\text{for all } (\mathrm{x},\mathrm{y})\in T_i^{-1}(\mathcal{T}[i]\setminus(\mathcal{K}[i-1]\cup\mathcal{K}[i])),
\end{aligned} \tag{6.48}$$

while by (6.40)

$$\begin{aligned}
&\left(\kappa_i^* u\right)\left((\widetilde{t}(\underline{t}),\theta\right) = \left(\underline{\kappa}_i^*\left(\mathcal{P}u\right)\right)(\underline{t},\theta)\\
&\quad\text{for all } (\underline{t},\theta)\in\mathbb{K}_{\underline{a}_i} \text{ (equivalently all } \left(\widetilde{t}(\underline{t}),\theta\right)\in\mathbb{K}_{a_i}).
\end{aligned} \tag{6.49}$$

The asserted bounds are now clear from (4.98), using (2.14), (4.14), (6.41), and Propositions 4.30 and 4.79. q.e.d.

The main theorem. We are ready to prove the main theorem.

Theorem 6.50 (The main theorem)**.** *Let $\alpha, \gamma \in (0,1)$. Given integers $N \geq 2$ and $\ell \geq k \geq 1$, there are real numbers $C, \underline{c}, m_0 > 0$ such that for every $m > m_0$ there exist parameters $\zeta, \xi \in [-\underline{c}, \underline{c}]^{N-1}$ and a function $u \in C^\infty_{\mathcal{G}}(\Sigma[N,k,\ell,m,\zeta,\xi])$ (recalling (2.26), (2.30), and Notation 4.5) such that $\|u\|_{2,\alpha,\gamma} \leq C\tau_1$ (recalling (2.14) and (4.98)) and the image of the normal deformation $\iota[u] : \Sigma \to \mathbb{S}^3$ (recalling (5.1)) by u of the inclusion $\iota : \Sigma \to \mathbb{S}^3$ is a closed embedded minimal surface invariant under $\mathcal{G}[k,\ell,m]$ and having genus $k\ell m^2(N-1)+1$.*

Proof. Fix $\alpha, \gamma \in (0,1)$ and integers $N \geq 2$ and $\ell \geq k \geq 1$. For each integer $m \geq 1$ set

$$B[N,k,\ell,m] := \Big\{ v \in C^{2,\alpha/2}_{\mathcal{G}[k,\ell,m]}(\Sigma[N,k,\ell,m,0,0],\chi) \\ : \|v\|_{2,\alpha,\gamma} \leq \underline{\tau}_1^{1+\gamma/3} \Big\} \tag{6.51}$$

(recalling (2.13)). Given $\zeta, \xi \in \mathbb{R}^{N-1}$ and assuming m sufficiently large, define also

$$\big(u_1, (\lambda_1, \ldots, \lambda_N), (\underline{\lambda}_2, \ldots, \underline{\lambda}_{N-1})\big) \\ := -\mathcal{R}\left(\rho^{-2}H - \sum_{i=2}^{N-1} \mathcal{D}_i \underline{w}_i\right)[N,k,\ell,m,\zeta,\xi], \tag{6.52}$$

as in Corollary 5.66 (recalling (3.32), (4.8), (4.14), (4.78), and Proposition 5.47), and for each $v \in B[N,k,\ell,m]$ define

$$\Big(v', (\mu_1[v], \ldots, \mu_N[v]), \Big(\underline{\mu}_2[v], \ldots, \underline{\mu}_{N-1}[v]\Big)\Big) \\ := -\mathcal{R}\left(\rho^{-2}\mathcal{Q}\left[u_1 + \mathcal{P}^{-1}v\right]\right)[N,k,\ell,m,\zeta,\xi] \tag{6.53}$$

(recalling (6.1) and (6.44)).

Thus, for all $\zeta, \xi \in \mathbb{R}^{N-1}$ and $v \in B[N,k,\ell,m]$, provided m is sufficiently large in terms of N, k, ℓ, and ζ, ξ, recalling (5.2), (5.4), (5.5), (6.1), and Proposition 5.47,

$$\begin{aligned} \rho^{-2}\mathcal{H}\left[u_1 + v', \zeta, \xi\right] &= \rho^{-2}H + \mathcal{L}_\chi(u_1 + v') + \rho^{-2}\mathcal{Q}[u_1 + v'] \\ &= \left[\mathcal{L}_\chi u_1 + \left(\rho^{-2}H - \sum_{i=2}^{N-1} \mathcal{D}_i \underline{w}_i\right)\right] \\ &\quad + \rho^{-2}\mathcal{Q}[u_1 + v'] + \mathcal{L}_\chi v' + \sum_{i=2}^{N-1} \mathcal{D}_i \underline{w}_i \\ &= \rho^{-2}\mathcal{Q}\left[u_1 + v'\right] - \rho^{-2}\mathcal{Q}\left[u_1 + \mathcal{P}^{-1}v\right] \\ &\quad + \sum_{i=1}^{N} \left(\lambda_i + \mu_i[v]\right) w_i \end{aligned} \tag{6.54}$$

$$+\sum_{i=2}^{N-1}\Big(\mathcal{D}_i+\underline{\lambda}_i+\underline{\mu}_i[v]\Big)\,\underline{w}_i.$$

Evidently we want to pick (v,ζ,ξ) so that $\mathcal{P}v'=v$ (to make the nonlinear terms cancel), $\lambda_i+\mu_i[v]=0$ for all $i\in\mathbb{Z}\cap[1,N]$ (to make the w_i terms vanish), and $\mathcal{D}_i+\underline{\lambda}_i+\underline{\mu}_i[v]=0$ for all $i\in\mathbb{Z}\cap[2,N-1]$ (to make the $\underline{w}_i$ terms vanish). Recalling (3.1), the unit normal ν for Σ specified just above (4.8), and (5.30), we observe that on the support of $w_i|_{\Omega_i}$ the function $(g_S\circ\iota)(K\circ\iota,\nu)$ has a sign (namely $(-1)^{N-1}$) and the function w_i itself is nonnegative. Consequently, recalling (6.16), if $\mathcal{P}v'=v$ and $\mathcal{D}_i+\underline{\lambda}_i+\underline{\mu}_i[v]=0$ for all $i\in\mathbb{Z}\cap[2,N-1]$, then, for any given $i\in\mathbb{Z}\cap[1,N]$, $\lambda_i+\mu_i[v]=0$ if and only if $\widetilde{\mathcal{F}}_i=0$. Accordingly, recalling (6.52), (6.53), and Lemma 3.33, we seek a fixed point for the map $\mathcal{J}:B[N,k,\ell,m]\times\mathbb{R}^{2N-2}\to C^{2,\alpha/2}_{\mathcal{G}[k,\ell,m]}(\Sigma[N,k,\ell,m,0,0],\chi)\times\mathbb{R}^{2N-2}$ given by

$$\mathcal{J}\left(v,\begin{pmatrix}\zeta_1\\ \vdots\\ \zeta_{N-1}\\ \xi_1\\ \vdots\\ \xi_{N-1}\end{pmatrix}\right)=\left(\mathcal{P}v',\begin{pmatrix}\zeta_1\\ \vdots\\ \zeta_{N-1}\\ \xi_1\\ \vdots\\ \xi_{N-1}\end{pmatrix}\right.$$
$$(6.55)\qquad \left.-\,\Theta^{-1}\tau_1^{-1}\begin{pmatrix}m^2\widetilde{\mathcal{F}}_1[N,k,\ell,m,\zeta,\xi,u_1+\mathcal{P}^{-1}v]\\ \vdots\\ m^2\widetilde{\mathcal{F}}_N[N,k,\ell,m,\zeta,\xi,u_1+\mathcal{P}^{-1}v]\\ \mathcal{D}_2[N,k,\ell,m,\zeta,\xi]+\underline{\lambda}_2+\underline{\mu}_2[v]\\ \vdots\\ \mathcal{D}_{N-1}[N,k,\ell,m,\zeta,\xi]+\underline{\lambda}_{N-1}+\underline{\mu}_{N-1}[v]\end{pmatrix}\right).$$

We will check that the hypotheses of the Schauder fixed-point theorem apply to $\mathcal{J}$, after restricting its domain as specified below. It is clear from definition (3.32) and from the continuity assertions made in Proposition 5.47, Corollary 5.66, Lemma 6.2, and Lemma 6.17 that $\mathcal{J}$ is continuous in the sense of Definition 4.101, with the product topology on the domain and target, the Euclidean topology on the $\mathbb{R}^{2N-2}$ factors, and the $C^{2,\alpha/2}$ topology on the function-space factors. Because each initial surface is compact, the topology of each Hölder space is independent of the underlying metric and $C^{2,\alpha}(\Sigma)$ embeds compactly in $C^{2,\alpha/2}(\Sigma)$ (as does the former's $\mathcal{G}$-equivariant subspace); therefore $B[N,k,\ell,m]$ is compact relative to the $C^{2,\alpha/2}$ topology and is clearly convex.

Now let C_R be the constant $C[N,k,\ell,\alpha,\gamma]$ from Proposition 5.47, let C_1 be the constant $C[N,k,\ell]$ from Corollary 5.66, let C_P be the constant

$C[N,k,\ell]$ from Lemma 6.45, let $\widetilde{c}$ be the constant $\widetilde{c}[N,k,\ell,2C_1]$ from Lemma 6.17, let C_Θ be the constant $C[N,k,\ell]$ from Lemma 3.33, let

$$\underline{c} := C_\Theta \left(C_\Theta + \sqrt{N}\sqrt{\widetilde{c}^2 + 4C_1^2} \right), \tag{6.56}$$

and let m_1 be the maximum of the quantities named $m_0[N,k,\ell,\underline{c}]$ from Proposition 2.31, Lemma 3.33, and Proposition 5.47 as well as the quantity named $m_0[N,k,\ell,\underline{c},\gamma]$ from Corollary 5.66 and the quantities named $m_0[N,k,\ell,m,2C_1,\underline{c}]$ from Lemma 6.2 and Lemma 6.17.

Suppose $m > m_1$, $\zeta,\xi \in [-\underline{c},\underline{c}]$, and $v \in B[N,k,\ell,m]$. Then by (6.51), (6.52), Corollary 5.66, and Lemma 6.45

$$\begin{aligned} &\sum_{i=1}^{N} |\lambda_i| + \sum_{i=2}^{N-1} |\underline{\lambda}_i| \leq C_1\tau_1 \qquad \text{and} \\ &\|u_1\|_{2,\alpha,\gamma} + \left\|\mathcal{P}^{-1}v\right\|_{2,\alpha,\gamma} \leq C_1\tau_1 + C_{\mathcal{P}} e^{2\underline{c}} \underline{\tau}_1^{1+\gamma/3} \leq 2C_1\tau_1, \end{aligned} \tag{6.57}$$

where for the last inequality we use (2.13), (2.14), and line 2 of (3.6) and we assume $m > m_2$ for some $m_2 = m_2[N,k,\ell,\gamma] \geq m_1$. It follows in turn, using (6.53), Proposition 5.47, Lemma 6.2, and Lemma 6.45, that

$$\left\|\mathcal{P}v'\right\|_{2,\alpha,\gamma} + \sum_{i=1}^{N} |\mu_i[v]| + \sum_{i=2}^{N-1} \left|\underline{\mu}_i[v]\right| \leq C_{\mathcal{P}} e^{2c} C_R \underline{\tau}_1^{1+\gamma/2} \leq \underline{\tau}_1^{1+\gamma/3}, \tag{6.58}$$

assuming, for the last inequality, that $m > m_3$ for some $m_3 = m_3[N,k,\ell,\gamma] \geq m_2$. In particular we have verified that

$$v \in B[N,k,\ell,m] \Rightarrow \mathcal{P}v' \in B[N,k,\ell,m]. \tag{6.59}$$

Continuing to assume $m > m_3$, from Lemma 3.33, Lemma 6.17, (6.56), (6.57), and (6.58) we find that for any $\zeta,\xi \in [-\underline{c},\underline{c}]^{N-1}$

$$\begin{aligned} &\left| \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{N-1} \\ \xi_1 \\ \vdots \\ \xi_{N-1} \end{pmatrix} - \Theta^{-1}\tau_1^{-1} \begin{pmatrix} m^2\widetilde{\mathcal{F}}_1[\zeta,\xi,u_1+\mathcal{P}^{-1}v] \\ \vdots \\ m^2\widetilde{\mathcal{F}}_N[\zeta,\xi,u_1+\mathcal{P}^{-1}v] \\ \mathcal{D}_2[\zeta,\xi] + \underline{\lambda}_2 + \underline{\mu}_2[v] \\ \vdots \\ \mathcal{D}_{N-1}[\zeta,\xi] + \underline{\lambda}_{N-1} + \underline{\mu}_{N-1}[v] \end{pmatrix} \right| \\ &\qquad \leq C_\Theta^2 + C_\Theta\sqrt{N\widetilde{c}^2 + 4(N-2)C_1^2} \leq \underline{c}, \end{aligned} \tag{6.60}$$

where the norm $|\cdot|$ is the Euclidean one on $\mathbb{R}^{2N-2}$ and we have suppressed the N, k, ℓ, and m arguments from each $\widetilde{\mathcal{F}}_i$ and $\mathcal{D}_i$. In conjunction with (6.59) this bound shows that $\mathcal{J}$ (defined in (6.55)) maps $B[N,k,\ell,m] \times [-\underline{c},\underline{c}]^{2N-2}$ to itself. Moreover, it is immediately clear from our observations in the paragraph following (6.55) that $\mathcal{J}$ is continuous and $B[N,k,\ell,m] \times [-\underline{c},\underline{c}]^{2N-2}$ is compact relative to the topology described there, and of course $B[N,k,\ell,m] \times [-\underline{c},\underline{c}]^{2N-2}$ is convex.

The Schauder fixed-point theorem therefore applies to guarantee the existence of a fixed point (v,ζ,ξ) for $\mathcal{J}$. If we set $u := u_1 + \mathcal{P}^{-1}v$, then, as discussed above in the paragraph containing (6.54), we get

$$\mathcal{H}[u,\zeta,\xi] = 0 \qquad \text{and} \qquad \|u\|_{2,\alpha,\gamma} \leq 2C_1\tau_1. \tag{6.61}$$

That u is actually smooth now follows from the minimality and standard regularity theory. We have already chosen m sufficiently large that $\iota[u]$ is an immersion. By taking m possibly even larger, we can guarantee embeddedness as follows. Recalling (4.14), consider in the initial surface Σ the overlapping subsets $K := \left\{\rho \geq m^2\right\}$ and $T := \left\{\rho \leq m^3\right\}$, so that K has $(N-1)k\ell m^2$ components, each contained in an isometric copy (under an element of $\mathcal{G}$) of some $\mathcal{K}[i]$, and T has N components, each a graph over $\mathbb{T}$. By scaling g_S it is clear that there exists $\epsilon = \epsilon[N,k,\ell,\underline{c}] > 0$ such that $\iota[u]|_K$ and $\iota[u]|_T$ are embeddings whenever (given that they are immersions) $\left\|u|_K : C^0(K)\right\| < \epsilon\tau_1$ and $\left\|u|_T : C^0(T)\right\| < \epsilon m^{-3}$. Both inequalities are ensured by the estimate for u in (6.61), assuming $m > m_4$ for some $m_4 = m_4[N,k,\ell,\gamma] \geq m_3$ (and, to get the first inequality, using the decay built into the norm $\|\cdot\|_{2,\alpha,\gamma}$ (4.98)). Moreover, there is a constant $\delta = \delta[N,k,\ell] > 0$ so that the distance between any two components of K is at least $\min\{\delta m^{-1}, \delta m^2\tau_1\}$, the distance between any two components of T is at least $\delta m^2\tau_1$, and the distance between any component of $K\backslash T$ and component of $T\backslash K$ is at least δm^{-2}. Of course $2C_1\tau_1 < m^2\tau_1 < m^{-2} < m^{-1}$ provided $m > m_0$ for some $m_0 = m_0[N,k,\ell,\gamma] \geq m_4$. Thus $\iota[u]$ is an embedding when $m > m_0$. In particular its image is diffeomorphic to Σ, so by Proposition 2.31 has the stated genus. This ends the proof. q.e.d.

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