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A PRIORI ESTIMATES OF TODA SYSTEMS, I: THE LIE ALGEBRAS OF $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ AND $\mathbf{G}_2$

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Abstract

It is well-known that the PDE (Partial Differential Equation) system arising from the infinitesimal Plücker formulas is a particular case of the Toda system of $\mathbf{A}_n$ type. In this paper, we prove an a priori estimate of solutions of the Toda systems associated with the simple Lie algebras $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ and $\mathbf{G}_2$. Previous results in this direction have been done only for the case of Lie algebras of rank two. Our result for $n \geq 3$ is new. The proof of this fundamental theorem is to combine techniques from PDE and the monodromy theory. One of the key steps is to calculate the local mass of a sequence of blowup solutions near each blowup point. At each blowup point, a sequence of bubbling steps (via scaling) are performed, and the local mass of the present step could be computed from the previous step. We find out that this transformation of the local mass of each step is related to the action of an element in the Weyl group of the Lie algebra. The correspondence of the Pohozaev identity and the Weyl group could reduce the complicated calculation of the local mass into a simpler one.

1. Introduction

Let (M, g) be a compact Riemann surface, and Δ_g be the Beltrami-Laplacian, and $u = (u_1, \cdots, u_n)$ be smooth functions on M which satisfies

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij} e^{u_j} = 4\pi \sum_{p_t \in S} \alpha_{t,i} \delta_{p_t} + \mathcal{F}_i, \; i \in I = \{1, 2, \cdots, n\}, \tag{1.1}$$

where $\alpha_{t,i} > -1$, δ_p stands for the Dirac measure at $p \in M$, S is a finite set of M and $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are smooth functions on M. Here (k_{ij}) is the Cartan matrix of a simple Lie algebra.

Let

$$\rho_i = 4\pi \sum_{p_t \in S} \gamma_{t,i} + \sum_{j=1}^{n} k^{ij} \int_M \mathcal{F}_j, \tag{1.2}$$

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where $\gamma_{t,i} = \sum_{j=1}^{n} k^{ij}\alpha_{t,j}$ and $(k^{ij})_{n\times n}$ is the inverse of $(k_{ij})_{n\times n}$. Our main theorem is the following a priori estimate:

Theorem 1.1. *Suppose that (k_{ij}) is the Cartan matrix of one of the simple Lie algebras $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ and $\mathbf{G}_2$, $\alpha_{t,i} \in \mathbb{N}$ and $\rho_i \notin 4\pi\mathbb{N}$ for any $i \in I$. Then for any compact subset $\mathcal{K} \subset M \setminus S$, there exists a constant $C > 0$ such that for any solution of (1.1) there holds*

$$|u_i(x)| \leq C,\ \forall x \in \mathcal{K},\ i \in I. \tag{1.3}$$

Equation (1.1) is closely related to the classical infinitesimal Plücker formula. Let f be an holomorphic curve from a simply connected domain D of $\mathbb{C}$ into $\mathbb{CP}^n$. Lift locally f to $\mathbb{C}^{n+1}$ and denote the lift by $\nu(z) = [\nu_0(z), \nu_1(z), \cdots, \nu_n(z)]$. The k-th associated curve of f is defined by $f_k : D \to G(k, n+1) \subset \mathbb{CP}(\Lambda^k\mathbb{C}^{n+1})$ and

$$f_k(z) = \left[\nu(z) \wedge \nu'(z) \wedge \cdots \wedge \nu^{(k-1)}(z)\right],$$

where $\nu^{(j)}$ is the j-th derivative of ν with respect to z. Let

$$\Lambda_k(z) = \nu(z) \wedge \cdots \wedge \nu^{(k-1)}(z),$$

and the well-known infinitesimal Plücker formula (see [**18**]) gives,

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k(z)\|^2 = \frac{\|\Lambda_{k-1}(z)\|^2 \|\Lambda_{k+1}(z)\|^2}{\|\Lambda_k(z)\|^4} \text{ for } k = 1, 2, \cdots, n, \tag{1.4}$$

where we define the norm $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ by the Fubini-Study metric in $\mathbb{CP}(\Lambda^k\mathbb{C}^{n+1})$ and put $\|\Lambda_0(z)\|^2 = 1$. We observe that (1.4) holds only for $\|\Lambda_k(z)\| > 0$, i.e., all the unramificated points z. Let us set $\|\Lambda_{n+1}(z)\| = 1$ by normalization (analytical extended at the ramificated points) and define U_k by

$$U_k(z) = -\log \|\Lambda_k(z)\|^2 + k(n-k+1)\log 2,\ 1 \leq k \leq n,$$

then, from (1.4) we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} U_i + \frac{1}{4} \exp\left(\sum_{j=1}^{n} k_{ij} U_j\right) = 0 \text{ in } D, \quad i \in I, \tag{1.5}$$

where

$$\mathbf{K} = (k_{ij})_{n\times n} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{1.6}$$

is the Cartan matrix of the simple Lie algebra $sl(n+1, \mathbb{C})$.

Set

$$u_i = \sum_{j=1}^{n} k_{ij} U_j, \tag{1.7}$$

(1.5) can be written as

$$\frac{\partial^2}{\partial z \partial \bar{z}} u_i + \frac{1}{4} \sum_{j=1}^{n} k_{ij} e^{u_j} = 0 \text{ in } D. \tag{1.8}$$

The equation (1.8) can be globally defined on any compact Riemann surface (M, g) if f is an holomorphic curve from M into $\mathbb{CP}^n$. Locally, we could introduce the complex structure z and let the metric $g = e^{2\psi(x)}|dz|^2$, where $\psi(x)$ satisfies

$$\Delta \psi(x) + K_0 e^{2\psi} = 0,$$

where Δ is the Laplacian with respect to the flat metric and K_0 is the Gaussian curvature with respect to the metric g. Then globally, the equation (1.8) can be rewritten as

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij} e^{u_j} - K_0 = 4\pi \sum_{p_t \in S} \alpha_{t,i} \delta_{p_t}, \ i \in I, \tag{1.9}$$

where S is a set of all the ramificated points and $\alpha_{t,i}$, $i \in I$ is the ramification index at the singular point p_t. This is the system (1.1) with $\mathcal{F}_i = K_0$, $i \in I$ and $\mathbf{K} = (k_{ij})_{n \times n}$ is the matrix (1.6).

The system (1.1) has been extensively studied in many disciplines of mathematics and physics. For example, when $n = 1$, (1.1) is reduced to a single equation, it is related to the Nirenberg problem of finding the prescribed Gaussian curvature if $S = \emptyset$, and the existence of the same curvature metric of problem (1.1) with conic singularities if $S \neq \emptyset$. While in physics, the Toda system is a well-known integrable system and closely related to the $\mathcal{W}$-algebra in conformal field theory, see [**1**, **17**] and references therein.

Integrating the equation (1.1), we can rewrite it as the following form

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij} \rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j} dV_g} - \frac{1}{|M|} \right) = \sum_{p_t \in S} 4\pi \alpha_{t,i} (\delta_{p_t} - 1) \text{ on } M, \tag{1.10}$$

with ρ_i given by (1.2), and

$$u_i \in \mathring{H}^1(M) = \left\{ f \in H^1(M) | \int_M f dV_g = 0 \right\}, \quad i \in I.$$

In this paper, we would like to consider (1.10) with positive C^1 functions h_i, $\alpha_{t,i} > -1$, $\rho_i \in \mathbb{R}^+$ and the matrix $\mathbf{K}$ is the Cartan matrix of the Lie algebras of $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$, where the Cartan matrices for $\mathbf{B}_n$ and

$\mathbf{C}_n$ respectively are

(1.11)
$$\mathbf{B}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{C}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

Given $\alpha_{t,i}$, we define

$$\mu_{t,i} = \alpha_{t,i} + 1 > 0. \tag{1.12}$$

For a given $\mu_1, \cdots, \mu_n$, in section 3 we will introduce the set of the local mass $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$

$$\Gamma(\mu_1, \cdots, \mu_n) = \Big\{ \boldsymbol{\sigma} \mid \sigma_i = 2 \sum_j a_{ij} \mu_j \text{ for } \mathcal{A} = [a_{ij}] \text{ given by (3.6)} \Big\}.$$

By Theorem 3.2, we will see any entry a_{ij} of $\mathcal{A}$ is a nonnegative integer. For example if the Lie algebra is $sl(3)$, then

$$\begin{aligned} \Gamma(\mu_1, \mu_2) = \Big\{ &(0,0), (2\mu_1, 0), (2\mu_1, 2\mu_1 + 2\mu_2), (0, 2\mu_2), \\ &(2\mu_1 + 2\mu_2, 2\mu_2), (2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2) \Big\}, \end{aligned}$$

and if the Lie algebra is $sl(4)$, then

$$\begin{aligned} \Gamma(\mu_1, \mu_2, \mu_3) = \Big\{ &(0,0,0), (2\mu_1, 0, 0), (0, 2\mu_2, 0), (0, 0, 2\mu_3), \\ &(2\mu_1 + 2\mu_2, 2\mu_2, 0), (2\mu_1, 2\mu_1 + 2\mu_2, 0), \\ &(2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2, 0), (2\mu_1, 0, 2\mu_3), \\ &(0, 2\mu_2 + 2\mu_3, 2\mu_3), (0, 2\mu_2, 2\mu_2 + 2\mu_3), \\ &(0, 2\mu_2 + 2\mu_3, 2\mu_2 + 2\mu_3), \\ &(2\mu_1, 2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2 + 2\mu_3), \\ &(2\mu_1 + 2\mu_2, 2\mu_2, 2\mu_2 + 2\mu_3), \\ &(2\mu_1, 2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_3), \\ &(2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_2 + 2\mu_3, 2\mu_3), \\ &(2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2, 2\mu_1 + 2\mu_2 + 2\mu_3), \\ &(2\mu_1, 2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_1 + 2\mu_2 + 2\mu_3), \\ &(2\mu_1 + 2\mu_2, 2\mu_1 + 4\mu_2 + 2\mu_3, 2\mu_2 + 2\mu_3), \\ &(2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_3), \\ &(2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_2 + 2\mu_3, 2\mu_2 + 2\mu_3), \\ &(2\mu_1 + 2\mu_2, 2\mu_1 + 4\mu_2 + 2\mu_3, 2\mu_1 + 2\mu_2 + 2\mu_3), \\ &(2\mu_1 + 2\mu_2 + 2\mu_3, 2\mu_1 + 4\mu_2 + 2\mu_3, 2\mu_2 + 2\mu_3), \end{aligned}$$

$$(2\mu_1+2\mu_2+2\mu_3, 2\mu_1+2\mu_2+2\mu_3, 2\mu_1+2\mu_2+2\mu_3),$$
$$(2\mu_1+2\mu_2+2\mu_3, 2\mu_1+4\mu_2+2\mu_3, 2\mu_1+2\mu_2+2\mu_3)\Big\}.$$

We introduce the critical parameter set Γ_i of (1.3) by

$$\Gamma_i=\Big\{2\pi\sum_{p_t\in R}\sigma_{t,i}+4\pi m \mid (\sigma_{t,1},\cdots,\sigma_{t,n})\in\Gamma(\mu_{t,1},\cdots,\mu_{t,n}),$$
$$R\subset S,\ m\in\mathbb{Z}\Big\}.$$

Our main theorem in this paper is the following a priori estimate for the system (1.10), which contains Theorem 1.1 as a particular case.

Theorem 1.2. *Suppose that $\rho_i\notin\Gamma_i$ for any $i\in I$. Then for any compact set $\mathcal{K}\subset M\setminus S$, there exists a constant $C>0$ such that for any solution u of* (1.10) *in $\mathring{H}^1(M)$,*

$$|u(x)|\leqslant C,\ \forall x\in\mathcal{K}. \tag{1.13}$$

If $\alpha_{t,i}\in\mathbb{N}$ then we will see $\sigma_i\in 2\mathbb{N}\cup\{0\}$ for any tuple $(\sigma_1,\cdots,\sigma_n)\in\Gamma(\mu_1,\cdots,\mu_n)$. Then Theorem 1.1 is a special case of the following corollary.

Corollary 1.3. *Suppose that $\alpha_{t,i}\in\mathbb{N}$ for any $i\in I$ and $p_t\in S$. If $\rho_i\notin 4\pi\mathbb{N}$, then the a priori estimate* (1.13) *holds for any solution of* (1.10) *in $\mathring{H}^1(M)$.*

In the past decades, there are a lot of works concerning the a priori estimate of the equation (1.10). In particular, for $n=1$, the study of a priori estimate began with the work of Brezis-Merle [**8**] and Theorem 1.2 was proved by Li [**23**] for the case without singularities, and Bartolucci and Tarantello [**2**] with singular sources. While for the case $n=2$, the problem becomes very difficult. In [**21**], the a priori estimate was first derived when (1.10) has no singular sources, and recently through a series of works by Lin-Wei-Zhang [**28**], Lin-Zhang [**29**], Lin-Wei-Yang-Zhang [**27**], a slightly weak version of Theorem 1.2 was obtained for the case of rank two simple Lie algebra. For $n\geq 3$, Theorem 1.2 is completely new.

To establish the a priori estimate, the most important issue is to understand the blow up phenomena of (1.10). A point p is called a blow up point if, along a subsequence, a sequence of solutions $\{u^k=(u_1^k,\cdots,u_n^k)\}$ satisfies

$$\max_i\max_{B(p,\delta)}u_i^k=\max_i u_i^k(p_k)\to+\infty,\quad p_k\to p.$$

The set of all the blow up points is called the blow up set.

Our proof of Theorem 1.2 is based on two results. The first one is

Theorem 1.4. *Suppose that $u^k=(u_1^k,\cdots,u_n^k)$ is a sequence of blowup solutions of* (1.10) *and B is the blow up set of u^k. Then at* *least one component* u_i^k *of* u^k *satisfies*

$$u_i^k(x) - \log \int_M h_i e^{u_i^k} dV_g \to -\infty \text{ for } x \notin B \cup S.$$

The second and harder part of the proof of Theorem 1.2 is to determine the local mass at each blow up point of u^k, which is defined as follows. Suppose that q is a blow up point of u^k, we define

$$\sigma_i(q) = \frac{1}{2\pi} \lim_{r\to 0} \lim_{k\to+\infty} \frac{\rho_i \int_{B(q,r)} h_i^k e^{u_i^k} \mathrm{d}x}{\int_M h_i^k e^{u_i^k} \mathrm{d}x}, \ i \in I, \tag{1.14}$$

where $B(q,r)$ is the ball centered at q with radius r. The main issue is how to calculate $\sigma_i(q)$. Of course the local mass σ_i can also be defined for a sequence of local solutions. Let $u^k = (u_1^k, \cdots, u_n^k)$ be a sequence of local solutions of

$$\Delta u_i^k + \sum_{j=1}^n k_{ij} h_j^k e^{u_j^k} = 4\pi\alpha_i\delta_0 \text{ in } B(0,1), \ i \in I, \tag{1.15}$$

where $\alpha_i > -1$ and $B(0,1)$ is the unit ball in $\mathbb{R}^2$. Throughout this paper, we assume h_i^k are smooth functions satisfying

$$h_i^k(0) = 1, \ \frac{1}{C} \le h_i^k \le C, \ \|h_i^k\|_{C^3(B(0,1))} \le C \text{ in } B(0,1), \ i \in I. \tag{1.16}$$

For solutions $u^k = (u_1^k, \cdots, u_n^k)$ of (1.15) we assume:

$$\begin{cases} (i) : 0 \text{ is the only blow up point of } u^k \text{ in } B(0,1), \\ (ii) : |u_i^k(x) - u_i^k(y)| \le C, \ \forall x, y \text{ on } \partial B(0,1), \ i \in I, \\ (iii) : \int_{B(0,1)} h_i^k e^{u_i^k} \le C, \ i \in I. \end{cases} \tag{1.17}$$

For this sequence of blowup solutions we define the local mass by

$$\sigma_i = \lim_{r\to 0} \lim_{k\to\infty} \frac{1}{2\pi} \int_{B(0,r)} h_i^k e^{u_i^k}, \ i \in I. \tag{1.18}$$

It is proved in [**28**] that $(\sigma_1, \cdots, \sigma_n)$ always satisfies the Pohozaev identity (P.I. in short):

$$\sum_{i=1}^n \sigma_i^2 - \sum_{i=1}^{n-1} \sigma_i\sigma_{i+1} = 2\sum_{i=1}^n \mu_i\sigma_i, \tag{1.19}$$

where $\mu_i = \alpha_i + 1$.

Theorem 1.5. *Suppose that* σ_i, $i \in I$ *are the local masses of a sequence of blowup local solutions of* (1.15) *such that the assumption* (1.17) *holds. Then there exists* $(\hat\sigma_1, \cdots, \hat\sigma_n) \in \Gamma(\mu_1, \cdots, \mu_n)$ *such that*

$$\sigma_i = \hat\sigma_i + 2m_i, \ m_i \in \mathbb{Z}, \ i \in I.$$

Our proof of Theorem 1.5 succeeds in combining two important features about (1.10): PDE and the integrable system. The PDE part is the bubbling analysis for selecting a finite set of blow up points $\{0, x_1^k, \cdots, x_m^k\}$ (if 0 is not singular point, then 0 can be deleted from this set) and balls $B(x_i^k, l_i^k)$. We have to analyze the behavior of u^k in these tiny balls by the scaling. The analysis shows that at each step of blowup in $B(x_i^k, l_i^k)$, only part of the re-scaling system will converge to a new system which can be decomposed into a decoupled $\mathbf{A}_n$-type Toda system. Part of this bubbling analysis has been carried out in [**27**, **28**]. To carry out the analysis [**27**, **28**], the authors in [**27**, **28**] introduces the notions of slow decay and fast decay. After [**27**, **28**], in order to complete the analysis, we have to prove that the fast decay does not increase the mass. For $n = 2$, it was proved in [**27**]. The original method in [**27**] can not be extended to the case $n \geq 3$. In any case, this important step is proved in section 5. Then it remains two questions, the first question is how to determine those total mass of the new $\mathbf{A}_n$-type Toda system from the previous step. One of the main discovery in this paper is that the *mass transformation from the one to the new one is related to the Weyl group of the Lie algebra* $\mathbf{A}_n$. Indeed this holds true also for $\mathbf{B}_n$ and $\mathbf{C}_n$. The second question is how to calculate the total mass outside the union of these balls $B(x_i^k, l_i^k)$. The computation of mass for this part seems not trivial at all from the analytic viewpoint. Indeed, this is where the integrability of the Toda system (1.9) comes to play an important role in our proof. See the main results in section 3. We remark that at the moment, we succeed only for Lie algebras $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$.

We say $(\alpha_1, \cdots, \alpha_n)$ satisfies $\mathbb{Q}$-condition if $\alpha_1, \cdots, \alpha_n$ are $\mathbb{Q}$-linearly independent. Theorem 1.5 has a sharper form if $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ satisfies the $\mathbb{Q}$-condition.

Theorem 1.6. *Suppose that $(\alpha_1, \cdots, \alpha_n)$ satisfies the $\mathbb{Q}$-condition and $u^k = (u_1^k, \cdots, u_n^k)$ is a sequence of blowup solutions to (1.15). If $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ is the local mass of u^k, then $\boldsymbol{\sigma} \in \Gamma(\mu_1, \cdots, \mu_n)$. Furthermore, the Harnack-type inequality holds:*

$$u_i^k(x) + 2\log|x| \leq C \text{ for } x \in B(0,1). \tag{1.20}$$

By Theorem 1.6, if $\boldsymbol{\alpha}_t = (\alpha_{t,1}, \cdots, \alpha_{t,n})$ satisfies the $\mathbb{Q}$-condition for any $p_t \in S$, then Theorem 1.2 can be strengthened. For $\boldsymbol{\sigma}_t = (\sigma_{t,1}, \cdots, \sigma_{t,n})$, we set

$$\Gamma_i^+ = \Big\{ 2\pi \sum_{p_t \in R} \sigma_{t,i} + 4\pi m \mid \boldsymbol{\sigma}_t \in \Gamma(\mu_{t,1}, \cdots, \mu_{t,n}),\ R \subset S,\ m \in \mathbb{N} \cup \{0\} \Big\},$$

then the following result holds:

Theorem 1.7. *If $\boldsymbol{\alpha}_t$ satisfy the $\mathbb{Q}$-condition for any $p_t \in S$ and $\rho_i \notin \Gamma_i^+$ for all $i \in I$, then the a priori estimate (1.13) holds for any solution u of (1.10) in $\mathring{H}^1(M)$.*

We conjecture that Γ_i^+ is the true critical parameter set for the system (1.10). But so far we could prove it is true for generic $\boldsymbol{\alpha}_t$ only.

Before we end this introduction, let us state some existence result from the a priori estimate obtained in this paper. The a priori bound of Theorem 1.2 could allow us to define the Leray-Schauder degree for equation (1.10). For equation (1.10) with $\rho = (\rho_1, \cdots, \rho_n)$, we denote the degree by d_ρ.

Theorem 1.8. *Suppose that $\alpha_{t,i}, i \in I$ are non-negative integers for any $p_t \in S$ and $\chi(M) \leq 0$. Let $\rho_i \notin 4\pi\mathbb{N}$ for some $i \in I$ and $\rho_j \in (0, 4\pi)$ for $j \in I \setminus \{i\}$. Then the topological degree of the equation (1.10) is non-zero.*

The counting degree formulas under the assumption of Theorem 1.8 could be obtained from the previous works on the mean field equation [**10**, **12**].

Recently, Battaglia [**5**] considered the general Liouville system (1.10) with $\alpha_{t,i}, i \in I$ are non-negative integers for any $p_t \in S$ and $\mathbf{K} = (k_{ij}),\ i, j \in I$ is symmetric, positive definite matrix with non-positive entries outside the diagonal (i.e., $k_{ij} \leq 0$ for any $i \neq j$). Assume the a priori bound holds and $\chi(M) \leq 0$, he has shown the existence result of (1.10), see [**5**, Corollary 1.3]. Through the following simple transformation:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ \rho_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ 2\rho_n \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ \rho_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -2 \\ 0 & \cdots & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-1} \\ \frac{1}{2}\rho_n \end{pmatrix},$$

we can write $\mathbf{B}_n$ and $\mathbf{C}_n$ Toda systems into a symmetric form. Together with [**5**, Corollary 1.3], we get the following existence theorem for $\mathbf{A}_n, \mathbf{B}_n$ and $\mathbf{C}_n$ Toda systems.

Theorem 1.9. *Suppose that $\chi(M) \leq 0$. Let $\rho_i \notin \Gamma_i$ for any $i \in I$. Then the equation (1.10) with $\mathbf{K} = \mathbf{A}_n$ always admits a solution. Similar results hold for $\mathbf{K} = \mathbf{B}_n$ or $\mathbf{C}_n$.*

The organization of this article is as follows. In section 2 we establish the global mass for the entire solutions of the $\mathbf{A}_n$ type Toda system with many singular sources defined in $\mathbb{R}^2$. This kind of result plays important roles in our bubbling analysis. In [**26**], it has been proved that any solution u is associated with a $(n+1)$-th order ODE in complex form. Our method is to apply the monodromy of this ODE to establish a formula of global mass. The aim of section 3 is to study the Pohozaev identity of all the possible solutions. As discussed above, we shall also explore the relation between the Pohozaev identity and the Weyl group. Then in section 4 we review some results of the bubbling analysis, which have been proved in the previous work [**28**]. In section 5 we present two crucial lemmas, which play the key role in the proof of main results. Then in section 6 and section 7 we discuss the local mass on each bubbling disk centered at 0 and not at 0 respectively, thereby we prove all the results of $\mathbf{A}_n$ type Toda system. In the last section, we provide all the counterpart results for $\mathbf{B}_n$ and $\mathbf{C}_n$ Toda system.

2. Total mass for Toda system

In this section, we shall consider the $SU(n+1)$ Toda system,

$$\begin{cases} \Delta u_i + \sum_{j=1}^n k_{ij} e^{u_j} = \sum_{t=1}^N 4\pi\alpha_{t,i}\delta_{p_t} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty,\ i = 1, 2, \cdots, n, \end{cases} \tag{2.1}$$

where $p_1, \cdots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_{t,i} > -1,\ \forall\ 1 \le i \le n,\ 1 \le t \le N$. We recall the Cartan matrix is given by (1.6). For any solution $u = (u_1, \cdots, u_n)$, we let

$$\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i}\mathrm{d}x,\ 1 \le i \le n, \tag{2.2}$$

and $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ be the total mass of the solution u. In this section we shall determine the total mass of u.

To investigate this problem, we shall regard (2.1) as an integrable system, that is, for any solution $u = (u_1, \cdots, u_n)$ there is an associated $(n+1)$-th order Fuchsian differential equation:

$$L(y) = y^{(n+1)}(z) + \sum_{j=0}^{n-1} W_j(z) y^{(j)}(z) = 0 \text{ in } \mathbb{C}, \tag{2.3}$$

whose singular points exactly are $p_t,\ t = 1, \cdots, N$. Here $W_j(z)$ are the so-called W-invariants and meromorphic functions defined on $\mathbb{C}$, see [**26**]. For a Fuchsian ODE, the most important data is the monodromy transformation $\mathcal{M}$. Fix a base point $z_0 \in \mathbb{C},\ z_0 \notin \{p_1, \cdots, p_N\}$, and a basis $\nu(z) = (\nu_0(z), \cdots, \nu_n(z))$ of solutions for (2.3). Let l_i (starting and ending at z_0) be a loop cycling around p_i once, $1 \le i \le N$ and $(l_i)^*\nu$ denote the analytic continuation of ν along l_i. Then there exists a matrix $\mathcal{M}_i$ such that

$$(l_i)^*(\nu(z))^t = (\nu(z))^t \mathcal{M}_i. \tag{2.4}$$

The matrix $\mathcal{M}_i$ is called the monodromy transformation. Without loss of generality, one may arrange the index i such that

$$\mathcal{M}_\infty \mathcal{M}_N \cdots \mathcal{M}_1 = \mathbb{I}_{n+1}. \tag{2.5}$$

The fundamental theorem about the Toda system is,

Theorem 2.A ([**26**]). *Let $u(z)$ be a solution of (2.1) and (2.3) be the associated ODE. Then there is a basis $(\nu_0, \cdots, \nu_n(z))$ of solutions to (2.3) such that*

$$e^{-\mathrm{U}_1(z)} = |\nu(z)|^2,$$

where $\mathrm{U}_1 = \sum_{j=1}^n k^{1j} u_j$.

Let $\beta_{t,i}$ be the local exponents of equation (2.3) at p_t, $1 \le t \le N$, then

$$\beta_{t,0} = -\alpha_t^1, \ \beta_{t,i} = \beta_{t,i-1} + \mu_{t,i}, \ 1 \le i \le n, \tag{2.6}$$

where $\alpha_t^i = \sum_{j=1}^n k^{ij}\alpha_{t,j}, \ 1 \le i \le n$. To compute the local exponent at ∞, we have

Lemma 2.1. *Suppose that the solution u satisfies*

$$u_i(x) = -2\alpha_{\infty,i} \log|x| + O(1) \text{ at } \infty. \tag{2.7}$$

Then the local exponent at ∞ is

$$\beta_{\infty,0} = -\alpha_\infty^1, \quad \beta_{\infty,j} = -\alpha_\infty^{j+1} + \alpha_\infty^j - j, \ 1 \le j \le n-1, \quad \beta_{\infty,n} = \alpha_\infty^n - n, \tag{2.8}$$

where $\alpha_\infty^i = \sum_{j=1}^n k^{ij}\alpha_{\infty,j}, \ 1 \le i \le n$.

Proof. By [**26**], the ODE (2.3) is written as

$$\begin{aligned} L &= (\partial_z - \mathrm{U}_{n,z})(\partial_z + \mathrm{U}_{n,z} - \mathrm{U}_{n-1,z}) \cdots (\partial_z + \mathrm{U}_{2,z} - \mathrm{U}_{1,z})(\partial_z + \mathrm{U}_{1,z}) \\ &= \partial_z^{n+1} + \sum_{j=0}^{n-1} W_j \partial_z^j, \end{aligned} \tag{2.9}$$

where $\mathrm{U}_i(z) = \sum_{i=1}^n k^{ij} u_j, \ 1 \le i \le n$. The exponents at $z = \infty$ can be computed by applying the transformation $w = \frac{1}{z}$ and the relation $\partial_z = -w^2 \partial_w$. Let

$$\tilde{\mathrm{U}}_i(w) = \mathrm{U}_i(\frac{1}{w}) = \mathrm{U}_i(z), \quad \text{and} \quad \tilde{V}_i(w) = \tilde{\mathrm{U}}_i(w) - 2\alpha_\infty^i \log|w|.$$

Then

$$\mathrm{U}_{i,z} = -w^2 \tilde{\mathrm{U}}_{i,w} = -w^2(\tilde{V}_{i,w} + \frac{\alpha_\infty^i}{w}),$$

and the linearized operator L is transformed to

$$\begin{aligned}\tilde{L} &= (-1)^{n+1}w^2(\partial_w - \frac{\alpha_\infty^n}{w} - \tilde{V}_{n,w})\cdots w^2(\partial_w + \frac{\alpha_\infty^1}{w} + \tilde{V}_{1,w}) \\ &= (-1)^{n+1}(\partial_w^{n+1} + \sum_{j=0}^{n-1}\tilde{W}_j\partial_w^j).\end{aligned} \tag{2.10}$$

It is not difficult to see that at point $w = 0$, the $\lim_{w\to 0} w^{n+1-j}\tilde{W}_j$ only depends on the terms of $\tilde{W}_j$ containing the $\frac{\alpha_\infty^i}{w}$ and their derivatives. Thus the terms $\tilde{V}_{i,w}$ can be neglected when we compute the local exponents of $\tilde{L}$ at $w = 0$. As a consequence, the exponents of L at ∞ are the exponents at $w = 0$ of the equation

$$(-1)^{n+1}w^2(\partial_w - \frac{\alpha_\infty^n}{w})\cdots w^2(\partial_w + \frac{\alpha_\infty^2 - \gamma_{\infty,1}}{w})w^2(\partial_w + \frac{\alpha_\infty^1}{w})\tilde{h} = 0,$$

which is the same as

$$(-1)^{n+1}w^{n+1}(w\partial_w - \alpha_\infty^n + n)\cdots(w\partial_w + \alpha_\infty^2 - \alpha_\infty^1 + 1)(w\partial_w + \alpha_\infty^1)\tilde{h} = 0.$$

Therefore the exponents are

$$-\alpha_\infty^1,\ -\alpha_\infty^2 + \alpha_\infty^1 - 1, \cdots, -\alpha_\infty^n + \alpha_\infty^{n-1} - (n-1),\ \alpha_\infty^n - n,$$

which is exactly (2.8). q.e.d.

In this section, the main result is to use the monodromy theory to calculate the mass (2.2) of the solution u to (2.1).

Theorem 2.2. *Suppose that $u = (u_1, \cdots, u_n)$ is a solution of (2.1) and $\alpha_{t,i} \in \mathbb{N} \cup \{0\},\ 2 \le t \le N,\ 1 \le i \le n$. Then there exists a permutation map f on $I_0 = \{0, 1, \cdots, n\}$ such that*

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i} = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\alpha_{1,l} - \sum_{l=1}^{j}\alpha_{1,l}\right) + 2N_i,\ i \in I,$$

where $N_i \in \mathbb{Z},\ 1 \le i \le n$.

Proof. Following the approach introduced by Chen-Li [**13**, **14**] for Liouville equations (based on potential analysis), we could get that any solution $u = (u_1, \cdots, u_n)$ of (2.1) has the following asymptotic behavior at infinity:

$$u_i(z) = -2\alpha_{\infty,i}\log|z| + O(1),\ \alpha_{\infty,i} > 1,\ i = 1, \cdots, n, \tag{2.11}$$

and the total mass for u_i satisfies

$$\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i}dx = \sum_{j=1}^{n} k^{ij}\left(2\sum_{l=1}^{N}\alpha_{l,j} + 2\alpha_{\infty,j}\right),\ i = 1, \cdots, n, \tag{2.12}$$

where $(k^{ij})_{n\times n}$ is the inverse of $\mathbf{K}$.

Let (2.3) be the associated ODE of the solution $u(z)$ and $\mathcal{M}_t$ be the monodromy transformation at $p_t,\ t=1,\cdots,N,\infty$. Then we have

$$\mathcal{M}_\infty\mathcal{M}_N\cdots\mathcal{M}_1=\mathbb{I}_{n+1}. \tag{2.13}$$

The local exponents of L at p_t are given in (2.6) provided $1\le t\le N$, and by (2.8) provided $t=\infty$. In addition, by Theorem 2.A, $\mathcal{M}_t$ is unitary. Hence the monodromy at each $p_t, 1\le t\le N$ and ∞ are the following

$$\mathcal{M}_t=C_t\begin{bmatrix} e^{2\pi i\beta_{t,0}} & 0 & \cdots & 0\\ 0 & e^{2\pi i\beta_{t,1}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{2\pi i\beta_{t,n}}\end{bmatrix}C_t^{-1},\ 1\le t\le N, \tag{2.14}$$

and

$$\mathcal{M}_\infty=C_\infty\begin{bmatrix} e^{2\pi i\beta_{\infty,0}} & 0 & \cdots & 0\\ 0 & e^{2\pi i\beta_{\infty,1}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{2\pi i\beta_{\infty,n}}\end{bmatrix}C_\infty^{-1}, \tag{2.15}$$

where C_t and C_∞ are constant invertible matrices. By the assumption

$$\alpha_{t,i}\in\mathbb{N}\cup\{0\},\ 2\le t\le N,\ 1\le i\le n,$$

we have $\mathcal{M}_t=e^{2\pi i\beta_{t,0}}\mathbb{I}_{n+1},\ 2\le t\le n$. Set $\Xi=\sum_{t=2}^N\beta_{t,0}$. Then (2.13) yields

$$\mathcal{M}_1=e^{-2\pi i\Xi}\mathcal{M}_\infty^{-1}. \tag{2.16}$$

Therefore, we can find a permutation map f on I_0 such that,

$$\beta_{\infty,j}+\Xi+\beta_{1,f(j)}+m_j=0,\ j\in I_0, \tag{2.17}$$

where $m_j\in\mathbb{Z}, 0\le j\le n$. By (2.6) and (2.8), (2.12) can be rewritten as

(2.18)

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathbb{R}^2}e^{u_i}&=\sum_{j=1}^n k^{ij}\left(\sum_{t=1}^N\alpha_{t,j}+\alpha_{\infty,j}\right)=\sum_{t=1}^N\alpha_t^i+\alpha_\infty^i\\ &=-\sum_{t=1}^N\sum_{\ell=0}^{i-1}(\beta_{t,\ell}-\ell)-\sum_{\ell=0}^{i-1}(\beta_{\infty,\ell}+\ell)\\ &=-\sum_{\ell=0}^{i-1}(\beta_{1,\ell}-\beta_{1,f(\ell)})+\sum_{\ell=0}^{i-1}\left(\sum_{t=2}^N\beta_{t,0}-\sum_{t=2}^N\beta_{t,\ell}\right)+m_{i,1}\\ &=-\sum_{\ell=0}^{i-1}(\beta_{1,\ell}-\beta_{1,f(\ell)})-\sum_{\ell=0}^{i-1}\sum_{t=2}^N\sum_{j=1}^{\ell}\alpha_{t,j}+m_{i,2}\\ &=-\sum_{\ell=0}^{i-1}(\beta_{1,\ell}-\beta_{1,f(\ell)})+m_{i,3},\end{aligned}$$

where $m_{i,1}, m_{i,2}, m_{i,3} \in \mathbb{Z},\ 1 \le i \le n$. Using (2.18), we finally get

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} e^{u_i} = \sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\alpha_{1,l} - \sum_{l=1}^{j}\alpha_{1,l}\right) + N_i,\ i \in I. \tag{2.19}$$

Combining with equation (2.1), we prove the conclusion. q.e.d.

A direct consequence of Theorem 2.2 is the following result:

Corollary 2.3. *Suppose that $u = (u_1, \cdots, u_n)$ is a solution of (2.1) and $\alpha_{t,i} \in \mathbb{N} \cup \{0\}$. Let $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ be the total mass of u. Then $\sigma_i,\ i \in I$, are positive even integers.*

3. The Pohozaev identity and Weyl group

3.1. Definition of $\Gamma(\boldsymbol{\mu})$. We recall the Pohozaev identity (P.I. in short)

$$\sum_{i=1}^{n}\sigma_i^2 - \sum_{i=1}^{n-1}\sigma_i\sigma_{i+1} = 2\sum_{i=1}^{n}\mu_i\sigma_i. \tag{3.1}$$

It has been known that the P.I. plays an important role for studying the bubbling behavior of solutions of the mean field equation. For example, see [**2**, **3**, **9**, **10**, **11**, **12**] and the references therein. The purpose of this section is to derive all possible values of the local mass for a local solution u of (1.15).

In this section, $\mu_1, \cdots, \mu_n$ are considered as independent variables and the P.I. is seen as a polynomial with degree 2 and coefficients in $\mathbb{C}[\mu_1, \cdots, \mu_n]$. For $(\mu_1, \cdots, \mu_n)$, the set $\Gamma(\mu_1, \cdots, \mu_n)$ ($\Gamma(\boldsymbol{\mu})$ in short) is defined as the minimal set satisfying the following conditions:

(i) $(0, \cdots, 0) \in \Gamma(\boldsymbol{\mu})$ and
(ii) if $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, then $\mathfrak{R}_i\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu}),\ i \in I$, where

$$\mathfrak{R}_i\boldsymbol{\sigma} = (\sigma_1, \cdots, 2\mu_i - \sum_{j=1}^{n}k_{ij}\sigma_j + \sigma_i, \cdots, \sigma_n). \tag{3.2}$$

It is not difficult to see that for each element $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, σ_i is a polynomial of $\mu_1, \cdots, \mu_n$ of degree 1. Indeed, we have $\sigma_i = 2\sum_{j=1}^{n} a_{ij}\mu_j$, where $a_{ij} \in \mathbb{Z}$. Set $\mathcal{A} = [a_{ij}]$.

We first prove each element in $\Gamma(\boldsymbol{\mu})$ satisfies the P.I.

Proposition 3.1. *For each element $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, we have $\boldsymbol{\sigma}$ satisfies the P.I.*

Proof. For any i, (3.1) can be written as

$$\sigma_i^2 - (2\mu_i - \sum_{j\neq i}k_{ij}\sigma_j)\sigma_i + \sum_{l\neq i}\sigma_l^2 - \sum_{l\in I_i}\sigma_l\sigma_{l+1} - 2\sum_{l\neq i}\mu_l\sigma_l = 0, \tag{3.3}$$

where $I_i = I \setminus \{i-1, i, n\}$, which shows σ_i is a root of the quadratic equation

$$(3.4)\quad x^2 - (2\mu_i - \sum_{j\neq i} k_{ij}\sigma_j)x + \sum_{l\neq i}\sigma_l^2 - \sum_{l\in I_i}\sigma_l\sigma_{l+1} - 2\sum_{l\neq i}\mu_l\sigma_l = 0.$$

Thus the other root of (3.4) is $2\mu_i - \sum_{j\neq i} k_{ij}\sigma_j - \sigma_i = 2\mu_i - \sum_{j=1}^n k_{ij}\sigma_j + \sigma_i$. Then $\mathfrak{R}_i\boldsymbol{\sigma}$ satisfies (3.3) which is equivalent to P.I. q.e.d.

Remark. We shall prove all the entries of $\mathcal{A}$ are non-negative integers. On the other hand, we set

$$(3.5)\quad \Gamma_N(\boldsymbol{\mu}) = \Big\{\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \text{ satisfies the P.I.},\ \sigma_i = 2\sum_{j=1}^n a_{ij}\mu_j,\ a_{ij}\in\mathbb{N}\cup\{0\}\Big\}.$$

In proposition 3.4 we will show that the two sets $\Gamma(\boldsymbol{\mu})$ and $\Gamma_N(\boldsymbol{\mu})$ are exactly the same.

Next, we state the main result in this section,

Theorem 3.2. *For each element $\boldsymbol{\sigma}\in\Gamma(\boldsymbol{\mu})$, there exists a permutation map f on $I_0 = I\cup\{0\}$ such that σ_i is expressed by*

$$(3.6)\qquad \sigma_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\mu_l - \sum_{l=1}^{j}\mu_l\right),\ i\in I.$$

Furthermore the correspondence $\sigma\to f$ is bijective from $\Gamma(\boldsymbol{\mu})$ onto the permutation group $\mathbb{S}_{n+1}$ of I_0. Consequently

$$|\Gamma(\boldsymbol{\mu})| = (n+1)!.$$

Proof. First, we note that when f is the identity map of I_0, equation (3.6) gives $\sigma_i = 0$, and $\boldsymbol{\sigma} = (0,\cdots,0)\in\Gamma(\boldsymbol{\mu})$. For some element $\boldsymbol{\sigma}\in\Gamma(\boldsymbol{\mu})$. Suppose that (3.6) holds for some permutation map f. According to the definition of $\mathfrak{R}_m\boldsymbol{\sigma}$, we have

$$(3.7)\quad (\mathfrak{R}_m\boldsymbol{\sigma})_i = \begin{cases} 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\mu_l - \sum_{l=1}^{j}\mu_l\right), & \text{if } i\in I\setminus\{m\},\\[2ex] 2\sum_{j=0}^{m-1}\left(\sum_{l=1}^{f(j)}\mu_l - \sum_{l=1}^{j}\mu_l\right) & \\ +2\sum_{j=1}^{f(m)}\mu_j - 2\sum_{j=1}^{f(m-1)}\mu_j, & \text{if } i=m.\end{cases}$$

Let g be $g(i)=f(i),\ i\in I\setminus\{m-1,m\}$ and $g(m-1)=f(m),\ g(m)=f(m-1)$. Then by using g, we have

$$(3.8)\qquad (\mathfrak{R}_m\boldsymbol{\sigma})_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{g(j)}\mu_l - \sum_{l=1}^{j}\mu_l\right),\ i\in I.$$

On the other hand, (3.8) also shows that if a permutation f yields an element in $\Gamma(\boldsymbol{\mu})$ by (3.8), then $f\circ s_m$ also yields another element of

$\Gamma(\boldsymbol{\mu})$, where s_m is the simple permutation only by exchange of $m-1$ and m. Since the permutation group is generated by all the simple permutations s_m, we conclude that any permutation f always gives an element in $\Gamma(\boldsymbol{\mu})$ via (3.6). Now it remains to prove different permutation map f gives different $\boldsymbol{\sigma}$.

Suppose that there are two permutations f, g give the same element $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, i.e.,

$$\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\mu_l - \sum_{l=1}^{j}\mu_l\right) = \sum_{j=0}^{i-1}\left(\sum_{l=1}^{g(j)}\mu_l - \sum_{l=1}^{j}\mu_l\right),\ i \in I. \tag{3.9}$$

If $i=1$, then we have $\sum_{l=1}^{f(0)}\mu_l = \sum_{l=1}^{g(0)}\mu_l$, so $f(0)=g(0)$. By induction on i, it is not difficult to see $f(i)=g(i),\ \forall i \in I$ by (3.9).

Since $f \to \boldsymbol{\sigma}$ is bijective, we have

$$|\Gamma(\boldsymbol{\mu})| = |\{f \mid f \text{ is a permutation map on } I_0\}| = (n+1)!.$$

q.e.d.

Corollary 3.3. *For any element $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, let $\mathcal{A}$ be the corresponding matrix. Then all the entries in $\mathcal{A}$ are non-negative integers.*

Proof. Since we have already seen that all the entries a_{ij} are integers, then it suffices to show $a_{ij} \geq 0$. For the i-th component σ_i, we rearrange $f(0), \cdots, f(i-1)$ such that

$$f(0) < f(1) < \cdots < f(i-1) \text{ without changing the value of } \sigma_i.$$

Therefore $l \leq f(l), 0 \leq l \leq i-1$. Thus it is easy to see that by (3.6), all the coefficients of $\mu_j, j \in I$ for σ_i are non-negative. q.e.d.

Proposition 3.4. *Let $\Gamma_N(\boldsymbol{\mu})$ be defined in (3.5). Then*

$$\Gamma_N(\boldsymbol{\mu}) = \Gamma(\boldsymbol{\mu}).$$

Proof. From Corollary 3.3 it is easy to see that $\Gamma(\boldsymbol{\mu}) \subset \Gamma_N(\boldsymbol{\mu})$. Hence it suffices to prove the other direction. For any $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n) \in \Gamma_N(\boldsymbol{\mu})$, in the proof of Proposition 3.1, we already proved that $\mathfrak{R}_i\boldsymbol{\sigma}$ also satisfies the P.I. We claim:

$$\text{If } \boldsymbol{\sigma} \in \Gamma_N(\boldsymbol{\mu}),\ \text{then } \mathfrak{R}_i\boldsymbol{\sigma} \in \Gamma_N(\boldsymbol{\mu}),\ \forall i \in I. \tag{3.10}$$

Let $\sigma_i = 2\sum_{j=1}^{n} a_{ij}\mu_j,\ i \in I$, where $a_{ij} \in \mathbb{N} \cup \{0\}$. The claim (3.10) is equivalent to show

$$b_{il} := \delta_{il} - \sum_{j=1}^{n} k_{ij}a_{jl} + a_{il} \geq 0,\ \forall l \in I. \tag{3.11}$$

Since $\boldsymbol{\sigma}$ satisfies the P.I., substituting the expression $\sigma_i = 2\sum_{j=1}^{n} a_{ij}\mu_j$ into the P.I., and comparing the coefficients of μ_l^2, we get

$$\sum_{j=1}^{n} a_{jl}^2 - \sum_{j=1}^{n-1} a_{jl}a_{j+1,l} = a_{ll},\ \ l \in I. \tag{3.12}$$

Similarly, for $\mathfrak{R}_i\sigma$ we have

$$\sum_{j\in I\setminus\{i\}} a_{jl}^2 + b_{il}^2 - \sum_{j\in I\setminus\{i,i-1\}} a_{jl}a_{j+1,l} - (a_{i-1,l}+a_{i+1,l})b_{il} = \hat{a}_{ll},\ \ l \in I, \tag{3.13}$$

where

$$a_{jl} = 0 \text{ if } j \le 0 \text{ or } j \ge n+1,\ \ \forall l \in I,$$

and

$$\hat{a}_{ll} = \begin{cases} a_{ll}, & \text{if } l \neq i, \\ b_{ii}, & \text{if } l = i. \end{cases}$$

From (3.12) and (3.13), we can view a_{il}, b_{il} $l \neq i$ as the solutions of the following quadratic equation

$$x^2 - (a_{i-1,l}+a_{i+1,l})x + \sum_{j\in I\setminus\{i\}} a_{jl}^2 - \sum_{j\in I\setminus\{i,i-1\}} a_{jl}a_{j+1,l} - a_{ll} = 0,\ l \in I\setminus\{i\}, \tag{3.14}$$

and a_{ii}, b_{ii} as the solutions of the following quadratic equation

$$x^2 - (a_{i-1,l}+a_{i+1,l}+1)x + \sum_{j\in I\setminus\{i\}} a_{ji}^2 - \sum_{j\in I\setminus\{i,i-1\}} a_{ji}a_{j+1,i} = 0. \tag{3.15}$$

At first, we prove that $b_{ii} \ge 0$. Note

$$\sum_{j\in I\setminus\{i\}} a_{ji}^2 - \sum_{j\in I\setminus\{i,i-1\}} a_{ji}a_{j+1,i} \ge \frac{1}{2} \sum_{j\in I\setminus\{i,i-1\}} (a_{ji}-a_{j+1,i})^2 \ge 0,$$

which together with the equation (3.15) yields $b_{ii} \ge 0$. It remains to show $b_{il} \ge 0$ for $l \neq i$. By (3.14), we see $b_{il} \ge 0$ if

$$\sum_{j\in I\setminus\{i\}} a_{jl}^2 - \sum_{j\in I\setminus\{i,i-1\}} a_{jl}a_{j+1,l} - a_{ll} \ge 0. \tag{3.16}$$

Set $c_{jl} = a_{jl},\ j \in I\setminus\{i\}$ and $c_{il} = 0$. Then we have

$$\begin{aligned} &\sum_{j\in I\setminus\{i\}} a_{jl}^2 - \sum_{j\in I\setminus\{i,i-1\}} a_{jl}a_{j+1,l} - a_{ll} \\ &= \sum_{j=1}^{n} c_{jl}^2 - \sum_{j=1}^{n-1} c_{jl}c_{j+1,l} - c_{ll} \\ &= \frac{1}{2}\left[\left(\sum_{j=0}^{l-1}(c_{jl}-c_{j+1,l})^2 - c_{ll}\right) + \left(\sum_{j=l}^{n}(c_{jl}-c_{j+1,l})^2 - c_{ll}\right)\right], \end{aligned} \tag{3.17}$$

where $c_{0l} = c_{n+1,l} = 0$. Since $c_{jl} \in \mathbb{N} \cup \{0\}$, it is easy to see

$$\sum_{j=0}^{l-1}(c_{jl} - c_{j+1,l})^2 - c_{ll} \geq 0 \text{ and } \sum_{j=l}^{n}(c_{jl} - c_{j+1,l})^2 - c_{ll} \geq 0. \tag{3.18}$$

As a consequence, (3.16) holds and we get $b_{il} \geq 0$ for $l \neq i$. Thus the claim (3.10) is proved, and $\mathfrak{R}_i\boldsymbol{\sigma} \in \Gamma_N(\boldsymbol{\mu})$.

Next we define a partial order $\preceq$ in $\Gamma_N(\boldsymbol{\mu})$, we say

$$\boldsymbol{\sigma}_1 \preceq \boldsymbol{\sigma}_2 \quad \text{provided} \quad (\boldsymbol{\sigma}_1)_i \leq (\boldsymbol{\sigma}_2)_i,\ i \in I.$$

It is easy to see that we have

$$\text{either} \quad \mathfrak{R}_i\boldsymbol{\sigma} \preceq \boldsymbol{\sigma} \quad \text{or} \quad \boldsymbol{\sigma} \preceq \mathfrak{R}_i\boldsymbol{\sigma}, \quad \forall i \in I.$$

For any $\boldsymbol{\sigma} \in \Gamma_N(\boldsymbol{\mu})$, we set

$$\Gamma_{\boldsymbol{\sigma}} = \{\mathfrak{R}_{i_1}\cdots\mathfrak{R}_{i_m}\boldsymbol{\sigma} \mid m \in \mathbb{N} \cup \{0\}\}. \tag{3.19}$$

It is easy to see that for any $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \Gamma_N(\boldsymbol{\mu})$, we have either

$$\Gamma_{\boldsymbol{\sigma}_1} = \Gamma_{\boldsymbol{\sigma}_2} \quad \text{or} \quad \Gamma_{\boldsymbol{\sigma}_1} \cap \Gamma_{\boldsymbol{\sigma}_2} = \emptyset. \tag{3.20}$$

Next we shall prove that $\mathbf{0} = (0, \cdots, 0) \in \Gamma_{\boldsymbol{\sigma}}$ for any $\boldsymbol{\sigma} \in \Gamma_N(\boldsymbol{\mu})$. An element $\hat{\boldsymbol{\sigma}} \in \Gamma_{\boldsymbol{\sigma}}$ is minimal if $\tilde{\boldsymbol{\sigma}} \preceq \hat{\boldsymbol{\sigma}}$ for some $\tilde{\boldsymbol{\sigma}} \in \Gamma_{\boldsymbol{\sigma}}$, then $\hat{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}$. It is not difficult to see that $\Gamma_{\boldsymbol{\sigma}}$ has a local minimal element $\boldsymbol{\sigma}_0 = (\sigma_{1,0}, \cdots, \sigma_{n,0})$, i.e.,

$$\boldsymbol{\sigma}_0 \preceq \mathfrak{R}_i\boldsymbol{\sigma}_0,\ \forall i \in I.$$

As a consequence, we have

$$2\mu_i - \sum_{j=1}^{n} k_{ij}\sigma_{j,0} \geq 0,\ i \in I.$$

On the other hand, since $\boldsymbol{\sigma}_0$ satisfies the P.I., we get

$$0 \leq \sum_{i=1}^{n}(2\mu_i - \sum_{j=1}^{n} k_{ij}\sigma_{j,0})\sigma_{i,0} = -2\sum_{i=1}^{n}\mu_i\sigma_{i,0} \leq 0. \tag{3.21}$$

Then $\sigma_{i,0} = 0$, $\forall i \in I$. Hence $\mathbf{0} \in \Gamma_{\boldsymbol{\sigma}}$. By (3.20), we obtain $\Gamma_{\boldsymbol{\sigma}} = \Gamma_{(0,\cdots,0)}$ for $\boldsymbol{\sigma} \in \Gamma_N(\boldsymbol{\mu})$ and it implies

$$\Gamma_N(\boldsymbol{\mu}) = \Gamma(\boldsymbol{\mu}).$$

q.e.d.

3.2. The Weyl group and $\Gamma(\boldsymbol{\mu})$. For any $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, we can define the matrix $\mathcal{A}$. Then we set

$$\mathcal{B} = \mathbb{I}_n - \mathbf{K}\mathcal{A}. \tag{3.22}$$

In the next theorem we will see that the matrix $\mathcal{B}$ is related to the Weyl group of the Lie algebra $\mathbf{A}_n$. It is known that the simple roots for the Cartan subalgebra of $sl(n+1)$ is $e_i - e_{i+1}$, $i \in I$, where e_i, $1 \leq i \leq n+1$ is the standard orthogonal basis in $\mathbb{R}^{n+1}$ and $\langle \cdot, \cdot \rangle$ is the inner product of

$\mathbb{R}^{n+1}$. Let V be the subspace spanned by simple roots. For $0 \neq \alpha \in V$, the orthogonal reflection $\mathcal{S}_\alpha$ is defined by

$$\mathcal{S}_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \text{ for } \beta \in V.$$

The subgroup W generated by the reflections $\mathcal{S}_{e_i - e_{i+1}}$, $i \in I$ is called the Weyl group of the root system R. Using the basis $e_{i+1} - e_i$, W can be embedded into $GL(n, \mathbb{R})$.

Theorem 3.5. *Let $\mathcal{B}$ be defined in (3.22), then we have the following conclusions.*

(a) *The matrix $\mathcal{B}$ for any $\mathcal{A} \in \Gamma(\boldsymbol{\mu})$ is non-singular. Furthermore, let $f \to \mathcal{A}_f$ be the correspondence by Theorem 3.2 and $\mathcal{B}_f$ is given by (3.22). Then $f \to \mathcal{B}_f$ is an anti-homomorphism from the permutation group $\mathbb{S}_{n+1}$ to $GL(n, \mathbb{Z})$. Consequently*

$$\{\mathcal{B} \mid \mathcal{B} \text{ is given by } (3.22),\ \mathcal{A} \in \Gamma(\boldsymbol{\mu})\} \text{ is a group}$$

denoted by $\mathbb{B}$.

(b) *The entries of any $\mathcal{B}$ in $\mathbb{B}$ consist only of $\{0, \pm 1\}$.*

(c) *The group $\mathbb{B}$ is the Weyl group of the root system for the Lie algebra $\mathbf{A}_n$.*

Proof. (a) We start the proof by showing $\mathcal{B}$ is invertible. Using the P.I. we get

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\mu_j\right)^2 - \sum_{i=1}^{n-1}\left(\sum_{j=1}^{n} a_{ij}\mu_j\right)\left(\sum_{j=1}^{n} a_{i+1,j}\mu_j\right) \\
&= \sum_{i=1}^{n}\mu_i\left(\sum_{j=1}^{n} a_{ij}\mu_j\right),
\end{aligned} \tag{3.23}$$

which implies

$$\boldsymbol{\mu}^t\mathcal{A}^t\mathcal{A}\boldsymbol{\mu} - \boldsymbol{\mu}^t\mathcal{A}^t\mathcal{R}\mathcal{A}\boldsymbol{\mu} = \boldsymbol{\mu}^t\mathcal{A}\boldsymbol{\mu}, \tag{3.24}$$

where

$$\mathcal{R} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)^t$. Since the equation (3.24) is independent of $\boldsymbol{\mu}$, we get

$$\frac{1}{2}\mathcal{A}^t\mathbf{K}\mathcal{A} = \mathcal{A}^t\mathcal{A} - \frac{1}{2}\mathcal{A}^t(\mathcal{R} + \mathcal{R}^t)\mathcal{A} = \frac{1}{2}(\mathcal{A} + \mathcal{A}^t), \tag{3.25}$$

where $\mathcal{A}^t$ denotes the transpose of $\mathcal{A}$.

On the other hand, we have by (3.22)

$$\mathcal{A} = \mathbf{K}^{-1}(\mathbb{I}_n - \mathcal{B}) \text{ and } \mathcal{A}^t = (\mathbb{I}_n - \mathcal{B}^t)\mathbf{K}^{-1}. \tag{3.26}$$

Substituting (3.26) into (3.25), we get

$$(\mathbb{I}_n - \mathcal{B}^t)\mathbf{K}^{-1}(\mathbb{I}_n - \mathcal{B}) = ((\mathbb{I}_n - \mathcal{B}^t)\mathbf{K}^{-1} + \mathbf{K}^{-1}(\mathbb{I}_n - \mathcal{B})) \tag{3.27}$$

which implies

$$\mathcal{B}^t\mathbf{K}^{-1}\mathcal{B} = \mathbf{K}^{-1}. \tag{3.28}$$

From the above equation, we deduce that $\mathcal{B}$ is non-singular.

Next we shall show the map $f \to \mathcal{B}_f$ is an anti-isomorphism from $\mathbb{S}_{n+1}$ to $\mathbb{B}$. It is easy to see $\mathcal{B}_f = \mathbb{I}_n$ provided f is the identity map. By Theorem 3.2 we get the map $f \to \mathcal{B}_f$ is onto, now we will prove

$$\mathcal{B}_{g\circ f} = \mathcal{B}_f\mathcal{B}_g \text{ for any permutation map } f \text{ and } g.$$

Since any permutation map f can be decomposed by some simple ones, which alternates only one couple of neighbor points. Then it suffices to show the anti-homomorphism holds with f is simple, i.e., f only change the places of $m-1$ and m for some $m \in I$. Suppose g corresponds to the element $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$, then $g \circ f$ corresponds to $\mathfrak{R}_m\boldsymbol{\sigma}$. We assume $\sigma_i = 2\sum_{j=1}^n a_{ij}\mu_j$, then the (i,j)-th entry of $\mathcal{B}_g$ and $\mathcal{B}_{g\circ f}$ can be represented respectively in the following

$$(\mathcal{B}_g)_{(i,j)} = \delta_{ij} - \sum_{l=1}^n k_{il}a_{lj} \tag{3.29}$$

and

$$(\mathcal{B}_{g\circ f})_{(i,j)} = \begin{cases} \delta_{ij} - \sum_{l=1}^n k_{il}a_{lj}, & \text{if } i \in I \setminus \{m \pm 1, m\}, \\ \delta_{ij} - k_{im}\delta_{jm} - \sum_{l=1}^n k_{il}a_{lj} \\ +k_{im}\sum_{l=1}^n k_{ml}a_{lj}, & \text{if } i = m \pm 1, \\ -\delta_{mj} + \sum_{l=1}^n k_{ml}a_{lj}, & \text{if } i = m. \end{cases} \tag{3.30}$$

On the other hand, for the permutation map f we have

$$(\mathcal{B}_f)_{(i,j)} = \begin{cases} \delta_{ij}, & \text{if } j \neq m, \\ \delta_{im} - k_{im}, & \text{if } j = m. \end{cases} \tag{3.31}$$

From (3.29) to (3.31) and straightforward computation we get

$$(\mathcal{B}_{g\circ f})_{(i,j)} = \sum_{l=1}^n (\mathcal{B}_f)_{(i,l)}(\mathcal{B}_g)_{(l,j)}, \ i,j \in I, \tag{3.32}$$

which implies $\mathcal{B}_{g\circ f} = \mathcal{B}_f\mathcal{B}_g$. Therefore we proved the map $f \to \mathcal{B}_f$ is anti-homomorphism and consequently $\mathbb{B}$ is a group. Now it remains to show the map is injective, i.e., the pre-image of $\mathbb{I}_n$ is the identity map. Suppose that there exists a permutation map f such that $\mathcal{B}_f = \mathbb{I}_n$.

Then $\mathcal{A}_f = 0$ and by Theorem 3.2 we have f is identity. Hence the map $f \to \mathcal{B}_f$ is anti-isomorphism.

(b) We have already known that the (i,j)-th entry of $\mathcal{B}_f$ is the coefficient of μ_j in $\sum_{l=1}^{f(i)} \mu_l - \sum_{l=1}^{f(i-1)} \mu_l$, $j \in I$. Then it is easy to see that the coefficient of $\mu_\ell, \ell \in I$ in $\mu_i - \frac{1}{2}\sum_j k_{ij}\sigma_j$ only takes the value in $\{-1, 0, 1\}$, therefore the second conclusion holds.

(c) We have already seen that the generators for the Weyl group of the Lie algebra $\mathbf{A}_n$ are

$$\mathcal{S}_{e_1-e_2}, \mathcal{S}_{e_2-e_3}, \cdots, \mathcal{S}_{e_n-e_{n+1}}. \tag{3.33}$$

It is known that

$$\mathcal{S}_{e_m-e_{m+1}}(e_i - e_{i+1}) = \begin{cases} e_i - e_{i+1}, & \text{if } i \notin \{m, m\pm 1\}, \\ e_{m-1} - e_{m+1}, & \text{if } i = m-1, \\ e_{m+1} - e_m, & \text{if } i = m, \\ e_m - e_{m+2}, & \text{if } i = m+1. \end{cases} \tag{3.34}$$

The corresponding matrix $M_{e_m-e_{m+1}}$ is

$$[M_{e_m-e_{m+1}}]_{ij} = \begin{cases} \delta_{ij}, & \text{if } j \neq m, \\ \delta_{im} - k_{im}, & \text{if } j = m. \end{cases} \tag{3.35}$$

On the other hand, from the proof of (a) we get the generators of the group $\mathbb{B}$ are $\mathcal{B}_f$, where f is the simple map which only change the places of $m-1$ and m for some $m \in I$. The matrix $\mathcal{B}_f$ has already been computed in (3.31). Combined with (3.35), we get

$$M_{e_m-e_{m+1}} = \mathcal{B}_f.$$

As a consequence, the generators of the Weyl group of $\mathbf{A}_n$ and the group $\mathbb{B}$ are the same. Therefore these two groups are the same. Hence we finish all the proof. q.e.d.

3.3. For any $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, we set

$$\bar{\mu}_i = \mu_i - \frac{1}{2}\sum_j k_{ij}\sigma_j, \ i \in I. \tag{3.36}$$

Then we state the following result

Theorem 3.6. *Assume $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, and let $\bar{\mu}_i$ be given in (3.36). For any permutation map f on $\{0, 1, \cdots, n\}$, we set $\boldsymbol{\sigma}_f = (\sigma_{f,1}, \cdots, \sigma_{f,n})$ by*

$$\sigma_{f,i} = \sigma_i + 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)} \bar{\mu}_l - \sum_{l=1}^{j} \bar{u}_l\right). \tag{3.37}$$

Then $\boldsymbol{\sigma}_f \in \Gamma(\boldsymbol{\mu})$.

Proof. Note that if f is an identity then $\boldsymbol{\sigma}_f = \boldsymbol{\sigma}$. Hence it suffices to prove that if f is any permutation such that $\boldsymbol{\sigma}_f \in \Gamma(\boldsymbol{\mu})$ and g is the simple permutation, then $\boldsymbol{\sigma}_{f\circ g} \in \Gamma(\boldsymbol{\mu})$. Let g be a simple permutation which only alternates the places of $\ell - 1$ and ℓ for some $\ell \in I$, then we have

(3.38)
$$(f\circ g)(\ell-1) = f(\ell),\ (f\circ g)(\ell) = f(\ell-1),\ (f\circ g)(i) = f(i),\ i \in I\backslash\{\ell-1, \ell\}.$$

According to the definition of $\sigma_{f\circ g,i}$, for $i \neq \ell$ we have that

(3.39)
$$\begin{aligned}\sigma_{f\circ g,i} &= \sigma_i + 2\sum_{j=0}^{i-1}\left(\sum_{m=1}^{(f\circ g)(j)} \bar{\mu}_m - \sum_{m=1}^{j} \bar{u}_m\right)\\ &= \sigma_i + 2\sum_{j=0}^{i-1}\left(\sum_{m=1}^{f(j)} \bar{\mu}_m - \sum_{m=1}^{j} \bar{\mu}_m\right) = \sigma_{f,i}.\end{aligned}$$

For $i = \ell$, we have

(3.40)
$$\begin{aligned}\sigma_{f\circ g,\ell} &= \sigma_\ell + 2\sum_{j=0}^{\ell-1}\left(\sum_{m=1}^{(f\circ g)(j)} \bar{\mu}_m - \sum_{m=1}^{j} \bar{\mu}_m\right)\\ &= \sigma_\ell + 2\sum_{j=0}^{\ell-2}\left(\sum_{m=1}^{f(j)} \bar{\mu}_m - \sum_{m=1}^{j} \bar{\mu}_m\right) + 2\left(\sum_{m=1}^{f(\ell)} \bar{\mu}_m - \sum_{m=1}^{\ell-1} \bar{\mu}_m\right)\\ &= \sigma_{f,\ell-1} + \sigma_{f,\ell+1} - \sigma_{f,\ell} + 2\bar{\mu}_\ell - \sigma_{\ell-1} - \sigma_{\ell+1} + 2\sigma_\ell\\ &= 2\mu_\ell - \sum_j k_{\ell j}\sigma_{f,j} + \sigma_{f,\ell}.\end{aligned}$$

Thus $\boldsymbol{\sigma}_{f\circ g} = \mathfrak{R}_\ell \boldsymbol{\sigma}_f$, where $\mathfrak{R}_\ell$ is given by (3.2), and then $\boldsymbol{\sigma}_{f\circ g} \in \Gamma(\boldsymbol{\mu})$. This proves Theorem 3.6. q.e.d.

An important consequence of Theorem 3.6 is the following proposition, which plays a crucial role in section 6 and section 7. Before we state the result, let us introduce the notion of consecutive indices. We say that $J \subset I$ consists of consecutive indices if $J = \{i, i+1, \cdots, i+m\} \subset I$ for some $m \geq 0$. We will see the concept of consecutive indices is needed in the partial blow-up phenomena of solutions of Toda system.

Theorem 3.7. *Assume* $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, $n \geq 2$. *Let* $J = J_1 \cup \cdots \cup J_l \subset I$, *where* $J_1, \cdots, J_l$ *are disjoint sets and* $J_t = \{i_t, \cdots, i_t + |J_t| - 1\}$, $t = 1, \cdots, l$ *consists of the maximal consecutive indices. We set* $\bar{\mu}_i = \mu_i - \frac{1}{2}\sum_{j=1}^n k_{ij}\sigma_j$, $i \in I$ *and*

(3.41)
$$\bar{\sigma}_i = \sigma_i \text{ for } i \in I \setminus J,\ \bar{\sigma}_i = \sigma_i + \sum_{j\in J_t} k_t^{ij}(2\bar{\mu}_j + 2\bar{\mu}_{2i_t+|J_t|-j-1}) \text{ for } i \in J_t,$$

where $(k_t^{ij})_{|J_t|\times|J_t|}$ *is the inverse of* $(k_{ij})_{|J_t|\times|J_t|}$, $i, j \in J_t$, $t = 1, \cdots, l$. *Then* $\bar{\boldsymbol{\sigma}} \in \Gamma(\boldsymbol{\mu})$.

Proof. For convenience we write

$$\bar{\sigma}_{i,0} = \sum_{j\in J_t} k_t^{ij}(2\bar{\mu}_j + 2\bar{\mu}_{2i_t+|J_t|-1-j}),\ i \in J_t. \tag{3.42}$$

For the matrix $(k_t^{ij})_{|J_t|\times|J_t|}$, we have the following equality

$$k_t^{ij} + k_t^{i,2i_t+|J_t|-1-j}$$
$$= \min\{i - i_t + 1, j - i_t + 1, |J_t| + i_t - i, |J_t| + i_t - j\},\ i,j \in J_t.$$

As a consequence, we can rewrite (3.42) as

$$\bar{\sigma}_{i,0} = 2\sum_{j=i_t-1}^{i-1}\left(\sum_{l=i_t}^{2i_t+|J_t|-2-j}\bar{\mu}_l - \sum_{l=i_t}^{j}\bar{\mu}_l\right),\ i \in J_t. \tag{3.43}$$

Next we define a permutation map $f_t,\ t = 1,\cdots,l$ on $J_t \cup \{i_t - 1\}$ such that

$$f_t(i) = 2i_t + |J_t| - 2 - i,\ i \in J_t \cup \{i_t - 1\} \tag{3.44}$$

and a permutation map f on $I_0 = I \cup \{0\}$ such that

$$f(i) = \begin{cases} f_t(i), & \text{if } i \in \bigcup_{t=1}^{l}(J_t \cup \{i_t - 1\}), \\ i, & \text{if } i \in I_0 \setminus (\bigcup_{t=1}^{l}(J_t \cup \{i_t - 1\})). \end{cases} \tag{3.45}$$

Then we can write $\bar{\sigma}_i$ as

$$\bar{\sigma}_i = \sigma_i + 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\bar{\mu}_l - \sum_{l=1}^{j}\bar{\mu}_l\right),\ i \in I. \tag{3.46}$$

By Theorem 3.6 we get $\bar{\boldsymbol{\sigma}} \in \Gamma(\boldsymbol{\mu})$ and it finishes the proof. q.e.d.

Theorem 3.6 and Theorem 3.7 play important roles for calculating the new local mass at each step of performing partial bubbling, see Proposition 6.1 and Lemma 6.3.

4. Bubbling Analysis for picking up "Bad points"

The main purpose of bubbling analysis for a local solution of equation (1.15) is to pick up those "bad points", $\{x_1^k,\cdots,x_N^k\}$ denoted by Σ_k. The set Σ_k is characterized by the Harnack-type inequality:

$$u_i^k(x) + 2\log \operatorname{dist}(x,\Sigma_k) \le C,\ \forall x \in B(0,1),\ i \in I, \tag{4.1}$$

for some constant C. The process for selecting those points has been done in [**28**]. In this section, we shall review this process for the convenience of readers. For the complete proof we refer to [**28**].

We construct Σ_k by induction. If (1.15) has no singularity at 0, we start with $\Sigma_k = \emptyset$. If (1.15) has a singularity, we start with $\Sigma_k = \{0\}$.

Suppose that we have defined $\Sigma_k = \{0, x_1^k, \cdots, x_{m-1}^k\}$ by induction, then we consider the function

$$\max_{1\le i\le n,\ x\in B_1}\left(u_i^k(x) + 2\log \mathrm{dist}(x,\Sigma_k)\right). \tag{4.2}$$

If the above function is bounded from above by a constant independent of k, then we stop the process and $\Sigma_k = \{0, x_1^k, \cdots, x_{m-1}^k\}$. Otherwise the maximum tends to infinity, let q_k be the point where (4.2) is achieved and we set

$$d_k = \frac{1}{2}\mathrm{dist}(q_k, \Sigma_k)$$

and

$$S_i^k(x) = u_i^k(x) + 2\log\left(d_k - |x-q_k|\right) \text{ in } B(q_k, d_k),\ i\in I.$$

Let i_0 be the index and p_k be the point such that

$$S_{i_0}^k(p_k) = \max_{1\le i\le n}\max_{x\in \bar B(q_k,d_k)} S_i^k(x).$$

Then we set

$$l_k = \frac{1}{2}(d_k - |p_k - q_k|)$$

and scale u_i^k by

$$v_i^k(y) = u_i^k(p_k + e^{-\frac{1}{2}u_{i_0}^k(p_k)}y) - u_{i_0}^k(p_k) \text{ for } |y|\le R_k \doteqdot e^{\frac{1}{2}u_{i_0}^k(p_k)} l_k. \tag{4.3}$$

It is not difficult to see that $R_k\to\infty$ and v_i^k is bounded from above on any fixed compact subset of $\mathbb{R}^2$. Thus by passing to a subsequence, v_i^k satisfies one of the following two alternatives:

(a) The sequence is fully bubbling: along a subsequence, $(v_1^k, \cdots, v_n^k)$ converges in $C^2_{loc}(\mathbb{R}^2)$ to $(v_1, \cdots, v_n)$ which satisfies

$$\Delta v_i + \sum_{j=1}^n a_{ij}e^{v_j} = 0 \text{ in } \mathbb{R}^2,\ i\in I. \tag{4.4}$$

(b) $I = J_1\cup J_2\cup\cdots\cup J_l\cup N$, where $J_1, J_2, \cdots, J_l$ and N are disjoint sets, $N\neq\emptyset$ and each J_i, $1\le i\le l$ consists of consecutive indices. For each $i\in N$, $v_i^k\to-\infty$ over any fixed compact subsets of $\mathbb{R}^2$. The components of $v^k = (v_1^k, \cdots, v_n^k)$ corresponding to each J_i $(i = 1, \cdots, l)$ converge in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a $SU(|J_i|+1)$ Toda system, where $|J_i|$ is the number of indices in J_i.

Therefore in either case, we could choose $l_k^*\to\infty$ such that

$$v_i^k(y) + 2\log|y| \le C \text{ for } |y|\le l_k^*,\ i\in I \tag{4.5}$$

and

$$\int_{B(0,l_k^*)} e^{v_i^k}\,\mathrm{d}y = \int_{\mathbb{R}^2} e^{v_i(y)} + o(1). \tag{4.6}$$

After scaling back to u_i^k, we set $l_m^k = e^{-\frac{1}{2}u_{i_0}^k(p_k)} l_k^*$. Let x_m^k be the point where $\max_{i\in I}\max_{B(p_k,l_m^k)} u_i^k$ is achieved and add x_m^k in Σ_k. We can continue in this way until the Harnack inequality (4.1) holds.

The inequality (4.1) is a Harnack type inequality, which has the following important consequence.

Proposition 4.A ([**28**, Lemma 2.4])**.** *Let u^k satisfy (1.15) in $B(x_0, 2r_k)$ such that (1.17) holds and*

$$u_i^k(x) + 2\log|x - x_0| \le C \text{ for } x \in B(x_0, r_k),\ 1 \le i \le n.$$

Then

$$|u_i^k(x_1) - u_i^k(x_2)| \le C_0 \text{ for } \frac{1}{2} \le \frac{|x_1 - x_0|}{|x_2 - x_0|} \le 2 \text{ and } x_1, x_2 \in B(x_0, r_k), \tag{4.7}$$

where C_0 depends on C only.

Now let us introduce the notions of fast decay or slow decay in below

Definition 4.1. Suppose that u^k satisfies the Harnack inequality in $B(x_k, 2r_k) \setminus B(x_k, \frac{1}{2}r_k)$. Then we say that u_i^k has fast decay on $\partial B(x_k, r_k)$ if along a subsequence,

$$u_i^k(x) + 2\log|x - x_k| \le -N_k, \quad \text{for } x \in \partial B(x_k, r_k)$$

for some $N_k \to \infty$ and u_i^k is said to have slow-decay if there is a constant C independent of k and

$$u_i^k(x) + 2\log|x - x_k| \ge -C, \quad \text{for } x \in \partial B(x_k, r_k).$$

As we have seen in [**27**] and [**28**], the fast decay is the necessary situation for applying Pohozaev identities. Based on the notation of fast decay and slow decay, we summarize the above discussion of induction process as the following.

Proposition 4.B. *Let $\mathbf{K} = (k_{ij})_{n\times n}$ be the Cartan matrix (1.6), h_i^k satisfy (1.16) and $u^k = (u_1^k, \cdots, u_n^k)$ be a sequence of solutions to (1.15). Then there exist a finite set $\Sigma_k := \{0, x_1^k, \cdots, x_m^k\}$ (if 0 is not singular point, then 0 can be deleted from Σ_k) and positive numbers $l_1^k, \cdots, l_m^k \to 0$ such that the following hold:*

1) *There exists $C > 0$ independent of k such that (4.1) holds.*
2) *At x_j^k, let $u_{i_0}^k(x_j^k) = \max_i \max_{B(x_j^k, l_j^k)} u_i^k(x) \to +\infty$ as $k \to +\infty$ and set*

$$v_i^k(y) = u_i^k(x_j^k + e^{-\frac{1}{2}u_{i_0}^k(x_j^k)} y) - u_{i_0}^k(x_j^k) \tag{4.8}$$

then $v^k = (v_1^k, \cdots, v_n^k)$ satisfies either (a) or (b). Furthermore there is a sequence $R_{j,k}$ such that u^k has fast decay on $\partial B(x_j^k, l_j^k)$, where $l_j^k = R_{j,k} e^{-\frac{1}{2}u_{i_0}^k(x_j^k)}$.

3) *$B(x_j^k, l_j^k) \cap B(x_i^k, l_i^k) = \emptyset,\ i \ne j$.*

We refer the readers to [**28**, Proposition 2.1] for the proof of this proposition. Let $x_l^k \in \Sigma_k$ and $\tau_l^k = \frac{1}{2}\mathrm{dist}(x_l^k, \Sigma_k \setminus \{x_l^k\})$, then we can derive from (4.7) that

$$u_i^k(x) = \bar{u}_{x_l^k,i}^k(r) + O(1), \quad x \in B(x_l^k, \tau_l^k), \tag{4.9}$$

where $r = |x_l^k - x|$ and $\bar{u}_{x_l^k,i}^k$ is the average of u_i^k on $\partial B(x_l^k, r)$:

$$\bar{u}_{x_l^k,i}^k(r) = \frac{1}{2\pi r}\int_{\partial B(x_l^k,r)} u_i^k dS, \tag{4.10}$$

and $O(1)$ is independent of r and k.

In the end of this section, we provide the following result which plays a crucial role in the later argument.

Proposition 4.C. *Let $B = B(x^k, r_k)$. If $x^k \neq 0$, then we assume $0 \notin B(x^k, 2r_k)$. Suppose that all the components of u^k have fast decay on ∂B. Then $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ satisfies the P.I., where*

$$\sigma_i = \lim_{k\to\infty} \frac{1}{2\pi}\int_{B(x^k,r_k)} h_i^k e^{u_i^k}.$$

The proof of Proposition 4.C requires some delicate analysis, see [**28**, Proposition 3.1].

5. Three technical lemmas

In this section, we will prove three crucial results which play the key role in sections 6 and 7. We consider the following equation

$$\Delta u_i^k(x) + \sum_{j=1}^n k_{ij} h_j^k e^{u_j^k} = 4\pi\alpha_i\delta_0 \text{ in } B(0,1),\ i \in I. \tag{5.1}$$

For Lemma 5.1, we assume

(i) The Harnack inequality

$$u_i^k(x) + 2\log|x| \le C, \quad \text{for } \frac{1}{2}l_k \le |x| \le 2s_k, \text{ and } i \in I.$$

(ii) All components of u^k have fast-decay on $\partial B(0, l_k)$ and

$$\lim_{r\to 0}\lim_{k\to+\infty} \sigma_i^k(B(0, rl_k)) = \lim_{k\to+\infty} \sigma_i^k(B(0, l_k)).$$

For simplicity, we denote the limit of $\sigma_i^k(B(0, l_k))$ by σ_i.

Lemma 5.1. *Let $\mu_i = \alpha_i + 1$. Assume (i) and (ii), then the following conclusions hold.*

(a) *If the i-th component u_i^k has slow-decay on $\partial B(0, s_k)$, then*

$$2\mu_i - \sum_{j=1}^n k_{ij}\sigma_j > 0.$$

(b) *At least one component of u^k has fast decay on $\partial B(0,s_k)$.*

Proof. (a) At first, notice that the following scaling

$$v_i^k(y) = u_i^k(s_k y) + 2\log s_k \text{ in } B_2, \quad i \in I,$$

gives

$$\Delta v_i^k(y) + \sum_{j=1}^n k_{ij} h_j^k(s_k y) e^{v_j^k(y)} = 4\pi\alpha_i\delta_0 \text{ in } B_2,$$

where α_i maybe zero. Let J be a (maximal) set of consecutive indices such that u_i^k has slow-decay on $\partial B(0,s_k)$, $i \in J$. Then $v_i, i \in J$ converges to the solution of a $SU(|J|+1)$ Toda system v_i, $i \in J$, which satisfies

$$\Delta v_i(y) + \sum_{j\in J} k_{ij} e^{v_j} = 0 \text{ in } B_2 \setminus \{0\},\ i \in J. \tag{5.2}$$

The strength of the Dirac measure at 0 for (5.2) can be expressed by

$$\begin{aligned}\lim_{r\to 0}\int_{\partial B(0,r)} \frac{\partial v_i(y)}{\partial \nu} dS &= \lim_{r\to 0}\lim_{k\to\infty}(4\pi\alpha_i - \sum_{j=1}^n \int_{B(0,r)} k_{ij} h_j^k e^{v_j^k} dy) \\ &= 4\pi\alpha_i - 2\pi\sum_{j=1}^n k_{ij}\sigma_j.\end{aligned}$$

The existence of a solution v_i to (5.2) implies $\alpha_i - \frac{1}{2}\sum k_{ij}\sigma_j > -1$ and then (a) is proved.

(b) Since all components have fast decay on $\partial B(0,l_k)$, the $\boldsymbol{\sigma}$ satisfies the P.I. by Proposition 4.C. By a simple manipulation, the P.I. can be written as

$$\sum_{i=1}^n \sigma_i(\sum_{j=1}^n k_{ij}\sigma_j - 2\mu_i) = 2\sum_{i=1}^n \mu_i\sigma_i, \tag{5.3}$$

which yields $\sum_{j=1}^n k_{ij}\sigma_j - 2\mu_i$ is positive for some $i \in I$. By (a), we get the i-th component u_i^k have fast decay on $\partial B(0,s_k)$. q.e.d.

The next result is about the fast-decay, which is a crucial step in the bubbling analysis.

Lemma 5.2. *Suppose that the Harnack-type inequality holds for $r \in [\frac{l_k}{2}, 2s_k]$ and all components of u^k have fast decay on all $r \in [l_k, s_k]$, then*

$$\sigma_i^k(B(0,s_k)) = \sigma_i^k(B(0,l_k)) + o(1),\ i \in I. \tag{5.4}$$

Proof. We prove it by contradiction. Suppose that this is not the case, then there exists $\ell \in I$ such that

$$\sigma_\ell^k(B(0,s_k)) > \sigma_\ell^k(B(0,l_k)) + \delta_1 \tag{5.5}$$

for some $\delta_1 > 0$. Let

$$\hat\sigma_i = \lim_{k\to+\infty} \sigma_i^k(B(0,l_k)),\ i \in I. \tag{5.6}$$

We first claim that:

$$\text{if } 2\mu_i - \sum_{j=1}^n k_{ij}\hat\sigma_j \neq 0 \text{ for all } i \in I, \text{ then (5.5) is impossible.} \tag{5.7}$$

Suppose that this claim is not true, i.e. (5.5) holds under the assumption

$$2\mu_i - \sum_{j=1}^n k_{ij}\hat\sigma_j \neq 0 \text{ for all } i.$$

Let

$$\delta_2 = \frac{1}{100n}\min\left\{\min_i |2\mu_i - \sum_{j=1}^n k_{ij}\hat\sigma_j|, \delta_1, 1\right\}$$

and $\tilde l_k \in (l_k, s_k)$ be such that

$$\max_i(\sigma_i^k(\tilde l_k) - \sigma_i^k(l_k)) = \delta_2. \tag{5.8}$$

We set

$$\hat I = \left\{ i \in I \,\middle|\, 2\mu_i - \sum_{j=1}^n k_{ij}\hat\sigma_j < 0\right\} \text{ and } \hat J = I \setminus \hat I. \tag{5.9}$$

The Harnack-type inequality (see Proposition 4.A) implies that $u_i^k(x) = \bar u_i^k(|x|) + O(1)$ for $|x| \in [\frac{1}{2}l_k, 2s_k]$. Thus we have from (1.15) that

$$\frac{d}{dr}(\bar u_i^k(r) + 2\log r) = \frac{2\mu_i - \sum_{j=1}^n k_{ij}\sigma_j^k(r)}{r},\ l_k \le r \le s_k, \tag{5.10}$$

where $\sigma_i^k(r) = \sigma_i^k(B(0,r)), i \in I$.

From the definition of δ_2, we have

$$\frac{d}{dr}(\bar u_i^k(r) + 2\log r) \le -\frac{\delta_2}{r} \text{ for } r \in [l_k, \tilde l_k],\ i \in \hat I. \tag{5.11}$$

By integrating the above equation from l_k up to $r \le \tilde l_k$, we have

$$\bar u_i^k(r) + 2\log r \le \bar u_i^k(l_k) + 2\log l_k + \delta_2 \log\frac{l_k}{r},\ i \in \hat I,$$

that is for $|x| = r$,

$$e^{u_i^k(x)} \le O(1)e^{\bar u_i^k(r)} \le e^{-N_k} l_k^{\delta_2} r^{-(2+\delta_2)},\ i \in \hat I,$$

where we used $\bar u_i^k(l_k) + 2\log l_k \le -N_k$ by the assumption of fast-decay. Thus

$$\int_{l_k\le|x|\le\tilde l_k} e^{u_i^k(x)}\mathrm{d}x \le 2\pi e^{-N_k} l_k^{\delta_2}\int_{l_k}^{\tilde l_k} r^{-(1+\delta_2)}dr \le 2\pi\frac{e^{-N_k}}{\delta_2} \to 0,\ i \in \hat I,$$

as $k \to +\infty$. Hence

$$\sigma_i^k(\tilde{l}_k) = \sigma_i^k(l_k) + o(1),\ i \in \hat{I}. \tag{5.12}$$

Since all the components have fast decay on $\partial B(0, l_k)$ and $\partial B(0, \tilde{l}_k)$, we have both $(\hat{\sigma}_1, \cdots, \hat{\sigma}_n)$ and $\lim_{k\to\infty}(\sigma_1^k(\tilde{l}_k), \cdots, \sigma_n^k(\tilde{l}_k))$ satisfy the Pohozaev identity, i.e.

$$\sum_{i=1}^{n} \hat{\sigma}_i^2 - \sum_{i=1}^{n-1} \hat{\sigma}_i \hat{\sigma}_{i+1} = 2 \sum_{i=1}^{n} \mu_i \hat{\sigma}_i, \tag{5.13}$$

and

$$\sum_{i=1}^{n} (\hat{\sigma}_i + \varepsilon_i)^2 - \sum_{i=1}^{n-1} (\hat{\sigma}_i + \varepsilon_i)(\hat{\sigma}_{i+1} + \varepsilon_{i+1}) = 2 \sum_{i=1}^{n} \mu_i (\hat{\sigma}_i + \varepsilon_i), \tag{5.14}$$

where $\varepsilon_i = \lim_{k\to+\infty} \sigma_i^k(\tilde{l}_k) - \hat{\sigma}_i$. Using (5.13) and (5.14), we get

$$\sum_{i\in\hat{J}} \varepsilon_i^2 - \sum_{i\in\hat{J}} \varepsilon_i \varepsilon_{i+1} + \sum_{i\in\hat{J}} (\sum_{j=1}^{n} k_{ij} \hat{\sigma}_j - 2\mu_i) \varepsilon_i = o(1), \tag{5.15}$$

where we used (5.12). By (5.8), we have $\max_{i\in\hat{J}} \varepsilon_i = \delta_2$. Then we get

$$\begin{aligned} 0 &= \sum_{i\in\hat{J}} \varepsilon_i^2 - \sum_{i\in\hat{J}} \varepsilon_i \varepsilon_{i+1} + \sum_{i\in\hat{J}} (\sum_{j=1}^{n} k_{ij} \hat{\sigma}_j - 2\mu_i) \varepsilon_i \\ &\le n\delta_2^2 - \min_i |\sum_{j=1}^{n} k_{ij} \hat{\sigma}_j - 2\mu_i| \delta_2 < 0, \end{aligned} \tag{5.16}$$

a contradiction. Thus we have proved the claim (5.7).

Next, we will show (5.5) is impossible by induction on the number N_e of ℓ such that $2\mu_\ell - \sum_{j=1}^{n} k_{\ell j} \hat{\sigma}_j = 0$. We assume

$$(5.5) \text{ is not true for } N_e = N, \tag{5.17}$$

where $0 \le N < n$. Next we will show (5.5) is also not true for $N+1$. We prove it by contradiction. Suppose that (5.5) is true when $N_e = N+1$. Decomposing

$$I = I_1 \cup I_2 \cup I_3,$$

where $I_1 := \{i \mid 2\mu_i - \sum_{j=1}^{n} k_{ij} \hat{\sigma}_j < 0\}$, $I_2 := \{i \mid 2\mu_i - \sum_{j=1}^{n} k_{ij} \hat{\sigma}_j > 0\}$ and $I_3 := \{i \mid 2\mu_i - \sum_{j=1}^{n} k_{ij} \hat{\sigma}_j = 0\}$, then $|I_3| = N+1$. Let

$$\delta_3 = \frac{1}{100n} \min \left\{ \min_{i\in I_1\cup I_2} |2\mu_i - \sum_{j=1}^{n} k_{ij} \hat{\sigma}_j|, \delta_1, 1 \right\},$$

and $\hat{l}_k \in (l_k, s_k)$ be such that

$$\max_i (\sigma_i^k(\hat{l}_k) - \sigma_i^k(l_k)) = \delta_3. \tag{5.18}$$

Then, by the same argument adopted above (see (5.12) and (5.15)) we conclude that

$$\sigma_i^k(\hat{l}_k) - \sigma_i^k(l_k) = o(1),\ i \in I_1, \tag{5.19}$$

and

$$\sum_{i\in I_2\cup I_3} \varepsilon_i^2 - \sum_{i\in I_2\cup I_3} \varepsilon_i\varepsilon_{i+1} + \sum_{i\in I_2}(\sum_{j=1}^n k_{ij}\hat{\sigma}_j - 2\mu_i)\varepsilon_i = o(1), \tag{5.20}$$

where $\varepsilon_i = \lim_{k\to+\infty} \sigma_i^k(\hat{l}_k) - \hat{\sigma}_i$. If there is some $i \in I_2$ such that $\varepsilon_i \geq \frac{\delta_3}{4}$, then we find that

$$\begin{aligned} 0 &= \sum_{i\in I_2\cup I_3} \varepsilon_i^2 - \sum_{i\in I_2\cup I_3} \varepsilon_i\varepsilon_{i+1} + \sum_{i\in I_2}(\sum_{j=1}^n k_{ij}\hat{\sigma}_j - 2\mu_i)\varepsilon_i \\ &\leq n\delta_3^2 - \frac{1}{4}\min_i |\sum_{j=1}^n k_{ij}\hat{\sigma}_j - 2\mu_i|\delta_3 < 0, \end{aligned} \tag{5.21}$$

a contradiction. As a consequence,

$$\delta_3 = \max_{i\in I_3} \varepsilon_i \text{ and } \delta_4 < \frac{1}{4}\delta_3,\ \text{where } \delta_4 := \max_{i\in I_2} \varepsilon_i.$$

Let

$$I_4 = \{i \in I_3,\ \varepsilon_i = \delta_3\}$$

and we pick out the smallest index $i_0 \in I_4$, it is easy to see that

$$2\mu_{i_0} - \sum_{j=1}^n k_{i_0 j}\sigma_j^k(\hat{l}_k) \leq \delta_4 - \delta_3 < 0. \tag{5.22}$$

On the other hand, we have for $i \in I_1 \cup I_2$, $2\mu_i - \sum_{j=1}^n k_{ij}\sigma_j^k(\hat{l}_k) \neq 0$ still holds. Then on $|x| = \hat{l}_k$, we have at most N components with $2\mu_i - \sum_{j=1}^n k_{ij}\sigma_j^k(\hat{l}_k) = o(1)$. In view of the assumption (5.17) with l_k replaced by $\hat{l}_k$, we can get

$$\sigma_i^k(B(0,s_k)) = \sigma_i^k(B(0,\hat{l}_k)) + o(1),\ i \in I. \tag{5.23}$$

On the other hand, we can easily get that

$$\begin{aligned} &\max_i \left(\sigma_i^k(B(0,s_k)) - \sigma_i^k(B(0,\hat{l}_k))\right) \\ &\geq \max_i \left(\sigma_i^k(B(0,s_k)) - \sigma_i^k(B(0,l_k))\right) \\ &\quad - \max_i \left(\sigma_i^k(B(0,\hat{l}_k)) - \sigma_i^k(B(0,l_k))\right) \\ &= \delta_1 - \delta_3 + o(1), \end{aligned}$$

which contradicts (5.23). Therefore, (5.5) is also not true for $N_e = N+1$ and we finish the induction process. Thus, in any case we have shown (5.5) can not hold and it proves the conclusion. q.e.d.

Before stating the last result in this section, we make the following preparation. Let the Harnack inequality hold for $r \in [\frac{1}{2}l_k, 2\tau_k]$. For a sequence $s_k \le \tau_k$, we define

(5.24)
$$\hat{\sigma}_i(B(\mathbf{x},\mathbf{s})) = \begin{cases} \lim\limits_{k\to+\infty} \sigma_i^k(B(x^k,s_k)) \text{ if } u_i^k \text{ has fast decay on } \partial B(x^k,s_k), \\ \\ \lim\limits_{r\to 0}\lim\limits_{k\to+\infty} \sigma_i^k(B(x^k,rs_k)) \text{ if } u_i^k \text{ has slow decay on } \partial B(x^k,s_k), \end{cases}$$

where $(\mathbf{x},\mathbf{s})$ stands for the sequence of the pair $\{(x^k,s_k)\}$. When $x^k=0$, we simply denote $\hat{\sigma}_i(B(\mathbf{x},\mathbf{s}))$ by $\hat{\sigma}_i(\mathbf{s})$.

The following lemma is very important in the bubbling analysis in section 6 and section 7.

Lemma 5.3. *Let $\hat{\sigma}_i(s)$ be defined as above and $r_k \in [l_k,\tau_k]$ satisfies the following conditions,*

(1) *u^k has fast decay on $\partial B(0,r_k)$,*
(2) *$\exists\ i \in I$, such that $\hat{\sigma}_i(\mathbf{r}) \ne \hat{\sigma}_i(\boldsymbol{\tau})$ ($\mathbf{r}$ and $\boldsymbol{\tau}$ stand for the sequence $\{r_k\}$ and $\{\tau_k\}$).*

Then there exists $s_k \in (r_k,\tau_k)$ such that

(i) *$s_k/r_k \to +\infty$, there is at least one component i such that u_i^k has slow decay on $\partial B(0,s_k)$,*
(ii) *$\hat{\sigma}_i(\mathbf{s}) = \hat{\sigma}_i(\mathbf{r}),\ i \in I$ ($\mathbf{s}$ stands for the sequence $\{s_k\}$).*

Proof. Since u^k has fast decay on $\partial B(0,r_k)$, it is easy to see that $(\hat{\sigma}_1(\mathbf{r}),\cdots,\hat{\sigma}_n(\mathbf{r}))$ satisfies the P.I. We set $\delta = \max_i(\hat{\sigma}_i(\boldsymbol{\tau}) - \hat{\sigma}_i(\mathbf{r}))$ and decompose
$$I = I_1 \cup I_2 \cup I_3,$$
where $I_1 := \{i \mid 2\mu_i - \sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{r}) < 0\}$, $I_2 := \{i \mid 2\mu_i - \sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{r}) > 0\}$ and $I_3 := \{i \mid 2\mu_i - \sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{r}) = 0\}$. Let

$$\delta_0 = \frac{1}{100n}\min\{\min_{i\in I_1\cup I_2} |2\mu_i - \sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{r})|, \delta, 1\} \tag{5.25}$$

and $\kappa_0 \in (0,\delta_0)$ be a small positive number. We choose $\ell_k \in [r_k,\tau_k]$ such that

$$\max_i(\sigma_i^k(\ell_k) - \sigma_i^k(r_k)) = \kappa_0. \tag{5.26}$$

Lemma 5.2 implies that u^k can not have fast decay on $[r_k,\ell_k]$. Thus there is a sequence $\ell_k \ge s_k \gg r_k$ such that some of u^k has slow decay on $\partial B(0,s_k)$. We claim

$$\hat{\sigma}_i(\mathbf{s}) = \hat{\sigma}_i(\mathbf{r}) \text{ for all } i \in I. \tag{5.27}$$

We prove it by contradiction. Suppose that (5.27) is not true, then again Lemma 5.2 says that there exists a sequence $r_k \ll \hat{s}_k \ll s_k$ such that

(i) some components of u^k have slow decay on $\partial B(0,\hat{s}_k)$,
(ii) $\max_i(\hat{\sigma}_i(\hat{\mathbf{s}})-\hat{\sigma}_i(\mathbf{r})) \in [\frac{1}{2}\varepsilon_0,\varepsilon_0]$ (where $\varepsilon_0 = \max_i(\hat{\sigma}_i(\mathbf{s})-\hat{\sigma}_i(\mathbf{r}))$ and $\hat{\mathbf{s}}$ stands for the sequence $\{\hat{s}_k\}$).

The next step is to re-scale u_i^k by

$$v_i^k(y) = u_i^k(x^k+\hat{s}_k y) + 2\log \hat{s}_k. \tag{5.28}$$

Due to the slow decay assumption on $\hat{s}_k$, some components of v_i^k converges and it implies that there is a sequence of $R_k \to +\infty$ such that

(iii) $R_k\hat{s}_k \ll s_k$, u^k has fast decay on $\partial B(0,R_k\hat{s}_k)$,
(iv) Set $\sigma_i = \lim_{k\to+\infty}\sigma_i^k(B(0,R_k\hat{s}_k))$, then $\boldsymbol{\sigma}=(\sigma_1,\cdots,\sigma_n)$ satisfies the P.I.

Let $\varepsilon_i = \sigma_i - \hat{\sigma}_i(\mathbf{r})$ and $\max_i \varepsilon_i \in [\frac{1}{2}\varepsilon_0,\varepsilon_0]$. Because both $\hat{\sigma}_i(\mathbf{r})$ and σ_i satisfies the P.I., we have

$$\sum_{i=1}^n \varepsilon_i^2 - \sum_{i=1}^n \varepsilon_i\varepsilon_{i+1} = \sum_{i=1}^n (2\mu_i - \sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{r}))\varepsilon_i. \tag{5.29}$$

Note if $i \in I_3$, we have $2\mu_i - \sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{r}) = 0$. We claim

$$\varepsilon_i = 0, \text{ for } i \in I_1 \cup I_2. \tag{5.30}$$

Once (5.30) is established, the R.H.S. of (5.29) vanishes and we have

$$\sum_{i=1}^n \varepsilon_i^2 - \sum_{i=1}^n \varepsilon_i\varepsilon_{i+1} = 0.$$

But the Cartan matrix is positive definite, the above identity yields $\varepsilon_i = 0,\ i \in I$, a contradiction, and the claim (5.27) holds.

So it remains to prove (5.30). We shall prove for $i \in I_2$ and the proof for $i \in I_1$ is similar. First, we claim all the components with index in I_2 are fast decay on $\partial B(0,\hat{s}_k)$. Otherwise, we could get some $i_0 \in I_2 \cap J$, where J is the maximal set with consecutive indices such that $v_i^k,\ i \in J$ converges to $SU(|J|+1)$ Toda system

$$\Delta v_i + \sum_{j\in J} k_{ij}e^{v_j} = 4\pi(\alpha_i - \frac{1}{2}\sum_{j=1}^n (k_{ij}\hat{\sigma}_j(\hat{\mathbf{s}})))\delta_0 \text{ in } \mathbb{R}^2,\ i \in J, \tag{5.31}$$

where $\alpha_i - \frac{1}{2}\sum_{j=1}^n (k_{ij}\hat{\sigma}_j(\hat{\mathbf{s}})) > -1,\ i \in J$. From the above equation, it is easy to see that

$$\begin{aligned}\sigma_{i_0}^k(R_k\hat{s}_k) &= \hat{\sigma}_{i_0}(\hat{\mathbf{s}}) + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_{i_0}} + o(1)\\ &\geq \hat{\sigma}_{i_0}(\hat{\mathbf{s}}) + \mu_{i_0} - \frac{1}{2}\sum_{j=1}^n (k_{i_0,j}\hat{\sigma}_j(\hat{\mathbf{s}})) + o(1)\\ &\geq \hat{\sigma}_{i_0}(\hat{\mathbf{s}}) + 2\delta_0 + o(1),\end{aligned}$$

and it contradicts (5.25) and (5.26). Therefore $u_i^k,\ i \in I_2$ are fast decay on $\partial B(0, \hat{s}_k)$. For the components $u_i^k,\ i \in I_2$, using (1.15) and (5.25), we have

(5.32)
$$\frac{d}{dr}(\bar{u}_i^k(r) + 2\log r) = \frac{2\mu_i - \sum_{j=1}^n k_{ij}\sigma_j^k(r)}{r} \geq \frac{\delta_0}{r} \text{ for } r \in [r_k, \hat{s}_k],\ i \in I_2.$$

Integrating the equation from $r \in [r_k, \hat{s}_k]$ to $\hat{s}_k$, we have

(5.33)
$$\bar{u}_i^k(r) + 2\log r \leq \bar{u}_i^k(\hat{s}_k) + 2\log\hat{s}_k + \delta_0 \log\frac{r}{\hat{s}_k} \text{ for } r \in [r_k, \hat{s}_k],\ i \in I_2.$$

Together with the fact $\bar{u}_i^k(\hat{s}_k) + 2\log\hat{s}_k \to -\infty$ as $k \to +\infty$, we get

$$\int_{r_k \leq |x| \leq \hat{s}_k} e^{u_i^k(x)}dx \leq 2\pi e^{\bar{u}_i^k(\hat{s}_k) + 2\log\hat{s}_k}\frac{1}{\hat{s}_k^{\delta_0}}\int_{r_k}^{\hat{s}_k} r^{-1+\delta_0}dr \to 0,\ i \in I_2.$$

Therefore

$$\hat{\sigma}_i(\mathbf{R}\hat{\mathbf{s}}) = \hat{\sigma}_i(\hat{\mathbf{s}}) = \hat{\sigma}_i(\mathbf{r}),\ i \in I_2, \tag{5.34}$$

where $\mathbf{R}\hat{\mathbf{s}}$ stands for the sequence $\{R_k\hat{s}_k\}$. Thus, the claim (5.30) holds and we finish the proof of the lemma. q.e.d.

6. Local masses on the bubbling disk centered at $x_j^k \neq 0$

6.1. Let $x_t^k \in \Sigma_k \setminus \{0\}$ and set

$$\tau_t^k = \frac{1}{2}\text{dist}(x_t^k, \Sigma_k \setminus \{x_t^k\}).$$

By Proposition 4.B, $l_t^k \ll \tau_t^k$. In this subsection, we study the local behavior of u^k in the ball $B(x_t^k, \tau_t^k)$. We recall that the Harnack inequality holds for $B(x_t^k, \tau_t^k) \setminus \{x_t^k\}$:

$$u_i^k(x) + 2\log|x - x_t^k| \leq C,\ \forall x \in B(x_t^k, \tau_t^k).$$

The local mass of the i-th component,

$$\sigma_{i,t}^k(r) = \frac{1}{2\pi}\int_{B(x_t^k, r)} h_i^k e^{u_i^k},\ i \in I.$$

Because $x_t^k \neq 0$ and $0 \notin B(x_t^k, \tau_t^k)$, equation (1.15) becomes

$$\Delta u_i^k + \sum_{j=1}^n k_{ij} h_j^k e^{u_j^k} = 0 \text{ in } B(x_t^k, \tau_t^k). \tag{6.1}$$

Throughout this subsection, we fix t and simplify our notation by dropping the index t. Recall that all u_i^k have fast decay on $\partial B(x^k, l^k)$. This is the starting point of the whole analysis in this section. Since $\alpha_i = 0$, we have $\mu_i = 1$. Hence in this section, $(\mu_1, \cdots, \mu_n)$ will be $(1, \cdots, 1)$.

For a sequence $s_k \leq \tau^k$, we recall $\hat{\sigma}_i(\mathbf{s})$ is defined by (5.24).

Proposition 6.1. *Let $u^k = (u_1^k, \cdots, u_n^k)$ be the solutions of (6.1) and $\hat{\sigma}_i(\boldsymbol{\tau})$ be defined in (5.24), the following hold:*

(1) *At least one component u^k has fast decay on $\partial B(x^k, \tau^k)$,*
(2) *$(\hat{\sigma}_1(\boldsymbol{\tau}), \cdots, \hat{\sigma}_n(\boldsymbol{\tau})) \in \Gamma(1, \cdots, 1)$.*

Proof. Basically (1) has been proved in Lemma 5.1. To prove (2), we divide our proofs into several steps.

Step 1. We prove that $(\hat{\sigma}_1(\mathbf{l}), \cdots, \hat{\sigma}_n(\mathbf{l})) \in \Gamma(1, \cdots, 1)$, where $\mathbf{l}$ stands for the sequence $\{l^k\}$. Recall that in Proposition 4.B, we set $u_{i_0}^k(x^k) = \max_i u_i^k(x^k)$ and let

$$v_i^k(y) = u_i^k(x^k + e^{-\frac{1}{2}u_{i_0}^k(x^k)} y) - u_{i_0}^k(x^k), \ i \in I. \tag{6.2}$$

After passing to a subsequence, we can find a set $J \subset I$ such that $v_i^k(y) \to -\infty$ over compact subsets of $\mathbb{R}^2$ for $i \in I \setminus J$, where $J = J_1 \cup \cdots \cup J_l$ and $J_t, t = 1, \cdots, l$ are disjoint sets, each $J_t = \{i_t, \cdots, i_t + |J_t| - 1\}, 1 \leq t \leq l$ consists of the maximal consecutive indices. While the components $v_i^k, i \in J_t$ converge in $C^2_{\text{loc}}(\mathbb{R}^2)$ to a $SU(|J_t| + 1)$ Toda system, i.e.,

$$\Delta v_i + \sum_{j \in J_t} k_{ij} e^{v_j} = 0, \ i \in J_t, \ t = 1, \cdots, l. \tag{6.3}$$

Applying the classification [**26**, Theorem 1.1], we can get

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{v_i} dy = 2 \sum_{j \in J_t} (k_t^{ij} + k_t^{ij_t^*}), \tag{6.4}$$

where $j_t^* = 2i_t + |J_t| - j - 1$ and $(k_t^{ij})_{|J_t| \times |J_t|}$ is the inverse of $(k_{ij})_{|J_t| \times |J_t|}$, $i, j \in J_t, \ t = 1, \cdots, l$. Since

$$v_i(y) = -2a_i \log |y| + O(1) \text{ at } \infty \text{ for some } a_i > 1,$$

it is easy to see there is a sequence of $R_k \to +\infty$ such that u^k has fast decay on $\partial B(0, l^k), \ l^k = R_k e^{-\frac{1}{2}u_{i_0}^k(x^k)}$.

According to the choice of l^k, we have

$$\hat{\sigma}_i(\mathbf{l}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_i}dy = 2\sum_{j\in J_t}(k_t^{ij} + k_t^{ij_t^*}),\ i\in J_t,\ t=1,\cdots,l, \tag{6.5}$$

and

$$\hat{\sigma}_i(\mathbf{l}) = 0,\ i\in I\setminus J. \tag{6.6}$$

Now we could apply Theorem 3.7. Set $\mu_i = 1$ and $\sigma_i = 0$ in (3.41). Hence

$$(\hat{\sigma}_1(\mathbf{l}),\cdots,\hat{\sigma}_n(\mathbf{l}))\in\Gamma(1,\cdots,1). \tag{6.7}$$

Step 2. If $\hat{\sigma}_i(\boldsymbol{\tau}) = \hat{\sigma}_i(\mathbf{l})$ for all $i\in I$, then Proposition 6.1 is proved. Otherwise, there exists $i\in I$ such that $\hat{\sigma}_i(\boldsymbol{\tau})\neq\hat{\sigma}_i(\mathbf{l})$, then we can apply Lemma 5.3 to find s_k such that $l^k\ll s_k\ll\tau^k$, some u_i^k has slow decay on $\partial B(x^k,s_k)$ and

$$\hat{\sigma}_i(\mathbf{s}) = \hat{\sigma}_i(\mathbf{l}),\ i\in I,$$

where $\mathbf{s}$ stands for the sequence $\{s_k\}$. Then we perform the same scaling as Step 1. But there is some difference for the limiting equation (6.3). For the sake of completeness, we sketch it below.

Let $v_i^k(y) = u_i^k(x^k+s_ky)+2\log s_k$. By a little abuse of notations, we still use J to collect the indices such that $u_i^k,\ i\in J$ has slow decay on $\partial B(x^k,s_k)$ and decompose $J=J_1\cup\cdots\cup J_l$ and $J_1,\cdots,J_l$ are disjoint sets, each $J_t=\{i_t,\cdots,i_t+|J_t|-1\}, 1\le t\le l$ consists of the maximal consecutive indices. Then $v_i^k\to-\infty$ in any compact set of $\mathbb{R}^2$ for $i\in I\setminus J$ and the components with indices $i\in J_t,\ t=1,\cdots,l$ converge to $SU(|J_t|+1)$ Toda system as $k\to+\infty$, i.e.,

$$\Delta v_i + \sum_{j\in J_t}k_{ij}e^{v_j} = 4\pi\sum_{j=1}^n(-\frac{1}{2}k_{ij}\hat{\sigma}_j(\mathbf{l}))\delta_0 \text{ in } \mathbb{R}^2,\quad i\in J_t, \tag{6.8}$$

where $-\frac{1}{2}\sum_{j=1}^n k_{ij}\hat{\sigma}_j(\mathbf{l}) > -1,\ i\in J$. Hence there are sequences $N_k^*, N_k\to+\infty$ as $k\to+\infty$ such that $N_k^*s_k\le\tau^k$ and satisfies

(i) $\int_{B(0,N_k^*)}e^{v_i}dy = \int_{\mathbb{R}^2}e^{v_i}dy + o(1),\quad i\in J$,

(ii) $v_i^k(y)+2\log|y|\le -N_k$ for $|y|=N_k^*,\quad i\in I$,

(iii) For $i\in I$, scaling back to u_i^k, we obtain u_i^k has fast decay on $\partial B(x^k,N_k^*s_k)$.

Then by Theorem 3.7, we claim that $\hat{\boldsymbol{\sigma}}(N_k^*s_k)\in\Gamma(1,\cdots,1)$. The proof of this claim will be given in Lemma 6.3 at the end of this section.

Let $s_{k,1} = N_k^*s_k$. If $\hat{\sigma}_i(\mathbf{s_1}) = \hat{\sigma}_i(\boldsymbol{\tau})$ for all $i\in I$ ($\mathbf{s_j}$ stands for the sequence $\{s_{k,j}\}$), then Proposition 6.1 is proved. If not, we could repeat the arguments of Step 2 to find $s_{k,1}\ll s_{k,j}\ll s_{k,j+1}$ such that

$\hat{\boldsymbol{\sigma}}(\mathbf{s_j}) \in \Gamma(1, \cdots, 1)$. Because $|\Gamma(1, \cdots, 1)|$ is finite, the gain for the energy of each step has a lower bound

$$\sum_{i=1}^{n} (\hat{\sigma}_i(\mathbf{s_{j+1}}) - \hat{\sigma}_i(\mathbf{s_j})) \geq \varepsilon_0 > 0,$$

and then after j steps, we have $\hat{\sigma}_i(\boldsymbol{\tau}) = \hat{\sigma}_i(\mathbf{s_j}),\ i \in I$. q.e.d.

6.2. Local mass in a group that does not contain 0**.** In this subsection we shall group together some of the points $x_i^k \in \Sigma_k$ provided the distance between these selected points is comparably smaller than the distance to the other points in Σ_k and 0. In other words, the subset S of Σ_k should satisfy the following S-conditions:

(1) $|S| \geq 2$, and $0 \notin S$ if equation (1.15) has singularity at 0.

(2) There is $x_1^k \in S$ such that for any two points $x_i^k, x_j^k \in S$

$$\operatorname{dist}(x_i^k, x_j^k) \leq C \max_{x^k \in S} \operatorname{dist}(x_1^k, x^k) \doteqdot C\ d(S)$$

for some constant C independent of k.

(3) The ratio $\operatorname{dist}(S, \Sigma_k \setminus S)/d(S) \to \infty$ as $k \to \infty$.

Suppose that $S = \{x_1^k, \cdots, x_{t_0}^k\}$ and let

$$l^k(S) = 2 \max_{1 \leq t \leq t_0} \operatorname{dist}(x_1^k, x_t^k).$$

Recall $\tau_t^k = \frac{1}{2}\operatorname{dist}(x_t^k, \Sigma_k \setminus \{x_t^k\})$, by (2) above we have $l^k(S) \sim \tau_t^k$ for $1 \leq t \leq t_0$. Let

$$\tau_S^k = \frac{1}{2}\operatorname{dist}(S, \Sigma_k \setminus S),$$

and it is easy to see

$$\tau_S^k \gg \tau_t^k, \text{ for } 1 \leq t \leq t_0.$$

By Proposition 6.1, the local mass $\hat{\sigma}_i(B(\mathbf{x_t}, \boldsymbol{\tau}_t)) = m_{t,i},\ i \in I$ satisfies

$$(m_{t,1}, \cdots, m_{t,n}) \in \Gamma(1, \cdots, 1),$$

where $(\mathbf{x_t}, \boldsymbol{\tau}_\mathbf{t})$ stands for the sequence of pair $\{(x_t^k, \tau_t^k)\}$. Furthermore, Theorem 3.2 yields that $m_{t,i} \in 2\mathbb{N} \cup \{0\},\ i \in I$.

Proposition 6.2. *The followings hold true:*

(i) *At least one component of u^k has fast decay on $\partial B(x_1^k, \tau_S^k)$,*

(ii) *Let $\sigma_i = \hat{\sigma}_i(B(\mathbf{x_1}, \boldsymbol{\tau}_\mathbf{S}))$, then $\sigma_i \in 2\mathbb{N} \cup \{0\}$.*

Proof. There are two cases to consider. The first one is all the components have fast decay on $\partial B(x_1^k, \tau_1^k)$, which implies u^k has fast decay on $\partial B(x_1^k, l^k(S))$ and the local mass

(6.9)

$$\sigma_i^k(B(x_1^k, l^k(S))) = \sum_{t=1}^{t_0} \sigma_i^k(B(x_t^k, \tau_t^k)) = \sum_{t=1}^{t_0} 2m_{t,i} + o(1) = 2M_i + o(1),$$

where $M_i \in \mathbb{N} \cup \{0\}$. Let $m_i(S) = \hat{\sigma}_i(B(\mathbf{x_1}, \boldsymbol{\tau}_\mathbf{S}))$. Suppose that

$$m_i(S) = \hat{\sigma}_i(B(\mathbf{x_1}, \mathbf{l}(\mathbf{S}))),$$

where $(\mathbf{x_1}, \mathbf{l}(\mathbf{S}))$ stands for the sequence of pair $\{(x_1^k, l^k(S))\}$. Then the proposition is proved. If $m_i(S) > \hat{\sigma}_i(B(\mathbf{x_1}, \mathbf{l}(\mathbf{S})))$ for some i, then we could apply Lemma 5.3 and follow the same argument of Proposition 6.1 to prove Proposition 6.2.

The second case is that some components of u^k have slow decay on $\partial B(x_1^k, \tau_1^k)$, which implies that some components of the re-scaled solution

$$v_i^k(y) = u_1^k(x_1^k + \tau_1^k y) + 2\log \tau_1^k$$

converges to $v_i(y)$, where $v_i(y)$ satisfy

$$\Delta v_i + \sum_{j \in J} k_{ij} e^{v_j} = 4\pi \sum_{t=1}^{t_0} n_{t,i} \delta_{q_t} \text{ in } \mathbb{R}^2,\ i \in J, \tag{6.10}$$

where $n_{t,i} = -\frac{1}{2}\sum_{j=1}^n k_{ij} m_{t,j} \in \mathbb{N} \cup \{0\}$. Then by Corollary 2.3, the total mass of v_i is $2\tilde{m}_i$, $i \in J$, where $\tilde{m}_i \in \mathbb{N}$. From it, there is a sequence of $R_k \to +\infty$ such that all the components of u^k have the fast decay on $\partial B(x_1^k, R_k l^k(S))$ and the local mass

$$\sigma_i^k(B(x_1^k, R_k l^k(S))) = 2(\sum_{t=1}^{t_0} m_{t,i} + \tilde{m}_i) + o(1),\ i \in J, \tag{6.11}$$

and

$$\sigma_i^k(B(x_1^k, R_k l^k(S))) = 2\sum_{t=1}^{t_0} m_{t,i} + o(1),\ i \in I \setminus J. \tag{6.12}$$

Thus the gain of mass at each step is at least 2. If we repeat this process, and Proposition 6.2 can be proved after finitely many steps. q.e.d.

We close this section by proving the lemma which is required at Step 2 of Proposition 6.1. In order to apply the lemma in general circumstance, we state the assumption first.

Suppose that there is a subset $J \subsetneq I$ such that u_i^k, $i \in J$ has slow decay on $\partial B(x^k, s_k)$. Let

$$v_i^k(y) = u_i^k(x^k + s_k y) + 2\log s_k.$$

Then $v_i^k(y) \to -\infty$ for $i \in I \setminus J$ and $v_i^k(y)$ converges to $v_i(y)$, $i \in J$. Decompose $J = J_1 \cup \cdots \cup J_l$, and $J_1, \cdots, J_l$ are disjoint sets, each $J_t = \{i_t, \cdots, i_t + |J_t| - 1\}, 1 \le t \le l$ consists of the maximal consecutive indices. For $i \in J_t$, $v_i(y)$ satisfies

$$\Delta v_i + \sum_{j \in J_t} k_{ij} e^{v_j} = 4\pi\alpha_i^* \delta_0 + 4\pi \sum_{l=1}^{N} m_{il} \delta_{q_l},\ i \in J_t, \tag{6.13}$$

where $0 \neq q_l \in \mathbb{R}^2$, $m_{il} \in \mathbb{N}$ and

$$\alpha_i^* = \alpha_i - \frac{1}{2}\sum_{j=1}^{n} k_{ij}\sigma_j > -1, \tag{6.14}$$

where α_i is given in (1.15). Set

$$\sigma_j^* = \begin{cases} \sigma_j, & \text{if } j \in I \setminus J, \\ \sigma_j + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_j}, & \text{if } j \in J. \end{cases} \tag{6.15}$$

Then we have the following result.

Lemma 6.3. *If there is* $\hat{\boldsymbol{\sigma}} \in \Gamma(\boldsymbol{\mu})$ *such that* $\sigma_i = \hat{\sigma}_i + 2n_i,\ n_i \in \mathbb{Z}$, *then* $\sigma_i^* = \hat{\sigma}_i^* + 2n_i^*$ *with* $\hat{\boldsymbol{\sigma}}^* \in \Gamma(\boldsymbol{\mu})$ *and* $n_i^* \in \mathbb{Z}$.

Proof. Let $\mathfrak{f}_t$ be a bijective map from $J_t \cup \{i_t - 1\}$ to itself, then by Theorem 2.2 we can compute the total mass of $v_i,\ i \in J_t$,

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_i} &= 2\sum_{j=i_t-1}^{i-1}\left(\sum_{l=i_t}^{\mathfrak{f}_t(j)} \alpha_l^* - \sum_{l=i_t}^{j} \alpha_l^*\right) + 2N_i \\ &= 2\sum_{j=i_t-1}^{i-1}\left(\sum_{l=i_t}^{\mathfrak{f}_t(j)} \mu_l^* - \sum_{l=i_t}^{j} \mu_l^*\right) + 2\bar{N}_i,\ i \in J_t, \end{aligned} \tag{6.16}$$

where

$$\mu_i^* = \alpha_i^* + 1 = \mu_i - \frac{1}{2}\sum_{l=1}^{n} k_{il}\sigma_l,\ i \in I,$$

and $N_i, \bar{N}_i \in \mathbb{Z},\ i \in J_t$. Since $\sigma_i = \hat{\sigma}_i + 2n_i$ for some $n_i \in \mathbb{Z}$, then we have $\mu_i^* = \mu_i - \frac{1}{2}\sum_{l=1}^{n} k_{il}\hat{\sigma}_l + \bar{n}_i,\ i \in I$ for some $\bar{n}_i \in \mathbb{Z}$, and (6.16) can be rewritten as

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{v_i} = 2\sum_{j=i_t-1}^{i-1}\left(\sum_{l=i_t}^{\mathfrak{f}_t(j)} \bar{\mu}_l - \sum_{l=i_t}^{j} \bar{\mu}_l\right) + 2\tilde{N}_i,\ i \in J_t, \tag{6.17}$$

with $\bar{\mu}_i = \mu_i - \frac{1}{2}\sum_{l=1}^{n} k_{il}\hat{\sigma}_l$ and $\tilde{N}_i \in \mathbb{Z}$.

Next we define an extension g of $\mathfrak{f}_t$ on $I_0 = I \cup \{0\}$:

$$g(i) = \begin{cases} \mathfrak{f}_t(i), & \text{if } i \in \bigcup_{t=1}^{l} (J_t \cup \{i_t - 1\}), \\ i, & \text{if } i \in I_0 \setminus \left(\bigcup_{t=1}^{l} (J_t \cup \{i_t - 1\})\right). \end{cases} \tag{6.18}$$

Then we can represent σ_i^* as the following

$$\sigma_i^* = \hat{\sigma}_i + 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{g(j)} \bar{\mu}_l - \sum_{l=1}^{j} \bar{\mu}_l\right) + 2n_i^* = \hat{\sigma}_i^* + 2n_i^*,\ n_i^* \in \mathbb{Z},\ i \in I, \tag{6.19}$$

where

$$\hat{\sigma}_i^* = \hat{\sigma}_i + 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{g(j)}\bar{\mu}_l - \sum_{l=1}^{j}\bar{\mu}_l\right).$$

Using Theorem 3.6 and $\hat{\boldsymbol{\sigma}} \in \Gamma(\boldsymbol{\mu})$, we get $\hat{\boldsymbol{\sigma}}^* \in \Gamma(\boldsymbol{\mu})$. Thus we finish the proof. q.e.d.

Remark 6.4. If $m_{il} = 0$ in (6.13) and $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu})$, then the proof together with [**26**, Theorem 1.1] and Theorem 3.7 shows $\boldsymbol{\sigma}^* \in \Gamma(\boldsymbol{\mu})$.

Denote the group S by S_1. Based on the discussion of Proposition 6.2, we could continue to select a new group S_2 such that S-condition holds.

Let

$$\tau_{S_2}^k = \frac{1}{2}\mathrm{dist}(x_2^k, \Sigma_k \setminus S_2) \text{ for } \; x_2^k \in S_2.$$

Then we can follow the arguments in Proposition 6.2 to obtain the same result.

If equation (1.15) does not contain singularity, the final step is to collect all x_i^k into one single biggest group and the conclusions in Proposition 6.2 hold. Then we get

$$(\sigma_1, \cdots, \sigma_n) = (2m_1, \cdots, 2m_n) \text{ satisfies the P.I.}$$

This proves Theorem 1.5 if (1.15) has no singularities.

If 0 is a singularity of (1.15) then we could decompose $\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_m$, where each S_i is the maximal collection of x_i^k in the following sense:

(i) $0 \notin S_i$ and there is $x_i^k \in S_i$ such that $\mathrm{dist}(x, y) \le d(S_i) \doteqdot \max_{x^k \in S_i} \mathrm{dist}(x_i^k, x^k)$ for any $x, y \in S_i$. Furthermore, we have $\mathrm{dist}(x_i^k, x^k) \ll |x_i^k|$ for all $x^k \in S_i$,

(ii) $\mathrm{dist}(x_i^k, x_j^k) \ge C \max\{|x_i^k|, |x_j^k|\}$ for some constant $C > 0$,

(iii) The local mass $\hat{\sigma}_i(B(x_j^k, \frac{1}{2}\tau_{S_j}^k)) \in 2\mathbb{N} \cup \{0\}$, $i \in I$.

7. Proof of Theorem 1.5, Theorem 1.6 and Theorem 1.7

In last section, we have decomposed

$$\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_m.$$

Let x_t^k be the point in S_t as described at the end of last section. Without loss of generality, we assume the sets $S_1, \cdots, S_{t_0}$ are the ones with $C^{-1}|x_1^k| \le |x_j^k| \le C|x_1^k|$ for $1 \le j \le t_0$ and $|x_j^k| \gg |x_1^k|$ for $j > t_0$.

Set $\tau_k = \frac{1}{2}$ if $m = 0$ and $\tau_k = \frac{1}{2}|x_1^k|$ if $m > 0$. Let $\sigma_i = \hat{\sigma}_i(\boldsymbol{\tau})$, then $(\sigma_1, \cdots, \sigma_n) \in \Gamma(\boldsymbol{\mu})$.

Lemma 7.1. $(\sigma_1, \cdots, \sigma_n) \in \Gamma(\boldsymbol{\mu})$.

Proof. We select $r_k \ll \tau_k$ such that $\max_i(\sigma_i^k(B(0,r_k))) = \frac{1}{k}$ and u^k has fast decay on $\partial B(0,r_k)$. We could use Lemma 5.3 to find a sequence of s_k such that u^k has slow decay on $\partial B(0,s_k)$ and $\hat{\sigma}_i(\mathbf{s}) = 0$. If $\tau_k/s_k \leq C$ or $\sigma_i = \hat{\sigma}_i(\mathbf{s})$, $i \in I$, then the claim is done, i.e., $\boldsymbol{\sigma} = (0,\cdots,0) \in \Gamma(\boldsymbol{\mu})$. If $\tau_k/s_k \to +\infty$. By performing the standard rescaling at s_k, there is a sequence $R_k \to +\infty$ as $k \to +\infty$ such that u^k has fast decay and $\hat{\sigma}_i(\mathbf{Rs}) \in \Gamma(\boldsymbol{\mu})$ ($\mathbf{Rs}$ stands for the sequence $\{R_k s_k\}$) by remark 6.4. After that, we could repeat the process to find $s_{k,j+1} \gg s_{k,j}$ and obtain $\hat{\sigma}_i(\mathbf{s_{j+1}}) \in \Gamma(\boldsymbol{\mu})$. ($\mathbf{s_{j+1}}$ stands for the sequence $\{s_{k,j+1}\}$) At each step, the total gain of the local mass at each partial blow-up has a lower bound, since $|\Gamma(\boldsymbol{\mu})|$ is finite, the process will stop after finite steps and we have

$$(\sigma_1,\cdots,\sigma_n) = (\hat{\sigma}_1(\boldsymbol{\tau}),\cdots,\hat{\sigma}_n(\boldsymbol{\tau})) \in \Gamma(\boldsymbol{\mu}), \tag{7.1}$$

and at least one component of u^k has fast decay on $\partial B(0,\tau_k)$. q.e.d.

Now we want to give a proof of Theorem 1.6.

Proof of Theorem 1.6. Clearly, it suffices to prove $m = 0$. Suppose $m \neq 0$ and we select $S_1,\cdots,S_{t_0}$ as above. As Proposition 6.2, there are two cases to consider.

The first one is all the components have fast decay on $\partial B(0,\frac{1}{2}\tau_k)$, which implies u^k has fast decay on $\partial B(0, l_{t_0}^k(S))$, where

$$l_{t_0}^k(S) := 4 \max_{1\leq j\leq t_0} \mathrm{dist}(0,S_j).$$

Using proposition 6.2, we have the local mass

$$\hat{\sigma}_i(\mathbf{l_{t_0}}(S)) = \hat{\sigma}_i(\boldsymbol{\tau}) + \sum_{j=1}^{t_0} \hat{\sigma}_i(B(\mathbf{x_j},\frac{\mathbf{1}}{\mathbf{2}}|\boldsymbol{\tau}_{\mathbf{S_j}}|)) = \hat{\sigma}_i(\boldsymbol{\tau}) + m_i,\ m_i \in 2\mathbb{Z},\ i \in I, \tag{7.2}$$

satisfies the P.I., where $\mathbf{l_{t_0}}(S)$ and $(\mathbf{x_j}, \frac{\mathbf{1}}{\mathbf{2}}|\boldsymbol{\tau}_{\mathbf{S_j}}|)$ stand for the sequence of $\{l_{t_0}^k(S)\}$ and the pair $\{(x_j^k, \frac{1}{2}|\tau_{S_j}^k|)\}$ respectively. For the simplicity of notations, we let $\sigma_i^* = \hat{\sigma}_i(\boldsymbol{\tau})$ and $\sigma_i = \hat{\sigma}_i(\mathbf{l_{t_0}}(S))$. By Lemma 7.1, we have $\boldsymbol{\sigma}^* = (\sigma_1^*,\cdots,\sigma_n^*) \in \Gamma(\boldsymbol{\mu})$ and both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ satisfies P.I. Hence

$$\sum_{i=1}^n (\sigma_i^*)^2 - \sum_{i=1}^{n-1} \sigma_i^*\sigma_{i+1}^* = 2\sum_{i=1}^n \mu_i\sigma_i^*, \tag{7.3}$$

and

$$\sum_{i=1}^n (\sigma_i^*+m_i)^2 - \sum_{i=1}^{n-1} (\sigma_i^*+m_i)(\sigma_{i+1}^*+m_{i+1}) = 2\sum_{i=1}^n \mu_i(\sigma_i^*+m_i). \tag{7.4}$$

Thus we have,

$$\sum_{i=1}^{n} 2m_i\sigma_i^* - \sum_{i=1}^{n-1}\sigma_i^* m_{i+1} - \sum_{i=1}^{n-1} m_i\sigma_{i+1}^* - \sum_{i=1}^{n} 2m_i\mu_i + \sum_{i=1}^{n} m_i^2 - \sum_{i=1}^{n-1} m_i m_{i+1} = 0. \tag{7.5}$$

Since $(\sigma_1^*, \cdots, \sigma_n^*) \in \Gamma(\mu_1, \cdots, \mu_n)$, we set

$$\sigma_i^* = 2\sum_{j=1}^{n} a_{ij}\mu_j, \qquad i \in I.$$

Then we can write (7.5) as

$$\begin{aligned} &4\sum_{i=1}^{n}\sum_{j=1}^{n} m_i a_{ij}\mu_j - 2\sum_{i=1}^{n-1}\sum_{j=1}^{n}(a_{ij}m_{i+1}\mu_j + a_{i+1,j}m_i\mu_j) \\ &= 2\sum_{i=1}^{n} m_i\mu_i + \sum_{i=1}^{n-1} m_i m_{i+1} - \sum_{i=1}^{n} m_i^2. \end{aligned} \tag{7.6}$$

Since $\alpha_1, \cdots, \alpha_n$ and 1 are $\mathbb{Q}$-linearly independent, we have $\mu_1, \cdots, \mu_n$ and 1 are $\mathbb{Q}$-linearly independent, which implies the coefficients of μ_i must vanish. Equivalently we have

$$\begin{bmatrix} 2a_{11} - a_{21} - 1 & 2a_{21} - a_{11} - a_{31} & \cdots & 2a_{n1} - a_{n-1,1} \\ 2a_{12} - a_{22} & 2a_{22} - a_{12} - a_{32} - 1 & \cdots & 2a_{n2} - a_{n-1,2} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{1n} - a_{2n} & 2a_{2n} - a_{1n} - a_{3n} & \cdots & 2a_{nn} - a_{n-1,n} - 1 \end{bmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} = 0. \tag{7.7}$$

We note the matrix on the left hand side of (7.7) is nothing but $-\mathcal{B}^t$, where $\mathcal{B}$ is introduced in (3.22) and is non-singular by Theorem 3.5. Therefore $m_i = 0$ and Theorem 1.6 is proved for this case.

If some components of u^k have slow decay on $\partial B(0, \tau_k)$, then we could perform the scaling:

$$v_i^k(y) = u_i^k(\tau_k y) + 2\log\tau_k.$$

Hence there is $J \subsetneq I$ such that $v_i^k(y) \to -\infty$ if $i \in I \setminus J$ and $v_i^k(y)$ converges to v_i for $i \in J$, where v_i satisfies (6.13). Furthermore, there is a sequence of $R_k \to +\infty$ as $k \to +\infty$ such that u^k has fast decay on $\partial B(0, R_k l_{t_0}^k(S))$. By Lemma 6.3,

$$\sigma_j := \lim_{k\to+\infty} \sigma_j^k(B(0, R_k l_{t_0}^k(S))) = \sigma_j^* + 2m_j$$

for some $\boldsymbol{\sigma}^* = (\sigma_1^*, \cdots, \sigma_n^*) \in \Gamma(\boldsymbol{\mu})$, and $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ satisfies the P.I. Then applying the same calculation as above, we have a contradiction. This completes the proof of Theorem 1.6. q.e.d.

Next, we prove the Theorem 1.5.

Proof of Theorem 1.5. Obviously if $m = 0$, then it was done by Lemma 7.1. So we assume $m \neq 0$, and we will group together $S = S_1 \cup \cdots \cup S_{t_0}$ and $\tilde{\tau}^k(S) = \mathrm{dist}(x_1^k, \Sigma_k \setminus S) \gg l_{t_0}^k(S)$. As the proof of Theorem 1.6, the first step is to find a sequence s_k: $l_{t_0}^k(S) \leq s_k \leq \tilde{\tau}^k(S)$ such that u^k has slow decay and the re-scaled v_i^k at s_k converges v_i which satisfies (6.13) for some $i \in J$ and goes to $-\infty$ for $i \in I \setminus J$. We note that for some situation, $m_{il} = 0$ for all $q_l \neq 0$. In any case, Lemma 6.3 tells that the new local mass $\lim_{k\to+\infty}(\sigma_i^k(R_k s_k), \cdots, \sigma_n^k(R_k s_k)) \in \Gamma(\boldsymbol{\mu}) + 2\mathbb{Z}$. We could repeat this process and to find $s_{k,j+1} \gg s_{k,j}$ such that $\hat{\boldsymbol{\sigma}}(\mathbf{s}_{j+1}) \in \Gamma(\boldsymbol{\mu}) + 2\mathbb{Z}$ and satisfies the P.I. Since all the local masses have an upper bound, the solution $\boldsymbol{\sigma}$ of P.I. such that $\boldsymbol{\sigma} \in \Gamma(\boldsymbol{\mu}) + 2\mathbb{Z}$ is finite. Thus the gain of local mass at each step has a lower bound, which implies after finite steps, we reach the conclusion

$$\boldsymbol{\sigma} = (\hat{\sigma}_1(\tilde{\boldsymbol{\tau}}(S)), \cdots, \hat{\sigma}_n(\tilde{\boldsymbol{\tau}}(S))) \in \Gamma(\boldsymbol{\mu}) + 2\mathbb{Z},$$

where $\tilde{\boldsymbol{\tau}}(S)$ stands for the sequence $\{\tilde{\tau}^k(S)\}$. This ends the process of grouping $\{0\}$ and $S_1, \cdots, S_{t_0}$. We could continue our grouping until S_m, and Theorem 1.5 is proved. q.e.d.

Proof of Theorem 1.2, Theorem 1.4 and Theorem 1.7. Let $(u_1^k, \cdots, u_n^k)$ be a sequence of blow up solutions of (1.10) with $(\rho_1, \cdots, \rho_n)$ replaced by $(\rho_1^k, \cdots, \rho_n^k)$. First, we prove Theorem 1.4. From the above discussion, we get that at least one component of u^k (say u_1^k) has the property that $u_1^k - \log \int_M h_1 e^{u_1^k}$ has fast decay on a small B near each blow up point q, which gives $u_1^k - \log \int_M h_1 e^{u_1^k} \to -\infty$ if x is neither in S, nor the blow up point. Hence Theorem 1.4 is proved.

We can interpret Theorem 1.2 as that the mass distribution of u_1^k is concentrated as $k \to +\infty$, we get that $\lim_{k\to+\infty} \rho_1^k$ equals to the summation of the local mass σ_1 at each blow up point q, which gives $\rho_1 \in \Gamma_1$, contradiction arises. Thus, we finish the proof of Theorem 1.2. Following the same argument we get Theorem 1.7. q.e.d.

8. The results on B, C and $\mathbf{G}_2$ type Toda system

In this section we shall discuss the corresponding result for **B**, **C** and $\mathbf{G}_2$ type Toda system. We divide this section into three subsections, the first one is about the estimation on the total mass for the **B**, **C** and $\mathbf{G}_2$ type Toda system, the second subsection is used for discussing the relation between Pohozaev identity and the Weyl group of **B**, **C** and

$\mathbf{G}_2$. In the last subsection we give the version of Corollary 1.3, Theorem 1.5 and Theorem 1.7 for **B** and **C** type Toda system.

It is well-known that the $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$ type Toda systems can be deduced from $\mathbf{A}_{2n}$, $\mathbf{A}_{2n-1}$ and $\mathbf{A}_6$ type Toda systems respectively. Precisely, we have the following results, see [**30**, Lemma 4.1 and Lemma 4.2] and [**31**, Example 3.4].

Lemma 8.A ($\mathbf{B}_n$ reduction)**.** *The u_i for $1 \le i \le n$ satisfy (1.15) for the $\mathbf{B}_n$ Toda system with parameters α_i for $1 \le i \le n$ if and only if the $\tilde{u}_i$ for $1 \le i \le 2n$ defined by*

$$\tilde{u}_i = \tilde{u}_{2n+1-i} = u_i + \delta_{in} \log 2, \quad 1 \le i \le n,$$

satisfy (1.15) for the $\mathbf{A}_{2n}$ Toda system with parameters $\tilde{\alpha}_i$ for $1 \le i \le 2n$ defined by

$$\tilde{\alpha}_i = \tilde{\alpha}_{2n+1-i} = \alpha_i, \quad 1 \le i \le n. \tag{8.1}$$

Lemma 8.B ($\mathbf{C}_n$ reduction)**.** *The u_i for $1 \le i \le n$ satisfy (1.15) for the $\mathbf{C}_n$ Toda system with parameters α_i for $1 \le i \le n$ if and only if the $\hat{u}_i$ for $1 \le i \le 2n-1$ defined by*

$$\hat{u}_i = \hat{u}_{2n-i} = u_i, \quad 1 \le i \le n,$$

satisfy (1.15) for the $\mathbf{A}_{2n-1}$ Toda system with parameters $\hat{\alpha}_i$ for $1 \le i \le 2n-1$ defined by

$$\hat{\alpha}_i = \hat{\alpha}_{2n-i} = \alpha_i, \quad 1 \le i \le n. \tag{8.2}$$

Lemma 8.C ($\mathbf{G}_2$ reduction)**.** *The u_i for $1 \le i \le 2$ satisfy (1.15) for the $\mathbf{G}_2$ Toda system with parameters α_i for $1 \le i \le 2$ if and only if the $\bar{u}_i$ for $1 \le i \le 6$ defined by*

$$\bar{u}_1 = \bar{u}_6 = u_1, \ \bar{u}_2 = \bar{u}_5 = u_2, \ \bar{u}_3 = \bar{u}_4 = u_1 + \log 2$$

satisfy (1.15) for the $\mathbf{A}_6$ Toda system with parameters $\bar{\alpha}_i$ for $1 \le i \le 6$ defined by

$$\bar{\alpha}_1 = \bar{\alpha}_3 = \bar{\alpha}_4 = \bar{\alpha}_6 = \alpha_1 \text{ and } \bar{\alpha}_2 = \bar{\alpha}_5 = \alpha_2. \tag{8.3}$$

We will see the following three groups play an important role in our discussion.

(1) Let $\mathbb{S}_{\mathbf{B}_n}$ be a subgroup of the permutation group for $\{0, 1, \cdots, 2n\}$ such that any element $f \in \mathbb{S}_{\mathbf{B}_n}$ satisfies

$$f(i) + f(2n - i) = 2n, \ 0 \le i \le 2n. \tag{8.4}$$

(2) Let $\mathbb{S}_{\mathbf{C}_n}$ be a subgroup of the permutation group for $\{0, 1, \cdots, 2n-1\}$ such that any element $f \in \mathbb{S}_{\mathbf{C}_n}$ satisfies

$$f(i) + f(2n - 1 - i) = 2n - 1, \ 0 \le i \le 2n - 1. \tag{8.5}$$

(3) Let $\mathbb{S}_{\mathbf{G}_2}$ be a subgroup of the permutation group for $\{0, 1, \cdots, 6\}$ such that any element $f \in \mathbb{S}_{\mathbf{G}_2}$ satisfies

$$f(i) + f(6 - i) = 6, \ 0 \le i \le 6 \text{ and } f(1) + f(2) = f(0) + 3. \tag{8.6}$$

8.1. Total mass for the $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$ Toda system. In this subsection, we shall get the estimate on the total mass for the solution of the $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$ (if $\mathbf{K} = \mathbf{G}_2$, then $n = 2$) Toda system,

$$\begin{cases} \Delta u_i + \sum_{j=1}^n k_{ij} e^{u_j} = 4\pi \sum_{t=1}^N \alpha_{t,i}\delta_{p_t} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty,\ i \in 1, 2 \cdots, n, \end{cases} \tag{8.7}$$

where $(k_{ij})_{n\times n}$ is the Cartan matrix for $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{G}_2$, $p_1, \cdots, p_N$ are distinct points in $\mathbb{R}^2$ and $\alpha_{t,i} > -1,\ 1 \le t \le N,\ 1 \le i \le n$. Let

$$\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i},\ 1 \le i \le n.$$

We will study $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ under the following case of (8.7), that is when all $\alpha_{t,i}$ are positive integers possibly except $\alpha_{1,i},\ 1 \le i \le n$. The main result in this subsection is the following

Theorem 8.1. *Suppose that $u = (u_1, \cdots, u_n)$ is a solution of* (8.7)*.*

(1) *If $(k_{ij})_{n\times n} = \mathbf{B}_n$, there exists a map $f \in \mathbb{S}_{\mathbf{B}_n}$ such that*

$$\sigma_i = \begin{cases} 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)} \tilde{\alpha}_{1,l} - \sum_{l=1}^{j} \tilde{\alpha}_{1,l}\right) + 2\tilde{N}_i, & \text{if } 1 \le i \le n-1, \\ 2\sum_{j=0}^{n-1}\sum_{l=f(2n-j)+1}^{n} \tilde{\alpha}_{1,l} + \tilde{N}_n, & \text{if } i = n, \end{cases} \tag{8.8}$$

where $\tilde{\alpha}_{t,i}$ is given by (8.1)*.*

(2) *If $(k_{ij})_{n\times n} = \mathbf{C}_n$, there exists a map $f \in \mathbb{S}_{\mathbf{C}_n}$ such that*

$$\sigma_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)} \hat{\alpha}_{1,l} - \sum_{l=1}^{j} \hat{\alpha}_{1,l}\right) + 2\hat{N}_i,\ 1 \le i \le n, \tag{8.9}$$

where $\hat{\alpha}_{t,i}$ is given by (8.2)*.*

(3) *If $(k_{ij})_{n\times n} = \mathbf{G}_2$, there exists a map $f \in \mathbb{S}_{\mathbf{G}_2}$ such that*

$$\sigma_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)} \bar{\alpha}_{1,l} - \sum_{l=1}^{j} \bar{\alpha}_{1,l}\right) + 2\bar{N}_i,\ 1 \le i \le 2, \tag{8.10}$$

where $\bar{\alpha}_{t,i}$ is given by (8.3)*.*

Proof. We only provide the proof for the $\mathbf{B}_n$ Toda system, the $\mathbf{C}_n$ and $\mathbf{G}_2$ cases can be argued similarly. By Lemma 8.A, we extend (8.7) to the following $\mathbf{A}_{2n}$ Toda system

$$\begin{cases} \Delta \tilde{u}_i + \sum_{j=1}^{2n} \tilde{k}_{ij} e^{\tilde{u}_j} = \sum_{t=1}^N 4\pi\tilde{\alpha}_{t,i}\delta_{p_t} \text{ in } \mathbb{R}^2, \\ \tilde{u}_i(x) = -2\tilde{\alpha}_{\infty,i}\log|x| + O(1),\ i \in 1, 2 \cdots, 2n, \end{cases} \tag{8.11}$$

where $(\tilde{k}_{ij})_{2n\times 2n}$ is the Cartan matrix of $\mathbf{A}_{2n}$,

$$\tilde{u}_i = \tilde{u}_{2n+1-i} = u_i + \delta_{i,n}\log 2,\ 1 \le i \le n, \tag{8.12}$$

and

$$\tilde{\alpha}_{t,i} = \tilde{\alpha}_{t,2n+1-i},\ 1 \le t \le N,\ 1 \le i \le n. \tag{8.13}$$

We set $\tilde\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\tilde u_i},\ 1 \le i \le 2n$. Then

$$\tilde\sigma_i = \tilde\sigma_{2n+1-i} = \sigma_i + \delta_{i,n}\sigma_n,\ 1 \le i \le n.$$

Let

$$\tilde L(y) = y^{(2n+1)} + \sum_{j=0}^{2n-1} \tilde a_j y^{(j)} = 0 \text{ in } \mathbb{C}, \tag{8.14}$$

be the corresponding ODE of the solution $\tilde u$ and the local exponents of (8.14) at p_t are $\tilde\beta_{t,i}$, defined by

$$\tilde\beta_{t,0} = -\tilde\gamma_{t,1},\ \tilde\beta_{t,i} = \tilde\beta_{t,i-1} + \tilde\alpha_{t,i} + 1,\ 1 \le i \le 2n. \tag{8.15}$$

The local exponents of (8.14) at ∞ are

$$\tilde\beta_{\infty,0} = -\tilde\gamma_{\infty,1},\ \tilde\beta_{\infty,i} = \tilde\beta_{\infty,i-1} + \tilde\alpha_{\infty,i} - 1,\ 1 \le i \le 2n, \tag{8.16}$$

where $\tilde\gamma_{\infty,1} = \sum_{j=1}^{2n} \tilde k^{1j}\tilde\alpha_{\infty,j}$.

Let $\mathcal{M}_t$ be the monodromy transformation at $p_t,\ t = 1, \cdots, N, \infty$. Then we have

$$\mathcal{M}_\infty \mathcal{M}_N \cdots \mathcal{M}_1 = \mathbb{I}_{2n+1}. \tag{8.17}$$

By our assumption, $e^{2\pi i\tilde\beta_{t,0}}$ is the only eigenvalue of $\mathcal{M}_t,\ 2 \le t \le N$. In addition, we can show $\tilde\beta_{t,0} \in \mathbb{Z}$. Indeed, it is known that

$$\tilde\beta_{t,0} = -\sum_{j=1}^{2n} \tilde k^{1j}\tilde\alpha_{t,j} = -\sum_{j=1}^{n} (\tilde k^{1j} + \tilde k^{1,2n+1-j})\alpha_{t,j},$$

where we used (8.13). For the matrix $(\tilde k^{ij})_{2n\times 2n}$, we have

$$\tilde k^{1j} + \tilde k^{1,2n+1-j} = 1,\ 1 \le j \le 2n.$$

Consequently $\tilde\beta_{t,0} \in \mathbb{Z}$ and $\mathcal{M}_t = \mathbb{I}_{2n+1},\ 2 \le t \le N$. While the monodromy at p_1 and ∞ are the following

$$\mathcal{M}_1 = \tilde C_1 \begin{bmatrix} e^{2\pi i\tilde\beta_{1,0}} & 0 & \cdots & 0 \\ 0 & e^{2\pi i\tilde\beta_{1,1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i\tilde\beta_{1,2n}} \end{bmatrix} \tilde C_1^{-1}, \tag{8.18}$$

and

$$\mathcal{M}_\infty = \tilde C_\infty \begin{bmatrix} e^{2\pi i\tilde\beta_{\infty,0}} & 0 & \cdots & 0 \\ 0 & e^{2\pi i\tilde\beta_{\infty,1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i\tilde\beta_{\infty,2n}} \end{bmatrix} \tilde C_\infty^{-1}, \tag{8.19}$$

where $\tilde C_0$ and $\tilde C_\infty$ are invertible matrices. By (8.15) and (8.16), we get

$$\tilde\beta_{1,i} + \tilde\beta_{1,2n-i} = -(\tilde\beta_{\infty,i} + \tilde\beta_{\infty,2n-i}) = 2n. \tag{8.20}$$

Using (8.17) and $\mathcal{M}_t = \mathbb{I}_{2n+1},\ 2 \le t \le N$, we get

$$\mathcal{M}_1 = \mathcal{M}_\infty^{-1}. \tag{8.21}$$

Therefore, we can find a permutation map f on $\{0, 1, \cdots, 2n\}$ such that,

$$\tilde{\beta}_{\infty,j} + \tilde{\beta}_{1,f(j)} + m_j = 0,\ 0 \le j \le 2n, \tag{8.22}$$

where $m_j \in \mathbb{Z},\ 0 \le j \le 2n$.

Next, we claim that there exists $g \in \mathbb{S}_{\mathbf{B}_n}$ (that depends on f) such that $\tilde{\beta}_{\infty,j}$ also can be written as

$$\tilde{\beta}_{\infty,j} + \tilde{\beta}_{1,g(j)} + \bar{m}_j = 0,\ 0 \le j \le 2n, \tag{8.23}$$

where $\bar{m}_j \in \mathbb{Z},\ 0 \le j \le 2n$. For the f introduced in (8.22), we set

$$\Lambda_f = \{i \mid f(i) + f(2n - i) = 2n,\ 0 \le i \le n\}. \tag{8.24}$$

If $|\Lambda_f| = n + 1$, then there is noting to prove and we can choose $g = f$. Suppose that the claim holds for $|\Lambda_f| \ge l,\ n + 1 \ge l > 0$, and we shall prove it holds also for $|\Lambda_f| = l - 1$. We choose $t \in \{0, 1, \cdots, n\} \setminus \Lambda_f$. According to the definition of Λ_f we get

$$f(t) + f(2n - t) \ne 2n. \tag{8.25}$$

Let t' be the index such that

$$f(t) + f(2n - t') = 2n. \tag{8.26}$$

Then it is easy to see that $t' \notin \Lambda_f$. We define a new map f' such that

$$f'(i) = \begin{cases} f(t'), & \text{if } i = t, \\ f(t), & \text{if } i = t', \\ f(i), & \text{if } i \notin \{t, t'\}, \end{cases} \tag{8.27}$$

and of course f' is a permutation map. By (8.20), (8.22) and (8.26), we have

$$\begin{aligned} \tilde{\beta}_{\infty,t'} + \tilde{\beta}_{1,f'(t')} &= -2n - \tilde{\beta}_{\infty,2n-t'} + \tilde{\beta}_{1,f(t)} \\ &= -2n + \tilde{\beta}_{1,f(2n-t')} + m_{2n-t'} + \tilde{\beta}_{1,f(t)} \\ &= m_{2n-t'} \end{aligned} \tag{8.28}$$

and

$$\begin{aligned} \tilde{\beta}_{\infty,t} + \tilde{\beta}_{1,f'(t)} = \tilde{\beta}_{\infty,t} + \tilde{\beta}_{1,f(t')} &= -m_t - \tilde{\beta}_{1,f(t)} - m_{t'} - \tilde{\beta}_{\infty,t'} \\ &= -m_t - m_{t'} - m_{2n-t'}. \end{aligned} \tag{8.29}$$

As a consequence of the above two equations and (8.22), we get

$$\tilde{\beta}_{\infty,j} + \tilde{\beta}_{1,f'(j)} + \tilde{m}_j = 0,\ \text{where } \tilde{m}_j \in \mathbb{Z},\ 0 \le j \le 2n. \tag{8.30}$$

On the other hand, it is easy to see that

$$|\Lambda_{f'}| \ge |\Lambda_f| + 1 = l.$$

From the induction assumption we get the claim holds for $|\Lambda_f| = l - 1$, i.e., we can find g such that (8.23) holds. Thus, we finish the induction process and the claim is proved. Then we follow the proof in Theorem 2.2 and get that

(8.31)
$$\tilde{\sigma}_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\tilde{\alpha}_{1,l} - \sum_{l=1}^{j}\tilde{\alpha}_{1,l}\right) + 2\tilde{N}_i,\ 1\le i\le n,\quad \text{where } f\in\mathbb{S}_{\mathbf{B}_n}.$$

Using $f\in\mathbb{S}_{\mathbf{B}_n}$ and after direct computation, we have

$$2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\tilde{\alpha}_{1,l} - \sum_{l=1}^{j}\tilde{\alpha}_{1,l}\right) = 2\sum_{j=0}^{2n-i}\left(\sum_{l=1}^{f(j)}\tilde{\alpha}_{1,l} - \sum_{l=1}^{j}\tilde{\alpha}_{1,l}\right),$$

which together with $\tilde{\sigma}_i = \tilde{\sigma}_{2n+1-i}$ gives $\tilde{N}_i = \tilde{N}_{2n+1-i}$. Therefore,

$$\sigma_i = \tilde{\sigma}_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\tilde{\alpha}_{1,l} - \sum_{l=1}^{j}\tilde{\alpha}_{1,l}\right) + 2\tilde{N}_i,\ 1\le i\le n-1. \tag{8.32}$$

When $i=n$, we have

$$\sum_{j=0}^{n-1}\left(\sum_{l=1}^{f(j)}\tilde{\alpha}_{1,l} - \sum_{l=1}^{j}\tilde{\alpha}_{1,l}\right) = 2\sum_{j=0}^{n-1}\sum_{l=f(2n-j)+1}^{n}\tilde{\alpha}_{1,l}.$$

Consequently, we get the σ_n can be written as

$$\sigma_n = \frac{1}{2}\tilde{\sigma}_n = 2\sum_{j=0}^{n-1}\sum_{l=f(2n-j)+1}^{n}\tilde{\alpha}_{1,l} + \tilde{N}_n.$$

Then we finish the proof. q.e.d.

When $\alpha_{t,i},\ 1\le t\le N,\ 1\le i\le n$ are integers, we get the following corollary.

Corollary 8.2. *Suppose that $u=(u_1,\cdots,u_n)$ is a solution of (8.7) and $\alpha_{t,i}\in\mathbb{N}\cup\{0\}$. Let $\boldsymbol{\sigma}=(\sigma_1,\cdots,\sigma_n)$ be the local mass of u. Then $\sigma_i,\ i\in I\setminus\{n\}$, are positive even integers, and σ_n is even positive integer for $\mathbf{C}_n$ and $\mathbf{G}_2$ cases and only an integer for $\mathbf{B}_n$ case.*

8.2. The Pohozaev identity and Weyl group for $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$. At first, we derive the Pohozaev identity for $\mathbf{B}_n$, $\mathbf{C}_n$ and $\mathbf{G}_2$ from $\mathbf{A}_{2n}$, $\mathbf{A}_{2n-1}$ and $\mathbf{A}_6$ respectively,

(8.33)
$$\mathbf{B}_n:\ \sum_{i=1}^{n-1}\sigma_i^2 + 2\sigma_n^2 - \sum_{i=1}^{n-2}\sigma_i\sigma_{i+1} - 2\sigma_{n-1}\sigma_n = 2\sum_{i=1}^{n-1}\mu_i\sigma_i + 4\mu_n\sigma_n,$$

$$\mathbf{C}_n:\ 2\sum_{i=1}^{n-1}\sigma_i^2 + \sigma_n^2 - 2\sum_{i=1}^{n-1}\sigma_i\sigma_{i+1} = 4\sum_{i=1}^{n-1}\mu_i\sigma_i + 2\mu_n\sigma_n, \tag{8.34}$$

and

$$\mathbf{G}_2: \ 3\sigma_1^2 - 3\sigma_1\sigma_2 + \sigma_2^2 = 6\mu_1\sigma_1 + 2\mu_2\sigma_2. \tag{8.35}$$

As $\mathbf{A}_n$ system, we define the corresponding set $\Gamma_{\mathbf{K}}(\boldsymbol{\mu})$ for $\mathbf{B}_n,\ \mathbf{C}_n$ and $\mathbf{G}_2$ by the same way: Let $(0,\cdots,0)\in\Gamma_{\mathbf{K}}(\boldsymbol{\mu})$, the other elements in this set are all generated from $(0,\cdots,0)$ by the principle:

$$\text{if } \boldsymbol{\sigma}\in\Gamma_{\mathbf{K}}(\boldsymbol{\mu}),\ \text{then } \mathfrak{R}_i\boldsymbol{\sigma}\in\Gamma_{\mathbf{K}}(\boldsymbol{\mu}),\ \forall i\in I,$$

where

$$\mathfrak{R}_i\boldsymbol{\sigma} = (\sigma_1,\cdots,2\mu_i - \sum_j k_{ij}\sigma_j + \sigma_i,\cdots,\sigma_n)\in\Gamma_{\mathbf{K}}(\boldsymbol{\mu}).$$

Regarding $\Gamma_{\mathbf{K}}(\boldsymbol{\mu})$ as a set of homogeneous polynomials of degree 1 in $\mathbb{C}[\boldsymbol{\mu}]$, we can represent each element $\boldsymbol{\sigma}$ as:

$$\sigma_i = \sum_{j=1}^{n} 2a_{ij}\mu_j,\ i\in I,$$

where $a_{ij}\in\mathbb{Z}\cup\{0\},\ i,j\in I$ and the matrix $\mathcal{A}_{\mathbf{K}}=[a_{ij}]$ is independent of $\boldsymbol{\mu}$.

Proposition 8.3. *For each element $\boldsymbol{\sigma}\in\Gamma_{\mathbf{K}}(\boldsymbol{\mu})$, we have $\boldsymbol{\sigma}$ satisfies (8.33), (8.34) and (8.35) for $\mathbf{K}=\mathbf{B}_n,\ \mathbf{C}_n$ and $\mathbf{G}_2$ respectively.*

Proof. For the $\mathbf{B}_n$ case, we can write the equation (8.33) as

$$\sigma_n^2 - (2\mu_n - \sum_{j\neq n} k_{nj}\sigma_j)\sigma_n + \sum_{j=1}^{n-1}\sigma_j^2 - \sum_{j=1}^{n-2}\sigma_j\sigma_{j+1} - 2\sum_{j=1}^{n-1}\mu_j\sigma_j = 0, \tag{8.36-n}$$

or

$$\sigma_i^2 - (2\mu_i - \sum_{j\neq i} k_{ij}\sigma_j)\sigma_i + \Pi_i = 0,\ 1\le i\le n-1, \tag{8.36-i}$$

where k_{ij} is the Cartan matrix for $\mathbf{B}_n$ and Π_i stands for the remainder terms in the equation (8.33). Since $\sigma_i + (2\mu_i - \sum_{j=1} k_{ij}\sigma_j + \sigma_i)$ is equal to the coefficients of σ_i in (8.36-i), we have $\mathfrak{R}_i\boldsymbol{\sigma}$ satisfies the P.I. if $\boldsymbol{\sigma}$ satisfies the P.I. So the conclusion holds for $\mathbf{B}_n$. The $\mathbf{C}_n$ and $\mathbf{G}_2$ cases can be proved similarly and we omit the details. q.e.d.

Theorem 8.4. *For each element $\boldsymbol{\sigma}\in\Gamma_{\mathbf{B}_n}(\boldsymbol{\mu})$, there exists a permutation map $f\in\mathbb{S}_{\mathbf{B}_n}$ such that $\sigma_i,\ i\in I$ admits the expression*

$$\sigma_i = \begin{cases} 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\tilde{\mu}_l - \sum_{l=1}^{j}\tilde{\mu}_l\right), & \text{if } 1\le i\le n-1,\\ 2\sum_{j=0}^{n-1}\sum_{l=f(2n-j)+1}^{n}\tilde{\mu}_l, & \text{if } i=n, \end{cases} \tag{8.37}$$

where $\tilde{\mu}_i = \tilde{\mu}_{2n+1-i} = \mu_i,\ i\in I.$

For each element $\boldsymbol{\sigma} \in \Gamma_{\mathbf{C}_n}(\boldsymbol{\mu})$*, there exists a permutation map* $f \in \mathbb{S}_{\mathbf{C}_n}$ *such that* $\sigma_i,\ i \in I$ *admits the expression*

$$\sigma_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\hat{\mu}_l - \sum_{l=1}^{j}\hat{\mu}_l\right),\ i \in I, \tag{8.38}$$

where $\hat{\mu}_i = \hat{\mu}_{2n-i} = \mu_i,\ i \in I$.

For each element $\boldsymbol{\sigma} \in \Gamma_{\mathbf{G}_2}(\boldsymbol{\mu})$*, there exists a permutation map* $f \in \mathbb{S}_{\mathbf{G}_2}$ *such that* σ_1, σ_2 *admit the expression*

$$\sigma_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\mu_{l,\mathbf{G}_2} - \sum_{l=1}^{j}\mu_{l,\mathbf{G}_2}\right),\ i = 1,2, \tag{8.39}$$

where $\mu_{1,\mathbf{G}_2} = \mu_{3,\mathbf{G}_2} = \mu_{4,\mathbf{G}_2} = \mu_{6,\mathbf{G}_2} = \mu_1$ *and* $\mu_{2,\mathbf{G}_2} = \mu_{5,\mathbf{G}_2} = \mu_2$. *Furthermore the correspondence* $\sigma \to f$ *is bijective for each case and consequently*

$$|\Gamma_{\mathbf{B}_n}(\boldsymbol{\mu})| = |\Gamma_{\mathbf{C}_n}(\boldsymbol{\mu})| = 2^n n! \text{ and } |\Gamma_{\mathbf{G}_2}(\boldsymbol{\mu})| = 12.$$

Proof. We only provide the proof for $\mathbf{K} = \mathbf{B}_n$, the $\mathbf{C}_n$ and $\mathbf{G}_2$ cases can be discussed similarly. We first extend the equation (1.15) to $\mathbf{A}_{2n}$ by Lemma 8.A. Then we get the corresponding local masses $(\tilde{\sigma}_1, \cdots, \tilde{\sigma}_{2n})$ for the extended $\mathbf{A}_{2n}$ satisfies

$$\sigma_n = \frac{1}{2}\tilde{\sigma}_n = \frac{1}{2}\tilde{\sigma}_{n+1} \quad \text{and} \quad \sigma_i = \tilde{\sigma}_i = \tilde{\sigma}_{2n+1-i},\ i \in I \setminus \{n\}. \tag{8.40}$$

It is not difficult to see

$$\tilde{\boldsymbol{\sigma}} \in \Gamma(\tilde{\boldsymbol{\mu}}),$$

where $\tilde{\mu}_j = \tilde{\mu}_{2n+1-j} = \mu_j,\ j \in I$. For any element of $\Gamma(\tilde{\boldsymbol{\mu}})$, using Theorem 3.2 we get there exists a permutation map f on $\{0, 1, \cdots, 2n\}$ such that

$$\tilde{\sigma}_i = 2\sum_{j=0}^{i-1}\left(\sum_{l=1}^{f(j)}\tilde{\mu}_l - \sum_{l=1}^{j}\tilde{\mu}_l\right),\ 1 \le i \le 2n. \tag{8.41}$$

From (8.40) and (8.41) we derive the following equality

$$\sum_{j=i}^{2n-i}\left(\sum_{l=1}^{f(j)}\tilde{\mu}_l - \sum_{l=1}^{j}\tilde{\mu}_l\right) = 0,\ i \in I, \tag{8.42}$$

which yields

$$\sum_{l=1}^{f(i)}\tilde{\mu}_l + \sum_{l=1}^{f(2n-i)}\tilde{\mu}_l = \sum_{l=1}^{i}\tilde{\mu}_l + \sum_{l=1}^{2n-i}\tilde{\mu}_l,\ i \in I. \tag{8.43}$$

From (8.42) and (8.43) we further obtain

$$f(i) + f(2n-i) = 2n,\ i \in I. \tag{8.44}$$

On the other hand, since f is a permutation map in $\{0, 1, \cdots, 2n\}$, we have

$$\sum_{i=0}^{2n} f(i) = \sum_{i=0}^{2n} i = n(2n+1),$$

which together with (8.44) gives

$$f(i) + f(2n-i) = 2n, \ i \in I \cup \{0\}.$$

Thus $f \in \mathbb{S}_{\mathbf{B}_n}$. Then we can represent the σ_i as

$$\sigma_i = \begin{cases} 2\sum_{j=0}^{i-1}\left(\sum_{l=0}^{f(j)}\tilde{\mu}_l - \sum_{l=0}^{j}\tilde{\mu}_l\right), & \text{if } 1 \le i \le n-1, \\ 2\sum_{j=0}^{n-1}\sum_{l=f(2n-j)+1}^{n}\tilde{\mu}_l, & \text{if } i = n, \end{cases} \tag{8.45}$$

where $f \in \mathbb{S}_{\mathbf{B}_n}$. Therefore we proved if $\boldsymbol{\sigma} \in \Gamma_{\mathbf{B}_n}(\boldsymbol{\mu})$, then σ_i admits the expression (8.45) with $f \in \mathbb{S}_{\mathbf{B}_n}$. It is not difficult to see that different element in $\mathbb{S}_{\mathbf{B}_n}$ gives different $\boldsymbol{\sigma}$. In addition, we can follow a similar argument of the proof for Theorem 3.2 to show the correspondence $\boldsymbol{\sigma} \to f$ is bijective. As a consequence,

$$|\Gamma_{\mathbf{B}_n}(\boldsymbol{\mu})| = 2^n n!.$$

This completes the proof. q.e.d.

For any $\boldsymbol{\sigma} \in \Gamma_{\mathbf{K}}$, we can define the corresponding matrix $\mathcal{A}_{\mathbf{K}}$. Then we set

$$\mathcal{B}_{\mathbf{K}} = \mathbb{I}_n - \mathbf{K}\mathcal{A}_{\mathbf{K}}. \tag{8.46}$$

Proposition 8.5. *For each f in $\mathbb{S}_{\mathbf{K}}$ we define the corresponding matrix $\mathcal{B}_{\mathbf{K},f}$. Then*

(a) *The matrix $\mathcal{B}_{\mathbf{K}}$ for any $\mathcal{A}_{\mathbf{K}}$ is non-singular. Furthermore, for any $f \in \mathbb{S}_{\mathbf{K}}$, let $f \to \mathcal{A}_{\mathbf{K},f}$ be the correspondence by Theorem 8.4 and $\mathcal{B}_{\mathbf{K},f}$ is given in (8.46). Then $f \to \mathcal{B}_{\mathbf{K},f}$ is an anti-homomorphism from $\mathbb{S}_{\mathbf{K}}$ to $GL(n, \mathbb{Z})$. Consequently*

$$\{\mathcal{B}_{\mathbf{K}} \mid \mathcal{B}_{\mathbf{K}} \text{ is given by } (8.46),\ \mathcal{A}_{\mathbf{K}} \in \Gamma_{\mathbf{K}}(\boldsymbol{\mu})\} \text{ is a group},$$

denoted by $\mathbb{B}_{\mathbf{K}}$.

(b) *The group $\mathbb{B}_{\mathbf{K}}$ is the Weyl group of the root system of the Lie algebra $\mathbf{K}$.*

Proof. The idea of the proof follows from Proposition 3.5. To show that $\mathcal{B}_{\mathbf{K}}$ is invertible, we note that the equation (8.33)–(8.35) can be rewritten into

$$\boldsymbol{\mu}^t \mathcal{A}_K^t M_{\mathbf{K}} \mathcal{A}_K \boldsymbol{\mu} = \boldsymbol{\mu}^t (D_{\mathbf{K}}^{-1}\mathcal{A}_{\mathbf{K}} + \mathcal{A}_{\mathbf{K}}^t D_{\mathbf{K}}^{-1})\boldsymbol{\mu}, \tag{8.47}$$

where $M_{\mathbf{K}} = D_{\mathbf{K}}^{-1}\mathbf{K}$ and

$$D_{\mathbf{K}} = \begin{cases} \text{diag}(1, 1, \cdots, 1, \frac{1}{2}), & \text{if } \mathbf{K} = \mathbf{B}_n, \\ \text{diag}(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, 1), & \text{if } \mathbf{K} = \mathbf{C}_n, \\ \text{diag}(\frac{2}{3}, 2), & \text{if } \mathbf{K} = \mathbf{G}_2. \end{cases} \tag{8.48}$$

Since the equation (8.47) is independent of μ, we get

$$\mathcal{A}_K^t M_{\mathbf{K}} \mathcal{A}_K = D_{\mathbf{K}}^{-1} \mathcal{A}_{\mathbf{K}} + \mathcal{A}_{\mathbf{K}}^t D_{\mathbf{K}}^{-1}. \tag{8.49}$$

On the other hand, $\mathbf{K}$ is invertible, we can rewrite $\mathcal{B}_{\mathbf{K}} = \mathbb{I}_n - \mathbf{K}\mathcal{A}_{\mathbf{K}}$ as

$$\mathcal{A}_{\mathbf{K}} = \mathbf{K}^{-1}(\mathbb{I}_n - \mathcal{B}_{\mathbf{K}}) \quad \text{and} \quad \mathcal{A}_{\mathbf{K}}^t = (\mathbb{I}_n - \mathcal{B}_{\mathbf{K}}^t)(\mathbf{K}^t)^{-1}. \tag{8.50}$$

Substituting (8.50) into (8.51), we get

$$\begin{aligned} &(\mathbb{I}_n - \mathcal{B}_{\mathbf{K}}^t) D_{\mathbf{K}}^{-1} M_{\mathbf{K}}^{-1} M_{\mathbf{K}} M_{\mathbf{K}}^{-1} D_{\mathbf{K}}^{-1} (\mathbb{I}_n - \mathcal{B}_{\mathbf{K}}) \\ &= D_{\mathbf{K}}^{-1} M_{\mathbf{K}}^{-1} D_{\mathbf{K}}^{-1} (\mathbb{I}_n - \mathcal{B}_{\mathbf{K}}) + (\mathbb{I}_n - \mathcal{B}_{\mathbf{K}}^t) D_{\mathbf{K}}^{-1} M_{\mathbf{K}}^{-1} D_{\mathbf{K}}^{-1}, \end{aligned} \tag{8.51}$$

which implies

$$\mathcal{B}_{\mathbf{K}}^t D_{\mathbf{K}}^{-1} M_{\mathbf{K}}^{-1} D_{\mathbf{K}}^{-1} \mathcal{B}_{\mathbf{K}} = D_{\mathbf{K}}^{-1} M_{\mathbf{K}}^{-1} D_{\mathbf{K}}^{-1}. \tag{8.52}$$

Therefore $\mathcal{B}_{\mathbf{K}}$ is invertible. Next we prove the left conclusion in (a), i.e., the map $f \to \mathcal{B}_{\mathbf{K},f}$ is an anti-isomorphism from $\mathbb{S}_{\mathbf{K}}$ to $\mathbb{B}_{\mathbf{K}}$, $\mathbf{K} = \mathbf{B}_n, \mathbf{C}_n$. Compared with the proof of Theorem 3.5, the differences for the proof of $\mathbf{K} = \mathbf{B}_n, \mathbf{C}_n$ are the choices of simple map f. When $\mathbf{K} = B_n$, we pick f from the following:

$$f_i(j) = \begin{cases} j+1, & \text{if } j = i, 2n-i-1, \\ j-1, & \text{if } j = i+1, 2n-i, \\ j, & \text{if } j \in I \setminus \{i, i+1, 2n-i-1, 2n-i\}, \end{cases} \quad 0 \le i \le n-2, \tag{8.53}$$

and

$$f_{n-1}(j) = \begin{cases} n+1, & \text{if } j = n-1, \\ n-1, & \text{if } j = n+1, \\ j, & \text{if } j \in I \setminus \{n-1, n+1\}. \end{cases} \tag{8.54}$$

We call f_i, $0 \le i \le n-1$ simple permutation. It is not difficult to see that $f_i \in \mathbb{S}_{\mathbf{B}_n}$ and any element in $\mathbb{S}_{\mathbf{B}_n}$ can be decomposed by the above simple maps. In addition, suppose that $g \in \mathbb{S}_{\mathbf{B}_n}$ corresponds to the element

$$\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n) \in \Gamma_{\mathbf{B}_n}(\boldsymbol{\mu}),$$

then $g \circ f_i$ corresponds to the following element $\mathfrak{R}_{i+1}\boldsymbol{\sigma}$, $i \in \{0, \cdots, n-1\}$.

While for $\mathbf{K} = \mathbf{C}_n$, the simple maps are the following ones,

$$f_i(j) = \begin{cases} j+1, & \text{if } j = i, 2n-i-2, \\ j-1, & \text{if } j = i+1, 2n-i-1, \\ j, & \text{if } j \in I \setminus \{i, i+1, 2n-i-1, 2n-i\}, \end{cases} \quad 0 \le i \le n-2, \tag{8.55}$$

and

$$f_{n-1}(j) = \begin{cases} n, & \text{if } j = n-1, \\ n-1, & \text{if } j = n, \\ j, & \text{if } j \in I \setminus \{n-1, n+1\}. \end{cases} \tag{8.56}$$

Similarly, it is easy to see that $f_i \in \mathbb{S}_{\mathbf{C}_n}$ and any element in $\mathbb{S}_{\mathbf{C}_n}$ can be decomposed by the simple permutations given in (8.55) and (8.56). Furthermore, if $g \in \mathbb{S}_{\mathbf{C}_n}$ corresponds to the element

$$\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n) \in \Gamma_{\mathbf{C}_n}(\boldsymbol{\mu}),$$

then $g \circ f_i$ corresponds to the following element $\mathfrak{R}_{i+1}\boldsymbol{\sigma},\ i \in \{0, \cdots, n-1\}$.

In the end, if $\mathbf{K} = \mathbf{G}_2$, the simple maps are the following two,

$$f_0(j) = \begin{cases} j+1, & \text{if } j = 0, 5, \\ j-1, & \text{if } j = 1, 6, \\ 6-j, & \text{if } j = 2, 3, 4, \end{cases} \tag{8.57}$$

and

$$f_1(j) = \begin{cases} j+1, & \text{if } j = 1, 4, \\ j-1, & \text{if } j = 2, 5, \\ j, & \text{if } j = 0, 3, 6. \end{cases} \tag{8.58}$$

As the above two cases, we find that $f_i \in \mathbb{S}_{\mathbf{G}_2},\ i = 0, 1$ and any element in $\mathbb{S}_{\mathbf{G}_2}$ can be decomposed by these two simple permutations. If $g \in \mathbb{S}_{\mathbf{G}_2}$ corresponds to the element

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \Gamma_{\mathbf{G}_2}(\boldsymbol{\mu}),$$

then $g \circ f_i$ corresponds to the element $\mathfrak{R}_{i+1}\boldsymbol{\sigma},\ i \in \{0, 1\}$.

Next we will show

$$\mathcal{B}_{\mathbf{K}, g \circ f} = \mathcal{B}_{\mathbf{K}, f}\mathcal{B}_{\mathbf{K}, g}$$

with f is the simple map given in (8.53)–(8.54), (8.55)–(8.56) and (8.57)–(8.58) for $\mathbf{K} = \mathbf{B}_n$, $\mathbf{K} = \mathbf{C}_n$ and $\mathbf{K} = \mathbf{G}_2$ respectively. The left argument goes almost the same as Theorem 3.5 and we omit it.

(b) It is known that the generators of the Weyl group for $\mathbf{B}_n$ and $\mathbf{C}_n$ are the following,

$$\mathbf{B}_n:\ \mathcal{S}_{e_1 - e_2},\ \mathcal{S}_{e_2 - e_3}, \cdots, \mathcal{S}_{e_{n-1} - e_n},\ \mathcal{S}_{e_n}, \tag{8.59}$$

$$\mathbf{C}_n:\ \mathcal{S}_{e_1 - e_2},\ \mathcal{S}_{e_2 - e_3}, \cdots, \mathcal{S}_{e_{n-1} - e_n},\ \mathcal{S}_{2e_n}, \tag{8.60}$$

and

$$\mathbf{G}_2:\ \mathcal{S}_{e_1 - e_2},\ \mathcal{S}_{2e_2 - e_1 - e_3}. \tag{8.61}$$

We refer the readers to [**20**] for the details.

As Theorem 3.5, we can obtain that the corresponding matrix of the generators given in (8.59) for $\mathbf{B}_n$, (8.60) for $\mathbf{C}_n$ and (8.61) for $\mathbf{G}_2$ are exactly the matrices $\mathcal{B}_{\mathbf{B}_n, f}$, $\mathcal{B}_{\mathbf{C}_n, f}$ and $\mathcal{B}_{\mathbf{G}_2, f}$ respectively, where f is the corresponding simple map of $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{G}_2$, see (8.53)-(8.58). Consequently, we get the conclusion (b). q.e.d.

Next, we shall establish the version of Theorem 3.6 for $\mathbf{K} = \mathbf{B}_n$ or $\mathbf{K} = \mathbf{C}_n$. (This result is used for doing the combination of the bubbling disks. However, for $\mathbf{G}_2$ case, we can avoid using it, see [**27**] for the details. Therefore we only study the cases for $\mathbf{B}_n$ and $\mathbf{C}_n$.) Since the proof is almost the same as Theorem 3.6, we will only give the statements without proof.

Theorem 8.6. *Assume $\boldsymbol{\sigma} \in \Gamma_{\mathbf{K}}(\boldsymbol{\mu})$, and let*

$$\bar{\mu}_i = \mu_i - \frac{1}{2}\sum_j k_{ij}\sigma_j, \ i \in I.$$

When $\mathbf{K} = \mathbf{B}_n$, for any $f \in \mathbb{S}_{\mathbf{B}_n}$ we set
(8.62)

$$\sigma_{f,i} = \begin{cases} \sigma_i + 2\sum_{j=0}^{i-1}\left(\sum_{l=0}^{f(j)}\tilde{\bar{\mu}}_l - \sum_{l=0}^{j}\tilde{\bar{\mu}}_l\right), & \text{if } 1 \le i \le n-1, \\ \sigma_n + 2\sum_{j=0}^{n-1}\sum_{l=f(2n-j)+1}^{n}\tilde{\bar{\mu}}_l, & \text{if } i = n, \end{cases}$$

where $\tilde{\bar{\mu}}_j = \tilde{\bar{\mu}}_{2n+1-j} = \bar{\mu}_j, \ j \in I$. When $\mathbf{K} = \mathbf{C}_n$, for any $f \in \mathbb{S}_{\mathbf{C}_n}$ we set

$$\sigma_{f,i} = \sigma_i + 2\sum_{j=0}^{i-1}\left(\sum_{l=0}^{f(j)}\hat{\bar{\mu}}_l - \sum_{l=0}^{j}\hat{\bar{\mu}}_l\right), \ i \in I, \tag{8.63}$$

where $\hat{\bar{\mu}}_j = \hat{\bar{\mu}}_{2n-j} = \bar{\mu}_j, \ j \in I$. Then

$$(\sigma_{f,1}, \cdots, \sigma_{f,n}) \in \Gamma_{\mathbf{K}}(\boldsymbol{\mu}).$$

8.3. The main results on B and C type Toda system. Let $u^k = (u_1^k, \cdots, u_n^k)$ be a sequence of solutions of the following equations

$$\Delta u_i^k + \sum_{j=1}^{n} k_{ij}h_j^k e^{u_j^k} = 4\pi\alpha_i\delta_0, \text{ in } B(0,1), \ i \in I, \tag{8.64}$$

where $\mathbf{K} = (k_{ij})_{n\times n}$ denotes the Cartan matrix for $\mathbf{B}_n$ or $\mathbf{C}_n$. As $\mathbf{A}_n$ system, we assume h_i^k satisfies (1.16) and $u^k = (u_1^k, \cdots, u_n^k)$ satisfies (1.17).

Let σ_i (defined in (1.18)) denotes the local mass for i-th component of the blow up solutions u^k.

Theorem 8.7. *Suppose that σ_i, $i \in I$ are local masses of a sequence of blowup solutions of (8.64) such that the assumption (1.16) and (1.17) hold. Then for $\mathbf{K} = \mathbf{B}_n$, σ_i can be written as*

$$\sigma_i = \begin{cases} 2(\sum_{j=1}^{n}\tilde{N}_{i,j}\mu_j + \tilde{N}_{i,n+1}), & \text{if } 1 \le i \le n-1, \\ 2(\sum_{j=1}^{n}\tilde{N}_{n,j}\mu_j) + \tilde{N}_{n,n+1}, & \text{if } i = n, \end{cases}$$

where $\tilde{N}_{i,j} \in \mathbb{Z}, \ i \in I$ and $j \in I \cup \{n+1\}$, while for $\mathbf{K} = \mathbf{C}_n$, σ_i can be written as

$$\sigma_i = 2(\sum_{j=1}^{n}\hat{N}_{i,j}\mu_j + \hat{N}_{i,n+1}), \ i = 1, 2, \cdots, n,$$

where $\hat{N}_{i,j} \in \mathbb{Z}, \ i \in I$ and *$j \in I \cup \{n+1\}$.*

Remark 8.1. By Theorem 8.7, if $\alpha_i,\ i \in I$ in (8.64) are non-negative integers and $\mathbf{K} = \mathbf{B}_n$, we only get $\sigma_i,\ 1 \le i \le n-1$ are even non-negative integers and σ_n is non-negative integer. However, using equation (8.33) we can also derive that σ_n is even.

From the Theorem 8.7 and the above remark, we can obtain the corresponding result of Corollary 1.3 for $\mathbf{B}$ and $\mathbf{C}$ type Toda system.

Suppose that $u = (u_1, \cdots, u_n)$ is the solutions of the following system defined on M:

$$\Delta_g u_i + \sum_{j=1}^{n} k_{ij}\rho_j\Big(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j} dV_g} - 1\Big) = \sum_{p_t \in S} 4\pi\alpha_{t,i}(\delta_{p_t} - 1), \tag{8.65}$$

where $\mathbf{K}$ denotes the Cartan matrix for $\mathbf{B}_n$ or $\mathbf{C}_n$, $\alpha_{t,i} > -1$ is the strength of the Dirac mass δ_{p_t} and S is finite subset of M.

Theorem 8.8. *Suppose that $\alpha_{t,i} \in \mathbb{N} \cup \{0\}$ for any $i \in I$ and $p_t \in S$. Let $\mathbf{K}$ be $\mathbf{B}_n$ or $\mathbf{C}_n$, h_i be positive C^1 functions on M and $\mathcal{K}$ be a compact subset of $M \setminus S$. If $\rho_i \notin 4\pi\mathbb{N},\ i \in I$, there exists a constant $C(\mathcal{K}, \rho_1, \cdots, \rho_n)$ such that for any solution $u = (u_1, \cdots, u_n)$ of (8.65) in $\mathring{H}^1(M)$,*

$$|u_i(x)| \le C, \quad \forall x \in \mathcal{K}, \quad i \in I.$$

As $\mathbf{A}_n$ Toda system, we can extend Theorem 8.8 if $\boldsymbol{\alpha} = (\alpha_{t,1}, \cdots, \alpha_{t,n})$ satisfies the $\mathbb{Q}$-condition. For equation (8.65), let $\mu_{t,i} = \alpha_{t,i} + 1,\ i \in I$ for the vortex point $p_t \in S$, and define

$$\Gamma^{+}_{i,\mathbf{K}} = \{2\pi(\Sigma_{t\in R}\sigma_{i,t} + 2n) \mid \boldsymbol{\sigma}_t \in \Gamma_{\mathbf{K}}(\boldsymbol{\mu}_t), R \subseteq S,\ n \in \mathbb{N} \cup \{0\}\}. \tag{8.66}$$

Theorem 8.9. *Let $h_i,\ i \in I$ be positive C^1 functions on M, and K be a compact set in M. For every point $p_t \in S$, if $\boldsymbol{\alpha}_t$ satisfy the $\mathbb{Q}$-condition for any $p_t \in S$ and $\rho_i \notin \Gamma^{+}_{i,\mathbf{K}}$ for $i \in I$, then there exists a constant C such that for any solution $u = (u_1, \cdots, u_n)$ of (8.65) in $\mathring{H}^1(M)$,*

$$|u_i(x)| \leqslant C, \quad \forall x \in \mathcal{K}, \quad i \in I.$$

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